A conjectural relation between crystals and WKB analysis

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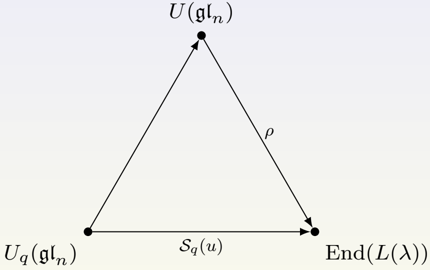
Main conjecture (quantum version)

Following Iwaki and Nakanishi, *WKB analysis, cluster algebra.*

- Associated to any $u \in \mathfrak{h}_{\text{reg}}$ and $\rho : U(\mathfrak{gl}_n) \to L(\lambda)$, we construct a representation $\mathcal{S}_q(u)$ of $U_q(\mathfrak{gl}_n)$ on the same space $L(\lambda)$.



Conjecture

The $q \to 0$ leading asymptotics of the image of the canonical basis under the map $\mathcal{S}_q(u)$ correspond to an eigenbasis $E(u;\lambda)$ of the action of the shift of argument subalgebra $\mathcal{A}(u) \subset U(\mathfrak{gl}_n)$ on $L(\lambda)$.

Main motivation

- "Solve" the linear system on z-plane $\frac{dF}{dz} = A(z) \cdot F$.
- Consider $\frac{dF}{dz} = \left(\frac{A}{z} + \frac{B}{z-1} \right) \cdot F$, then

$$\begin{cases} F(z) \sim z^{[A]} C_0, & as\ z \to 0, \\ F(z) \sim (z-1)^{[B]} C_1, & as\ z \to 1. \end{cases}$$

Connection problem: knowing C_0 to write down explicitly C_1.
Monodromy in Ishibashi's talk.

Example (2 × 2 case)

$$\begin{aligned} {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b,a+b+1-c;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a,c-b,1+c-a-b;1-z) \end{aligned}$$

The Stokes phenomenon

- Consider $\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right) \cdot F$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

There exits two opposite sectoral regions $\mathrm{Sect}_\pm$

$$\begin{cases} F(z) \sim e^{-u/z} z^{[A]} C_0, \ as \ z \to 0 \ in \ \mathrm{Sect}_+, \\ F(z) \sim e^{-u/z} z^{[A]} C_1, \ as \ z \to 0 \ in \ \mathrm{Sect}_-. \end{cases}$$

Connection problem: knowing C_0 to write down C_1.

- Normalize $C_0 = 1$, then $S_+(u, A) := C_1$. Similarly we introduce $S_-(u, A)$. The so called Stokes matrices.

Example (2 × 2 case)

Set ${}_1F_1(\alpha, \beta; z) = \sum_{n=0}^\infty \frac{\alpha^{(n)} z^{-n}}{\beta^{(n)} n!}$, *where* $\alpha^{(n)} = \alpha \cdots (\alpha + n - 1)$,

$${}_1F_1(\alpha, \beta; z) \sim \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (-z)^\alpha (1 + O(z)) + \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^{-\frac{1}{z}} z^{-\alpha+\beta} (1 + O(z))$$

Example: 2 by 2

- We consider $\frac{dF}{dz} = \left(\frac{1}{z^2} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} t_1 & a \\ b & t_2 \end{pmatrix} \right) F.$

Then

$$S_+(A,u) = \begin{pmatrix} e^{t_1} & \frac{a((u_2-u_1))^{t_1-t_2}}{\Gamma(1-\lambda_1+t_1)\Gamma(1-\lambda_2+t_1)} \\ 0 & e^{t_2} \end{pmatrix}$$

Here λ_1, λ_2 are eigenvalues of $\begin{pmatrix} t_1 & a \\ b & t_2 \end{pmatrix}$.

- Black box:

$$(u,A) \mapsto \Big(S_+(u,A), S_-(u,A) \Big) \in B_+ \times B_-.$$

Here the entries of $S_\pm(u,A)$ are "new" transcendental functions on the linear space (u,A).

Idea

• The aim of our project is to first develop algebraic understanding of various aspects of the Stokes phenomenon (solving the connection problem of meromorphic ODEs).

• By pursuing the heuristics of the algebraic understanding, we are led to rather interesting conjectures:

- a characterization of the Stokes phenomenon in the WKB approximation via the theory of crystal basis (Kashiwara and Lusztig);
- an algebraic characterization of confluent hypergeometric type equations which have soliton solutions (Jimbo-Miwa-Ueno).

