2022. 9. 22

Trends in Cluster Algebras 2022

Wilson lines & the $\mathcal{A}=\mathcal{U}$ problem for the moduli spaces of G-local systems

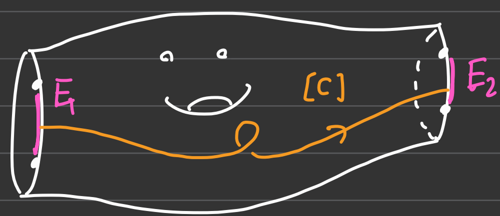
joint work w/ Hironori Oya & Linhui Shen

(arXiv: 2202.03168)

Wilson loops

$$\rho_{[\gamma]} : \mathcal{L}oc_{G,\Sigma} \longrightarrow [G/\mathrm{Ad}G] \xrightarrow{tr_V} \mathbb{C}$$

Wilson lines



$$g_{[c]} : \mathcal{A}^{\times}_{G,\Sigma} \longrightarrow G \xrightarrow{c^{V}_{f,v}} \mathbb{C}$$

§1 Statement.

$$\begin{cases} \Sigma : \text{a marked surface} \quad (\emptyset \neq \mathbb{M} \subset \partial\Sigma \text{ fin. set}) \\ G : \text{a simply-connected semisimple alg. group}/\mathbb{C} \end{cases}$$

[FG'06] §2

$\rightsquigarrow$ $\mathcal{A}_{G,\Sigma}$: moduli space of decorated twisted G-local systems on Σ (stack)

open $\cup$

$\mathcal{A}^{\times}_{G,\Sigma}$ (variety)

[FG'06, Le'16, GS'19] §3

$\rightsquigarrow$ $\mathscr{A}_{\mathfrak{g},\Sigma} \subseteq \mathscr{U}_{\mathfrak{g},\Sigma} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma})$

cluster alg — upper cluster alg. — field of rational functions

<u>Main Theorem</u> (I.-Oya-Shen'22) §4

If $|\mathbb{M}| \geq 2$ & G has a minuscule rep.,

then $\boxed{\mathscr{A}_{\mathfrak{g},\Sigma} = \mathscr{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}).}$

<u>Cor</u> The quantum CA $\mathcal{A}^q_{\mathfrak{g},\Sigma}$ gives a non-comm. deformation of $\mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$.

<u>Preceding results</u>:

- Muller'13: $\mathcal{A} = \mathcal{U}$ for $\mathfrak{g} = \mathfrak{sl}_2$, $|\mathbb{M}| \geq 2$
- Canakci-Lee-Schiffler'15: $\mathfrak{g} = \mathfrak{sl}_2$, $|\mathbb{M}| = 1$.
- Shen-Weng'21: $\mathfrak{g}$: arbitrary, Σ = polygon
- When Σ has punctures, it tends to hold

$$\mathcal{A}_{\mathfrak{g},\Sigma} \neq \mathcal{U}_{\mathfrak{g},\Sigma} \neq \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$$

[BFZ'05, Ladkani'13, Moon-Wong'22]

New technique: generation of $\mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$ by matrix coefficients of Wilson lines.

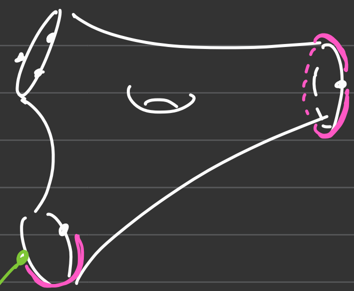
(another proof for $\mathfrak{g} = \mathfrak{sl}_2$)

[Plan]

§2. Wilson lines on the moduli space $\mathcal{A}^{\times}_{G,\Sigma}$

1/ Topology side

A marked surface $(\Sigma, \mathbb{M})$ is a compact ori. surface Σ equipped w/ a fin. set $\mathbb{M} \subset \Sigma$ of "marked pts".



