Relative Cluster Categories and Higgs Categories

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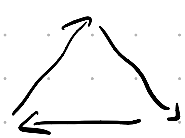
Plan :

1. Cluster algebras and additive Categorification
2. Ginzburg functors
3. Relative Cluster Categories and Higgs Categories

1. Cluster algebras and additive Categorification

Let Q be a finite quiver without loops nor 2-cycles

Example :

$A_2 : 1 \longrightarrow 2$, 

Cluster algebra $\mathcal{A}_Q$ = Commutative $\mathbb{Q}$-algebra endowed with

- A distinguished set of generators

  <u>the Cluster Variables</u>

- grouped into subsets of fixed size

  <u>the Clusters</u>

Constructed recursively from Q using interated seed mutations.

If $Q_0 = \{1, 2, \cdots n\}$, then $\mathcal{A}_Q$ is the subalgebra of $\mathbb{Q}(x_1, x_2, \cdots x_n)$ generated by the Cluster Variables.

Example:

$Q : 1 \longrightarrow 2$. $\qquad \mathcal{A}_Q \subseteq \mathbb{Q}(x_1, x_2)$

Cluster variables $= \{ x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2} \}$

More generally, a cluster algebra with coeff. is associated with an ice quiver, i.e. a finite quiver Q with a frozen subquiver F.

Example:

(1) $N = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \; a, b, c \in \mathbb{C} \right\} \subseteq SL_3(\mathbb{C})$

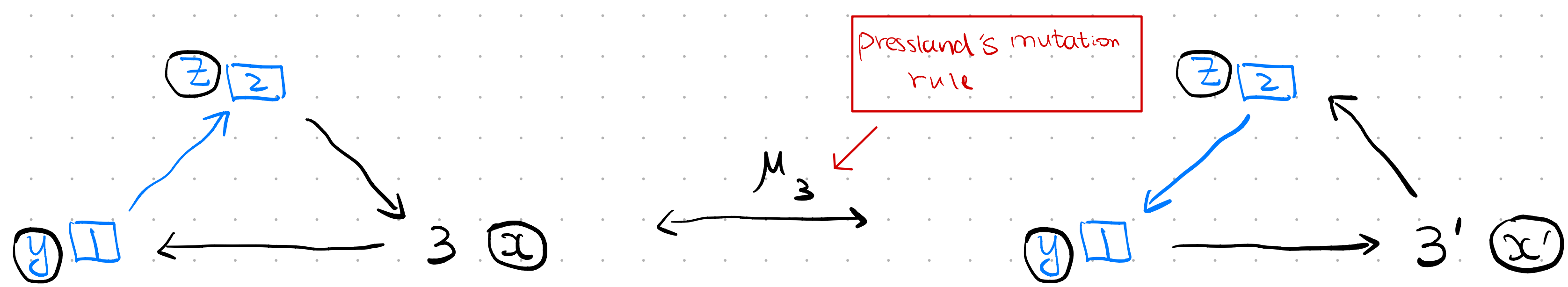
Consider the following functions on N

$$x = a : M \longmapsto \Delta_{1,2}(M), \quad x' = c : M \longmapsto \Delta_{12,13}(M)$$

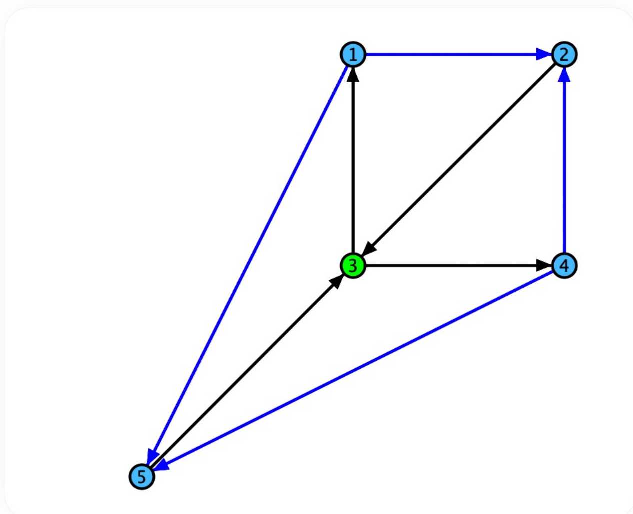
$$y = ac - b : M \longmapsto \Delta_{12,23}(M), \quad z = b : M \longmapsto \Delta_{1,3}(M)$$