Part I

The quantum setting: WKB approximation of Stokes matrices and $\mathfrak{gl}_n$-crystals

A $\mathfrak{gl}_n$-crystal is a finite set which models a weight basis for a representation of $\mathfrak{gl}_n$, and Kashiwara/crystal operators indicate the leading order behaviour of the simple root vectors on the basis under the crystal limit $q \to 0$ in quantum group $U_q(\mathfrak{gl}_n)$.

Stokes matrices of ODEs in the quantum case

- $U(\mathfrak{gl}_n)$: generator $\{E_{ij}\}$, relation $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$.
- $n \times n$ matrix $T = (T_{ij})$ with entries valued in $U(\mathfrak{gl}_n)$

$$T_{ij} = E_{ij}, \qquad for\ 1 \leq i, j \leq n.$$

For any $u \in \mathfrak{h}_{\mathrm{reg}}$ n by n diagonal matrices with distinct eigenvalues and a representation $L(\lambda)$, consider

$$\frac{1}{h}\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F,$$

for a $n \times n$ matrix function $F(z)$ with entries in $\mathrm{End}(L(\lambda))$.

- Stokes matrices $S_{h\pm}(u) = (s_{ij}^{(\pm)})$, with entries $s_{ij}^{(\pm)}$ in $\mathrm{End}(L(\lambda))$, i.e., $S_{h\pm} \in \mathrm{End}(L(\lambda)) \otimes \mathrm{End}(\mathbb{C}^n)$.

Representations of quantum group from Stokes matrices

Theorem (Xu)

For any fixed $h \in \mathbb{C}^*$ *and* $u \in \mathfrak{h}_{\mathrm{reg}}$, *the map (with* $q = e^{\pi i h}$*)*

$$\mathcal{S}_q(u) : U_q(\mathfrak{gl}_n) \to \mathrm{End}(L(\lambda)) \ ; \ e_i \mapsto S_{h+}(u)_{i,i+1}, \ f_i \mapsto S_{h-}(u)_{i+1,i}$$

defines a representation of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_n)$ *on the vector space* $L(\lambda)$. *Here* $U_q(\mathfrak{gl}_n)$ *is an associative algebra with generators* $q^{\pi\sqrt{-1}h_i}, e_j, f_j$, $1 \leq j \leq n-1, 1 \leq i \leq n$ *and:*

- *for each* $1 \leq i \leq n$, $1 \leq j \leq n-1$,

$$q^{h_i} e_j q^{-h_i} = q^{\delta_{ij}} q^{-\delta_{i,j+1}} e_j, \quad q^{h_i} f_j q^{-h_i} = q^{-\delta_{ij}} q^{\delta_{i,j+1}} f_j;$$

- *for each* $1 \leq i, j \leq n-1$,

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i - h_{i+1}} - q^{-h_i + h_{i+1}}}{q - q^{-1}};$$

- *for* $|i-j| = 1$,

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0,$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0,$$

and for $|i-j| \neq 1$, $[e_i, e_j] = 0 = [f_i, f_j]$.

WKB analysis and crystals

• WKB analysis: $\frac{1}{h^2}\frac{d^2\psi}{dz^2} - V\psi = 0$ (Wentzel-Kramers-Brillouin).
• The algebraic characterization of the $h \to -i\infty$ asymptotics of $S_{h\pm}(u) \in \mathrm{End}(L(\lambda)) \otimes \mathrm{End}(\mathbb{C}^n)$ of $\frac{dF}{dz} = h\left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F$.

Conjecture (Xu, proved in a special case)

For any $u \in \mathfrak{h}_{\mathrm{reg}}$, there exists a "canonical" basis $\{v_I(u)\}$ of $L(\lambda)$, operators $\tilde{e}_k(u)$ and $\tilde{f}_k(u)$ for $k = 1, ..., n-1$ such that there exists constants c, c'

$$\lim_{q=e^{\pi i h}\to 0} q^c s^{(+)}_{k,k+1}(u) \cdot v_I(u) = \tilde{e}_k(v_I(u)),$$

$$\lim_{q=e^{\pi i h}\to 0} q^{c'} s^{(-)}_{k+1,k}(u) \cdot v_I(u) = \tilde{f}_k(v_I(u)).$$

Furthermore, the datum $(\{v_I(u)\}, \tilde{e}_k(u), \tilde{f}_k(u))$ is a $\mathfrak{gl}_n-$crystal.