Today Assume $\mathbb{M} \subset \partial\Sigma$

$\mathbb{B} := \{ \text{conn. comp's of } \partial\Sigma \setminus \mathbb{M} \}$

(boundary intervals)

2/ Rep theory side

G: a $\underbrace{\text{semisimple}}_{\text{simply-connected (i.e. maximal center)}}$ alg. group $/\mathbb{C}$ (type $A \sim G$)

Choose: $G \supset B^{\pm} \supset H$ $\quad U^{\pm} := [B^{\pm}, B^{\pm}]$

$B^{\pm}$: opposite Borels, H: Cartan, $U^{\pm}$: unipotent

<u>Def</u> $\mathcal{A}_G := G/U^+ \ni g.[U^+]$: decorated flag variety

$$\text{H-b'dl} \downarrow$$

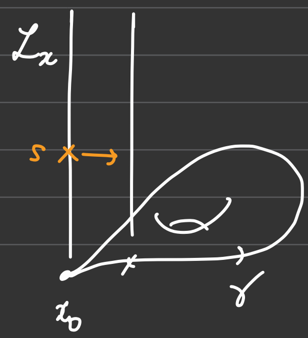
$\mathcal{B}_G := G/B^+$: flag variety

<u>Example</u>
$$\mathcal{A}_{SL_{N+1}} = \left\{ \begin{array}{l} \overbrace{0 \subset F_1 \subset \cdots \subset F_N \subset \mathbb{C}^{N+1}, \ \dim F_i = i}^{\in\, \mathcal{B}_{SL_N}} \\ f_i \in \Lambda^i F_i \simeq \mathbb{C} \ , \ i = 1, \cdots, N \end{array} \right\}$$

3/ Local systems.

Recall that a G-local system on Σ

is $\left\{ \begin{array}{l} \triangleright \text{ a principal } G\text{-b'dl } \mathcal{L} \text{ \& a flat conn. } \nabla \\ \qquad \text{or} \\ \triangleright \text{ a group hom. } \rho : \pi_1(\Sigma, x_0) \longrightarrow G \end{array} \right.$ $\Big\updownarrow$ 1:1 up to isom. / conj.



(iii) Moduli space $\mathcal{A}_{G,\Sigma}$ [Fock–Goncharov '06]

$\mathcal{L}$: a G-loc. sys. on Σ $\rightsquigarrow$ $\mathcal{L}_{\mathcal{A}} := \mathcal{L} \times_G \mathcal{A}_G$

A <u>decoration</u> of $\mathcal{L}$ is a flat section α of $\mathcal{L}_{\mathcal{A}}$ defined near $\mathbb{M}$

$\mathcal{A}_{G,\Sigma}$ " := " moduli space of decorated G-local systems $(\mathcal{L}, \alpha)$.

Slight modification : twistings (for positivity)

▷ $T'\Sigma := T\Sigma \setminus (0\text{-section})$: punctured tangent bundle.

<u>Twisted</u> G-local systems $\mathcal{L}$ on Σ :

defined on $T'\Sigma$ s.t. $\rho(\text{fiber}) = s_G$

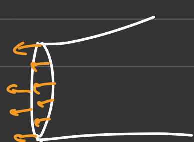
a special central element

 $= s_G \cdot \longrightarrow$

<u>e.g.</u> $s_{SL_N} = (-1)^{N-1}$

▷ Lift $\partial\Sigma \hookrightarrow T'\Sigma$ by <u>outward</u> tangent vectors.

$x \longmapsto (x, v_{\mathrm{out}}(x))$

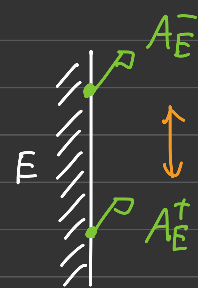


<u>Def</u> $\mathcal{A}_{G,\Sigma}$:= moduli space of decorated, twisted G-local systems on Σ.