$\Longrightarrow \mathbb{C}[N]$ is generated by $\{x, x', y, z\}$ with relation

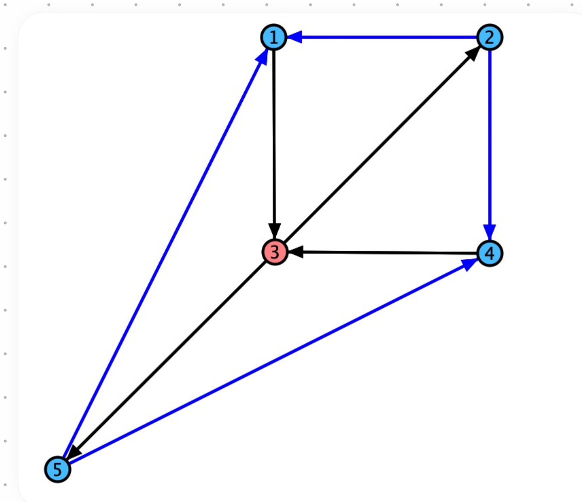
$$x x' = y + z$$



(2) Coordinate ring of $\widehat{Gr(2,4)}$



$$\xleftrightarrow{\quad \mu_3 \quad}$$



Addictive Categorification

Rough idea : triangulated category C with a decategorification map

$$CC: \mathrm{Obj}(\mathcal{C})/_\sim \longrightarrow \mathcal{A}_Q \; , \text{ such that}$$

(1) $\{$ idec. rigid object of $\mathcal{C}\}/_\sim \longrightarrow \{$Cluster Variables$\}$

(2) $\{\text{basic cluster-tilting objects}\}/\sim \longrightarrow \{\text{Clusters}\}$

Constructed by :

(1) $\mathcal{C} = \mathcal{C}_Q$: Buan-Marsh-Reineke-Reiten-Todorov (2006) for acyclic quiver.

CC for such $\mathcal{C}_Q$: Caldero-Chapoton (2006)

(2) $\mathcal{C}_{Q,W}$: Amiot (2009) for Jacobi-finite quiver with potential.

CC : Derksen-Weyman-Zelevinsky (2008), Palu (2008)

(3) $\mathcal{D}_{Q,W} \subseteq \mathcal{C}_{Q,W}$: Plamondon (2011) for any quiver with potential.

He also constructed the CC map in this setting.

(4) If Q is "Lie theoretic", $\mathcal{C}$ is a Frobenius Category (or, its stable category) contained in the category of modules over a preprojective algebra (Geiß-Leclerc-Schröer)

Remark:

(1), (2) and (3) : without coeff.

(4) : with coeff.

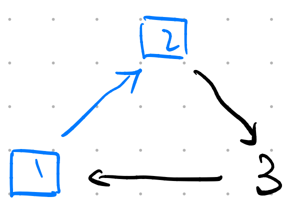
Aim:

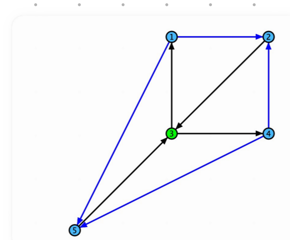
Generalize these constructions to ice quiver with potential.
Then $\mathcal{C}$ is replaced by a Frobenius extriangulated category, namely, the Higgs category.

2. Ginzburg functors

Let (Q, F) be a finite ice quiver.
Let k be a field, $k=\bar{k}$, $\mathrm{Char}(k)=0$

Example:

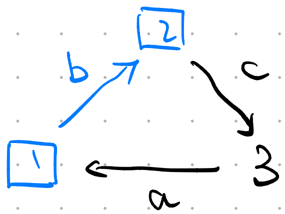




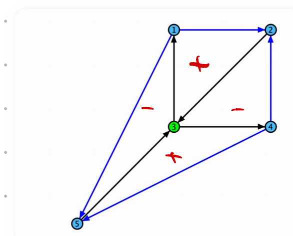
Let W be a potential on Q, i.e. an element in

$$HH_0(\widehat{kQ}) = \frac{\widehat{kQ}}{[\widehat{kQ}, \widehat{kQ}]}$$

Example:



$W = cba$



$W = \sum \circlearrowright - \sum \circlearrowleft$

Let (Q, F, W) be an ice quiver with potential.