• Crystal limit $q \to 0$ in $U_q(\mathfrak{gl}_n)$. • $\{v_I(u)\}$ eigenbasis of the shfit of argument subalgebra (Feigin-Frenkel-Rybnikov)

Part II

The semiclassical setting: WKB approximation of Stokes matrices and cluster algebras

The geometry in the WKB approximation

• WKB analysis: $\frac{1}{\hbar^2}\frac{d^2\psi}{dz^2} - V(z)\psi = 0$. The $\hbar \to 0$ hehavior is related to the Stokes graphs on z-plane determined by $V(x)$. (many viewpoints: Voros, Delabaere-Dillinger-Pham, Gaiotto-Moore-Neitzke, Iwaki-Nakanishi, Bridgeland-Smith, Aoki-Honda-Kawai- Koike-Nishikawa-Sasaki-Shudo-Takei ...)

• Set ε a small real parameter, consider $\varepsilon\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right)F$.

• Problem: between the asymptotics of $S(u, A; \varepsilon)$ as $\varepsilon \to 0$ and the geometry of the spectral curve

$$\det\left[\lambda - \left(\frac{u}{z^2} + \frac{A}{z}\right)\right] = 0.$$

Example (2 by 2 case)

$$S(A, u; \varepsilon) = \begin{pmatrix} e^{\frac{t_1}{\varepsilon}} & \frac{a\left(\frac{i(u_2-u_1)}{\varepsilon}\right)^{\frac{t_1-t_2}{2\pi i\varepsilon}}}{\Gamma(1-\frac{\lambda_1-t_1}{2\pi i\varepsilon})\Gamma(1-\frac{\lambda_2-t_1}{2\pi i})} \\ 0 & e^{\frac{t_2}{\varepsilon}} \end{pmatrix} \sim \begin{pmatrix} e^{\frac{t_1}{\varepsilon}} & e^{\frac{\max(\lambda_1,\lambda_2)}{\varepsilon}} \\ 0 & e^{\frac{t_2}{\varepsilon}} \end{pmatrix}$$

A fake analysis

- solutions of $\varepsilon \frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z} \right) F$ have the WKB type expansion,

$$F(z, \varepsilon) \sim e^{\frac{\omega(z)}{\varepsilon}} (v(z) + \sum \phi_k \varepsilon^k), \quad as \ \ \varepsilon \to 0, \tag{1}$$

where $\omega(z) = \mathrm{diag}(\omega_1, ..., \omega_n)$ is a diagonal matrix, and $v(z)$ is a $n \times n$ matrix with columns v_k.

- Leading asymptotics: $d\omega_k/dz$ and $v_k(z)$ satisfy

$$\frac{d\omega_k}{dz} \cdot v_k(z) = \left(\frac{u}{z^2} + \frac{A}{z} \right) \cdot v_k(z),$$

i.e., $\omega(z) = \mathrm{diag}(\int_{p_1}^{z} \lambda dt, ..., \int_{p_n}^{z} \lambda dt)$.

- Stokes phenomenon takes place in the asymptotics $\varepsilon \to 0$ in a way that the approximation in (1) is not uniformly valid w.r.t z around 0. Then the asymptotics of Stokes matrices as $\varepsilon \to 0$ should be encoded by certain periods on the spectral curve.

Main conjecture (semiclassical case)

- A coordinate chart on the space of upper triangular matrices from cluster algebra theory: $\{\Delta_i^{(j)}\}_{1\leq i\leq j\leq n}$ the minor formed by intersecting columns $i-j+1$ to i and the first j rows.
- Spectral curve $\Gamma(u,A)$ of genus $\frac{(n-1)(n-2)}{2}$

$$\det\left[\lambda - \left(\frac{u}{z^2}+\frac{A}{z}\right)\right] = 0.$$

Conjecture (Alekseev-X-Zhou, proved in a special case)

For generic u and A, there exists a canonical set of cycles $\{C_i^{(k)}\}_{1\leq i\leq k\leq n}$ on $\Gamma(u,A)$ such that

$$\lim_{\varepsilon\to 0}\left(\varepsilon\log\left|\Delta_i^{(k)}(S(A,u,\varepsilon))\right|\right) = \int_{C_i^{(k)}(u,A)}\omega.$$

- analytic difficult (left); • the discrete choice of cycles (right).
- K. Takasaki, *Dual isomonodromic problem, Whitham equation*

Soliton solutions

- From the representation of $U_q(\mathfrak{gl}_n)$ at roots of unity

Conjecture

The equation $\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right)F$ *has a soliton solution of the form*

$$F(z) = (1 + H_1 z^{-1} + \cdots + H_m z^{-m}) \cdot e^{-u/z} z^{[A]},$$

if and only if the set of periods $\{\int_{C_i^{(k)}} \omega\}_{1 \leq k \leq n}$ *constitutes a Gelfand-Tsetlin pattern, i.e.,*

$$\int_{C_i^{(k)}} \omega - \int_{C_{i-1}^{(k+1)}} \omega \in \mathbb{Z}_+, \quad \int_{C_i^{(k+1)}} \omega - \int_{C_i^{(k)}} \omega \in \mathbb{Z}_+.$$

- The name of soliton is after Jimbo-Miwa-Ueno.