<u>Remark</u> As a quotient stack:

$$\mathcal{A}_{G,\Sigma} = [A_{G,\Sigma}/G], \qquad A_{G,\Sigma} \simeq \mathrm{Hom}^{tw}(\pi_1(T'\Sigma), G) \times \mathcal{A}_G^{\mathbb{M}}$$

$\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$: open subspace obtained by imposing the <u>"genericity"</u> on the pairs of decorated flags assigned to each $E \in \mathbb{B}$



<u>Fact</u> $\mathcal{A}^{\times}_{G,\Sigma}$ is a representable stack.

<u>i.e.</u> can be viewed as a variety.

★ Want to study $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \mathcal{O}(A^{\times}_{G,\Sigma})^G$,

⑪ Wilson lines

Fundamental groupoid Fix $x_E \in E$, $E \in B$

Let $\Pi_1(T'\Sigma, B^{\pm})$ be the groupoid,

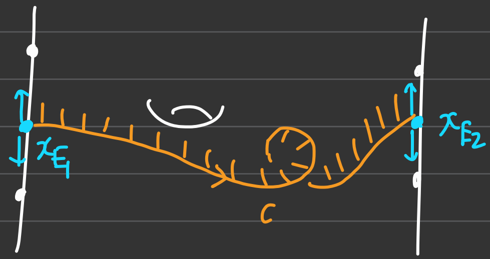
where obj. $E^{\pm} := (x_E, \pm v_{ori}(x_E)) \in \partial(T'\Sigma)$

positive v.f. along $\partial\Sigma$

morph. $[c] : E_1^{\varepsilon_1} \longrightarrow E_2^{\varepsilon_2}$.

homotopy classes of paths in $T'\Sigma$

("framed arc classes")



$[c] : E_1^{-} \longrightarrow E_2^{+}$

General idea :

$\mathcal{L}$: a twisted G-local system on Σ

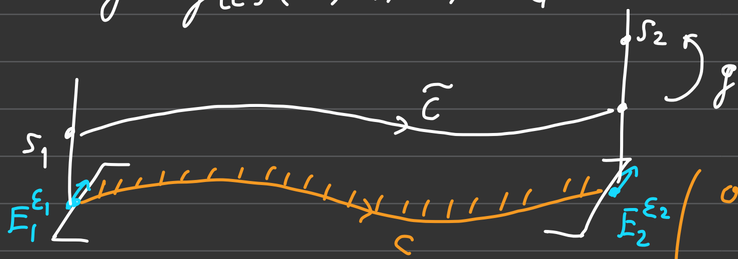
(defined on $T'\Sigma$)

$[c] : E_1^{\varepsilon_1} \longrightarrow E_2^{\varepsilon_2}$

Choose a local trivialization s_j of $\mathcal{L}$ at $E_j^{\varepsilon_j}$.

Then $\exists!\ g = g_{[c]}(\mathcal{L}; s_1, s_2) \in G$

s.t.



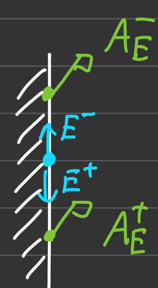
(open analogue of monodromy)

$[\mathcal{L}, \alpha] \in \mathcal{A}^{\times}_{G,\Sigma}$

$\rightsquigarrow$ "pinnings" [GS'19]

$$\begin{cases} p_{E^-} := (A_{E^-}, \underset{\sim}{B_{E^+}}) & \text{at } E^- \\ p_{E^+} := (A_{E^+}, \underset{\sim}{B_{E^-}}) & \text{at } E^+ \end{cases}$$

underlying flag



<u>Lemma</u> $G \curvearrowright \{(A_1, B_2) \in \mathcal{A}_G \times \mathcal{B}_G \text{ generic}\}$

is free & transitive.

⇝ pinnings determine local trivializations !