$\Longrightarrow$ Ginzburg functor:

$$G : \underbrace{\Pi_2(\widehat{kF})}_{\text{2-CY completion of } \widehat{kF}} \longrightarrow \underbrace{\Gamma_{rel}(Q, F, W)}_{\text{relative Ginzburg functor}}$$

Both are complete dg path alg

Example.

$$
\begin{tikzcd}
\boxed{1} \arrow[loop left, "s_1"] \arrow[r, shift left, "b"] & \boxed{2} \arrow[l, shift left, "b^*"] \arrow[loop right, "s_2"]
\end{tikzcd}
\xrightarrow{\quad G \quad}
\begin{tikzcd}
 & \boxed{2} \arrow[dr, shift left, "c"] & \\
\boxed{1} \arrow[ur, "b"] \arrow[rr, shift right, "a^\vee"'] & & 3 \arrow[ll, shift right, "a"'] \arrow[ul, shift left, "c^\vee"] \arrow[loop right, "t_3"]
\end{tikzcd}
$$

$|b| = |b^*| = 0$

$|s_1| = |s_2| = -1$

$d(s_1) = -b^* b$

$d(s_2) = b b^*$

$|a| = |b| = |c| = 0$

$|a^\vee| = |c^\vee| = -1$, $|t_3| = -2$

$d(a^\vee) = \partial_a W = cb$

$d(c^\vee) = \partial_c W = ba$

$d(t_3) = cc^\vee - a^\vee a$

Remark :

1. The Ginzburg functor $G : \Pi_2(\widehat{kF}) \longrightarrow \Gamma_{rel}(Q, F, W)$ is the relative 3-CY completion of $\widehat{kF} \hookrightarrow \widehat{kQ}$ with respect to W. (Yeung '16)

2. Thus, G has a canonical left 3-CY structure in the

Sense of Brav-Dyckerhoff '19.

We say that (Q, F, W) is relative Jacobi-finite if

$$J(Q, F, W) = H^0(\Gamma_{rel}(Q, F, W)) \text{ is f.d.}$$

For simplicity, we assum that (Q, F, W) is relative Jacobi-finite.

3. Relative Cluster Categories and Higgs Categories

For a dg alg A, we denote by

$D(A) = C(A)[qis^{-1}]$, Derived category of A.

$per(A) = thick(A_A) \subseteq D(A)$, Perfect derived category of A

$pvd(A) = \{M \in D(A) \mid M|_k \in per\, k\}$, Perfectly valued derived category of A.

Def: The relative cluster category $\mathcal{C}(Q,F,W)$ is defined by

$$\mathcal{C}(Q,F,W) = \frac{\operatorname{per} \Gamma_{rel}}{\operatorname{pvd}_F(\Gamma_{rel})}$$

where

$$\operatorname{pvd}_F(\Gamma_{rel}) = \{ M \in \operatorname{pvd}(\Gamma_{rel}) \mid M|_F \simeq 0 \}$$
$$= \operatorname{thick}\langle S_i \mid i \in F_0 \rangle .$$

We define the following subcategories of $\operatorname{per} \Gamma_{rel}$

$$\operatorname{pr}^F(\Gamma_{rel}) = \{ \operatorname{Cone}(X_1 \xrightarrow{f} X_0) \mid X_i \in \operatorname{add} \Gamma_{rel},\ \begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d] & X_0 \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd} \ \forall I \in \operatorname{per}(e_F \Gamma_{rel}) \}$$

$$\operatorname{copr}^F(\Gamma_{rel}) = \{ \Sigma^{-1} \operatorname{Cone}(X^0 \xrightarrow{f} X^1) \mid X^i \in \operatorname{add} \Gamma_{rel},\ \begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ X^0 \arrow[r, "f"'] & X^1 \end{tikzcd} \ \forall P \in \operatorname{per}(e_F \Gamma_{rel}) \}$$

Proposition

The quotient functor $\pi_{rel} : \mathrm{per}\, \Gamma_{rel} \longrightarrow \mathcal{C}(Q, F, W)$ induces the following fully faithful functors

$$\pi_{rel}\big|_{\mathrm{pr}^F(\Gamma_{rel})} : \mathrm{pr}^F(\Gamma_{rel}) \hookrightarrow \mathcal{C}(Q, F, W)$$

$$\pi_{rel}\big|_{\mathrm{copr}^F(\Gamma_{rel})} : \mathrm{copr}^F(\Gamma_{rel}) \hookrightarrow \mathcal{C}(Q, F, W)$$

Definition

We define Higgs category $\mathcal{H}(Q, F, W)$ as

$$\mathcal{H}(Q, F, W) = \pi_{rel}\left(\mathrm{pr}^F(\Gamma_{rel}) \cap \mathrm{copr}^F(\Gamma_{rel})\right).$$

Theorem

(1) The Higgs category is an extension closed subcategory of $\mathcal{C}(Q,F,W)$. Hence it becomes an extriangulated category in the sense of Nakaoka-Palu.

(2) Moreover, $\mathcal{H}(Q,F,W)$ is a Frobenius extriangulated category with Proj-Inj objs $\mathcal{P} = \mathrm{add}(e_F \Gamma_{rel})$. And the relative Ginzburg alg Γ_{rel} itself is a canonical cluster-tilting object with endomorphism alg

$$\mathrm{End}_{\mathcal{H}}(\Gamma_{rel}) \cong H^0(\Gamma_{rel}) = J_{rel}(Q,F,W)$$

(3) The stable category

$$\frac{\mathcal{H}(Q,F,W)}{[\mathcal{P}]}$$

is equivalent to

$\mathcal{C}(\bar{Q}, \bar{W})$, the Amiot's cluster category,

where $(\bar{Q}, \bar{W})$ is obtained from (Q,F,W) by deleting frozen part.