Part III

The technical tool: expression of Stokes matrices via the asymptotics of isomonodromy deformation equations

Isomonodromy deformation equations

- Jimbo-Miwa-Môri-Sato, Jimbo-Miwa-Ueno ...

Set $u = \mathrm{diag}(u_1, ..., u_n)$. The system of equations for $F(z,u) \in GL(n)$ is compatible

$$\begin{cases} \frac{\partial F}{\partial z} = (\frac{u}{z^2} + \frac{A(u)}{z})F, \\ \frac{\partial F}{\partial u_i} = (\frac{E_{ii}}{z} + V_i(u))F, \end{cases}$$

if and only if

$$\frac{\partial A(u)}{\partial u_i} = [A(u), V_i(u)], \text{ for all } i = 1, ..., n.$$

Here V_i is off-diagonal determined by $[u, V_i] = [E_{ii}, A(u)]$.

- Isomonodromy: $S_\pm(u, A(u))$ is locally constant.
- Inverse scattering: Stokes matrices are scattering data.
- As $n = 3$, Painlevé VI function, connection formula (Jimbo).

Asymptotic solution to Riemann-Hilbert problem

$$\text{Scattering}: \ (u, A(u)) \mapsto S_+(u, A(u)) = \begin{pmatrix} e^{t_1} & \frac{a((u_2-u_1))^{t_1-t_2}}{\Gamma(1-\lambda_1+t_1)\Gamma(1-\lambda_2+t_1)} \\ 0 & e^{t_2} \end{pmatrix}.$$

Theorem (Xu)

The sub-diagonals of $S_+(u, A(u))$ are

$$S_{k,k+1} = \sum_{i=1}^{k} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(\lambda_l^{(k)} - \lambda_i^{(k)})}{\prod_{l=1}^{k+1} \Gamma(\lambda_l^{(k+1)} - \lambda_i^{(k)})} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(\lambda_l^{(k)} - \lambda_i^{(k)})}{\prod_{l=1}^{k-1} \Gamma(\lambda_l^{(k-1)} - \lambda_i^{(k)})} \cdot m_i^{(k)}, \ k = 1, ..., n-1$$

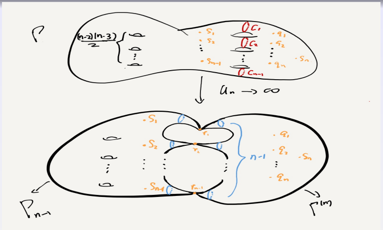
where $\{\lambda_i^{(k)}\}_{i=1,\dots,k}$ are the eigenvalues of the k-th principal submatrix of $A_\infty \in \mathfrak{gl}_n$, and A_∞ is such that

$$A(u) = \Big(\overrightarrow{\prod_k}(\frac{u_k}{u_{k+1}})^{\frac{\delta_k(A_\infty)}{2\pi\iota}}\Big)^{-1} \cdot A_\infty \cdot \Big(\overrightarrow{\prod_k}(\frac{u_k}{u_{k+1}})^{\frac{\delta_k(A_\infty)}{2\pi\iota}}\Big) + O\Big(\frac{1}{u_2-u_1}\Big),$$

as $\frac{u_k-u_{k-1}}{u_{k+1}-u_k} \to 0$ for all $k = 1, ..., n-1$. The other entries are given by explicit algebraic combinations of the sub-diagonal ones.

- By the isomonodromy property, the leading term gives a

Proof of the second conjecture near a limiting point



- Fixing $u_1, ..., u_{n-1}$, let $u_n \to \infty$,
- As $u_1 << \cdots << u_n$, set $\{V_i^{(k)}\}_{1 \leq i \leq k \leq n}$ the vanishing cycles.

Theorem (Alekseev-X-Zhou)

As u_k/u_{k+1} sufficiently small, the cycles $C_i^{(k)} = \sum_{j=1}^{i} V_{k-j}^{(k)}$ on $\Gamma(u, A)$ such that

$$\lim_{\varepsilon \to 0} \left(\varepsilon \lim_{u_1 \ll \cdots \ll u_n} (\log |\Delta_i^{(k)}(S(A, u; \varepsilon))|) \right) = \lim_{u_1 \ll \cdots \ll u_n} \int_{C_i^{(k)}} \omega$$

- Conjecture $\lim_{\varepsilon \to 0} \left(\varepsilon (\log |\Delta_i^{(k)}(S(A, u; \varepsilon))|) \right) = \int_{C_i^{(k)}} \omega.$

Thank you very much!