Fix $p_{std} := ([U^+], B^-) \in (RHS)$

<u>Def</u> The <u>Wilson line</u> along $[c] : E_1^{\varepsilon_1} \longrightarrow E_2^{\varepsilon_2}$

is defined by $g_{[c]}([\mathcal{L}, \alpha]) := g_{[c]}(\mathcal{L}; p_{E_1^{\varepsilon_1}}, p_{E_2^{\varepsilon_2}}^*)$.

Here $(g.p_{std})^* := g\underbrace{\overline{w_0}}_{\text{a lift} \in N_G(H) \text{ of } w_0 \in W(G)}^{-1}. p_{std}$: opposite pinning

e.g. $\overline{w_0} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ (-1)^n & & & \end{pmatrix} \in SL_n$

<u>Prop</u> It defines

a morphism $g_{[c]} : \mathcal{A}_{G,\Sigma}^\times \longrightarrow G$ of alg. var's.

- "Twisted" Wilson lines $g^{tw}_{[C]} := g_{[C]} \overline{w_0}^{-1}$

are multiplicative : $\boxed{g^{tw}_{[C_1] * [C_2]} = g^{tw}_{[C_1]} \cdot g^{tw}_{[C_2]}}$

<u>Theorem</u> For any unpunctured marked surface Σ, Wilson lines give a closed embedding

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \hookrightarrow & \mathrm{Hom}(\Pi_1(T'\Sigma, \mathbb{B}^{\pm}), G) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\mathcal{L}, \alpha] & \longmapsto & g^{tw}_{\bullet}([\mathcal{L}, \alpha]) \end{array}$$

$\leftarrow$ affine variety

<u>Cor</u> $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is generated by the matrix coefficients of (twisted) Wilson lines.

<u>cf.</u> Peter-Weyl : $\bigoplus_{\lambda} (V(\lambda)^* \otimes V(\lambda)) \xrightarrow{\sim} \mathcal{O}(G)$

$$(f, v) \longmapsto c^{\lambda}_{f,v}(g) := \langle f, g.v \rangle$$

§3. Cluster algebras $\mathcal{A}_{\mathfrak{g},\Sigma}$

History

Type A_n, special word : Fock-Goncharov '06

Type $B_n \sim D_n + G_2$, special word : Le '16

General case : Goncharov-Shen '19

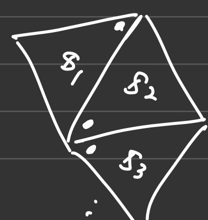
Construction

$\boldsymbol{\Delta} = (\Delta, m_\Delta, s_\Delta)$: a decorated triangulation,

i.e. Δ : an ideal triangulation of Σ

$m_\Delta(T) \in \mathbb{M}$: choice of a vertex, $\forall T \in t(\Delta)$

$s_\Delta(T)$: a reduced word of $w_0 \in W(G)$, $\forall T \in t(\Delta)$



① Δ gives a decomposition

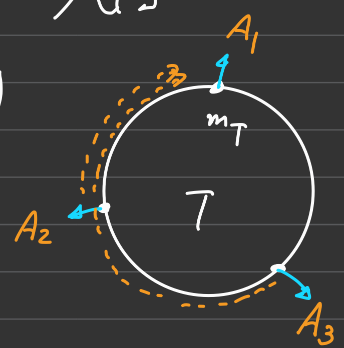
$$\psi_\Delta : \mathcal{A}_{G,\Sigma} \longrightarrow \prod_{T \in t(\Delta)} \mathcal{A}_{G,T}$$

by restrictions of local systems & sections

② $m_\Delta(T) =: m_T$ gives an isomorphism

$$f_{m_T} : \mathcal{A}_{G,T} \longrightarrow \mathrm{Conf}_3 \mathcal{A}_G = [\mathcal{A}_G^3 / G]$$