Picture

$$
\begin{tikzcd}
 & \mathrm{Per}(e_F T_{rel} e_F) \arrow[r, equal] \arrow[d, hook] & \mathrm{Per}(e_F T_{rel} e_F) \arrow[d, hook] \\
\mathrm{pvd}_F(T_{rel}) \arrow[r] \arrow[d, "\wr", equal] & \mathrm{per}\, T_{rel} \arrow[r] \arrow[d] & \mathcal{C}(Q,F,W) \arrow[d] \\
\mathrm{pvd}(\Pi(\bar{Q},\bar{W})) \arrow[r] & \mathrm{per}(\Pi(\bar{Q},\bar{W})) \arrow[r] & \mathcal{C}(\bar{Q},\bar{W})
\end{tikzcd}
$$

$$\mathrm{per}\, T_{rel} \supseteq F^{rel} \xrightarrow{\sim} \mathcal{H} \subseteq \mathcal{C}(Q,F,W)$$

$$\mathrm{per}(\Pi(\bar{Q},\bar{W})) \supseteq F \xrightarrow{\sim} \mathcal{C}(\bar{Q},\bar{W})$$

Let $\mathcal{E}$: Frobenius Cat, idemp split.

$\mathcal{E}$: 2-CY ; P = Inj-proj of $\mathcal{E}$.

T : Cluster-tilting object of $\mathcal{E}$

$$
\begin{tikzcd}
\mathcal{M} \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathrm{add}\, T \arrow[r, phantom, "\subset"] & \underline{\mathcal{E}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[d, hook] & \mathcal{H}^b(\mathcal{P}) \arrow[d, hook] \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}\text{-ac}}(\mathcal{M}) \arrow[r] \arrow[d, equal] & \mathcal{H}^b(\mathcal{M}) \arrow[r] \arrow[d] & D^b(\mathcal{E}) \arrow[d] \\
0 \arrow[r] & \mathcal{H}^b_{\mathcal{E}\text{-ac}}(\mathcal{M}) \arrow[r] & \dfrac{\mathcal{H}^b(\mathcal{M})}{\mathcal{H}^b(\mathcal{P})} \arrow[r] & \underline{\mathcal{E}}
\end{tikzcd}
$$

$$\mathcal{H}^b(\mathcal{M}) \supseteq F^{rel} \xrightarrow{\sim} \mathcal{E} \subseteq D^b(\mathcal{E})$$

$$F \xrightarrow{\sim} \underline{\mathcal{E}}$$

Remarks

(1) Under some technical assumptions, we also have these constructions in the relative Jacobi-infinite setting.

(2) Let $\mathcal{H}_{dg} = \tau_{\leq 0}($ dg subcategory of $\mathcal{C}_{dg}(Q,F,W)$ the same objects as $\mathcal{H}(Q,F,W))$

$\longrightarrow$ (Frobenius) exact dg catgory

Then

$$\mathcal{D}^b_{dg}(\mathcal{H}_{dg}) \xrightarrow{\sim} \mathcal{C}_{dg}(Q,F,W) \qquad (\text{Xiao-fa Chen } ' 2022)$$

(3) Assume that $Q_0 = \{1, 2, \cdots n\} \supseteq F_0 = \{r+1, r+2, \cdots n\}$. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}(Q,F,W) \arrow[r, hook] \arrow[d] \arrow[drr, "CC"'] & \mathcal{C}(Q,F,W) \arrow[rr, "CC_{loc}"] \arrow[dd] & & \mathbb{Q}[x_1^{\pm 1}, \cdots x_r^{\pm 1}, x_{r+1}^{\pm 1}, \cdots x_n^{\pm}] \arrow[dd, "x_i \mapsto 1,\ i > r"] \\
 & & \mathbb{Q}[x_1^{\pm 1}, \cdots x_r^{\pm 1}, x_{r+1}, \cdots x_n] \arrow[ur] \arrow[dr] & \\
\underline{\mathcal{H}} \simeq \mathcal{C}(\bar{Q}, \bar{W}) \arrow[r, Rightarrow] & \mathcal{C}(\bar{Q}, \bar{W}) \arrow[rr, "\overline{CC}"'] & & \mathbb{Q}[x_1^{\pm 1}, \cdots x_r^{\pm 1}]
\end{tikzcd}$$

Thanks !