$\left(f_{m'} \circ f_m^{-1} = \text{twisted cyclic shift} \right)$



③ $s_\Delta(T) =: s_T$ gives a cluster

$$A_i^{s_T} \in \mathcal{O}(\mathrm{Conf}_3 \mathcal{A}_G) \quad , \; i \in I(s_T)$$

$$\| $$

$$\bigoplus_{\lambda,\mu,\nu \in P_+} (V(\lambda) \otimes V(\mu) \otimes V(\nu))^G$$

$\rightsquigarrow$ a seed $(\varepsilon^\Delta, \{ \psi_\Delta^* f_{m_T}^* A_i^{s_T} \})$ in $\mathcal{K}(\mathcal{A}_{G,\Sigma})$

Theorem (Goncharov–Shen '19)

These seeds are mutation-equivalent to each other.

$\rightsquigarrow$ $\exists$ canonically defined CAs

$$\boxed{\mathcal{A}_{\mathfrak{g},\Sigma} \subset \mathcal{U}_{\mathfrak{g},\Sigma} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma}).}$$

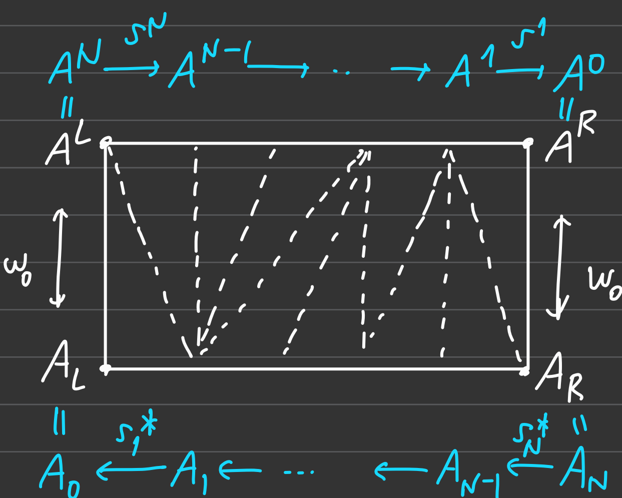
$$\cup$$

$$\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$$

Clusters on $\mathrm{Conf}_4 \mathcal{A}_G$

$\mathbf{s}^\circ = (s^1, \dots, s^N)$
$\mathbf{s}_\circ = (s_1, \dots, s_N)$ } red. words of w_0

$\rightsquigarrow$ double red. word of (w_0, w_0)



$$\left(V(\varpi_s) \otimes V(\varpi_s^*)\right)^G$$

$$\rotatebox{90}{$\in$}$$

To each $\begin{array}{c} A^k \\ \vdots \\ A_\ell \end{array}$, assign the function $\Delta_s(A^k, A_\ell)$.

§4. Proof of the main theorem.

Main Theorem (I.–Oya–Shen '22)

$G \neq E_8, F_4, G_2$

If $|\mathbb{M}| \geq 2$ & G has a minuscule rep.,

then $\boxed{\mathcal{A}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}).}$ as $\mathbb{C}$-alg's

Step 1: $\mathcal{U}_{\mathfrak{g},\Sigma} \cong \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ by

a "covering" argument.

generic along $E' \in e(\Delta) \setminus \{E\}$

$$\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \bigcap_{E \in e(\Delta)} \mathcal{O}(\mathcal{A}^{\Delta;E}_{G,\Sigma}) = \bigcap_{E} \mathcal{U}^{\Delta;E}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma}$$

3-gon, 4-gon cases [BFZ'05, SW'21]

upper bound thm

Step 2: Show $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \subseteq \mathcal{A}_{\mathfrak{g},\Sigma}$, as follows

Recall: gen'ed by matrix coeff's of Wilson lines.

Claim: Special kind of matrix coefficients of Wilson lines are cluster variables (up to frozen).

1) Generalized minors. [BFZ'07]

$w, w' \in W(G)$, λ : dominant weight

$\rightsquigarrow \Delta_{w\lambda, w'\lambda}(g) := \langle \overline{w}.f_{\lambda^*}, g\overline{w'}.v_\lambda \rangle_{V_\lambda}$

Lem $\mathcal{O}(G)$ is generated by generalized minors if G admits a minuscule rep. (i.e. $\neq E_8, F_4, G_2$.)

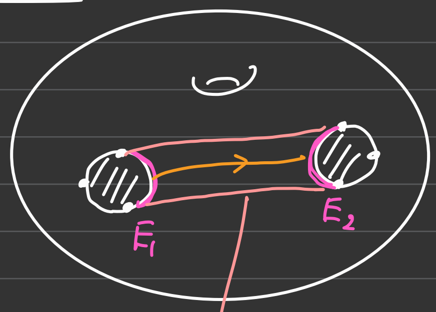
2) Simple Wilson lines

$[c] : E_1^- \longrightarrow E_2^-$ w/ • $E_1 \neq E_2$

no self-int. • "standard" framing

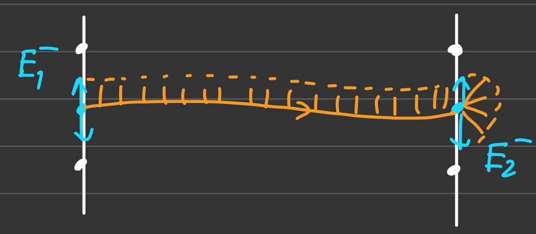
$\Rightarrow g_{[c]}$ is called a simple Wilson line.

$E_1 \neq E_2$



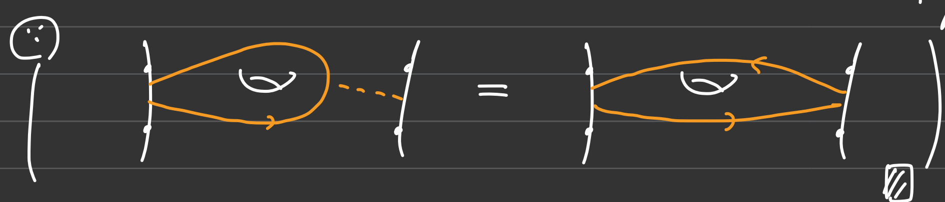
$\exists$ strip nbd $B_{[c]}$

std. framing



cf. "good lift" of Costantino–Lê '19

Lem If Σ has ≥ 2 marked points, then $\pi_1(T'\Sigma, \mathbb{B}^{\Sigma})$ is generated by simple classes & fiber loops.



Combining two lemmas:

Cor If $G \neq E_8, F_4, G_2$ & $|\mathbb{M}| \geq 2$, then $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is gen'd by generalized minors of simple Wilson lines.

Proposition For a simple class $[c] : E_1^- \longrightarrow E_2^-$,

GS variable on $\mathrm{Conf}_4 \mathcal{A}_G$

$$\Delta_{w\varpi_s, w'\varpi_s}(g_{[c]}) = \frac{\Delta_s(A^k, A_\ell)}{\text{frozen var's on } E_1 \,\&\, E_2}$$

☺ (BFZ variables $\propto$ GS variables

$H^2 \times G^{w_0, w_0} \cong \mathcal{A}^{\times}_{G, [\,]}$ $\square$)

Then we get $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \subseteq \mathscr{A}_{\mathfrak{g},\Sigma}$

(under the assumption above).

$$\mathscr{A}_{\mathfrak{g},\Sigma} \underset{\substack{\uparrow \\ \text{Laurent phenomenon}}}{\subseteq} \mathscr{U}_{\mathfrak{g},\Sigma} \underset{\substack{\uparrow \\ \text{alg. geom.}}}{=} \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \underset{\substack{\uparrow \\ \text{Wilson lines}}}{\subseteq} \mathscr{A}_{\mathfrak{g},\Sigma}$$ $\blacksquare$