Earthquake theorem for cluster algebras of finite type

Shunsuke Kano (Tohoku Univ, RACMaS)

joint work w/

Takeru Asaka (Univ of Tokyo)

Tsukasa Ishibashi (Tohoku Univ)

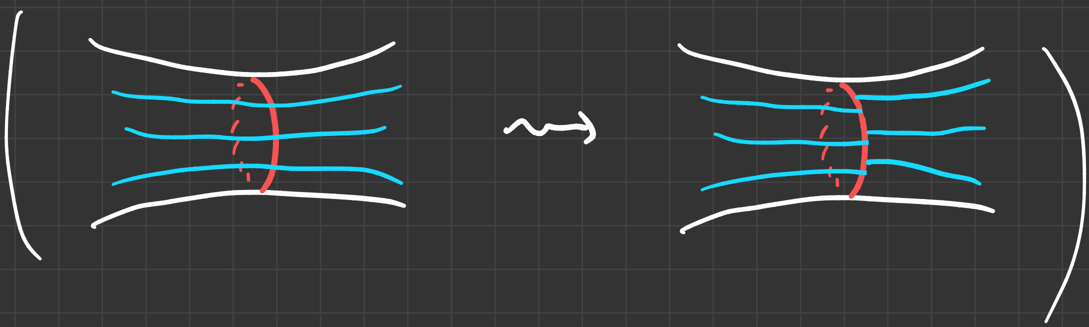
[arXiv: 2206.15226]

§0. Intro

What is the earthquake?

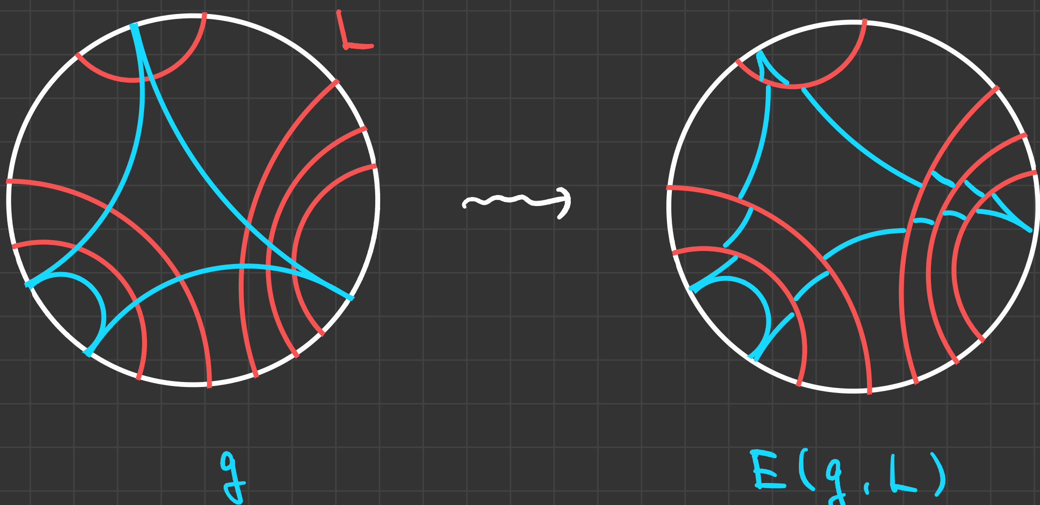
$\rightsquigarrow$ A generalization of

the Fenchel-Nielsen twist.



i.e. The FN twist is the deformation of hyperbolic str. along a (weighted) simple closed curve.

On the other hand, the earthquake is the deformation of hyp. str. along a measured lamination.



<u>Theorem</u> (Thurston's earthquake theorem)

Σ : a surface

$g_0 = [X_0, f_0] \in \mathcal{T}(\Sigma)$: fix.

Then, for $\forall g \in \mathcal{T}(\Sigma)$, $\exists !\, L \in \mathcal{ML}(X_0)$

s.t. $g = E(g_0, L)$.

More precisely,

$$\begin{aligned} E(g_0, -) : \mathcal{ML}(X_0) &\longrightarrow \mathcal{T}(\Sigma) \\ L &\longmapsto E(g_0, L) \end{aligned}$$

is a homeomorphism.

<u>Remark</u>

By considering the measured lamination bundle

$$\begin{tikzcd} \mathcal{ML}(X) \arrow[r, hook] \arrow[d] & \mathcal{ML}_\Sigma \arrow[d] \\ \{[X, f]\} \arrow[r, hook] & \mathcal{T}(\Sigma) \end{tikzcd}$$

one can think the earthquake is the continuous map

$$\begin{aligned} E : \mathcal{ML}_\Sigma &\longrightarrow \mathcal{T}(\Sigma) \\ (g, L) &\longmapsto E(g, L) \end{aligned}$$

and one can verify that E is

$MC(\Sigma)$ - equivariant.
$\quad \rotatebox{90}{$\in$}$
$\quad \phi$

$\left(\text{i.e., } E(\phi(g), \phi(L)) = E(g, L) \right)$.

- The popular application of this thm. is solving the Nielsen realization problem by Kerckhoff ('83).

$\mathcal{X}_{s}$: the cluster variety associated with a mutation class s.

<u>Fact</u>.

If s is obtained from a marked surf Σ

enhanced Teichmüller sp

$T(\Sigma) \subset \mathcal{X}_{\Sigma}(\mathbb{R}_{>0})$

$\mathcal{ML}(X) \subset \mathcal{X}_{\Sigma}(\mathbb{R}^{trop})$

the set of semifield valued points

Problem

Does the earthquake theorem hold for any mutation class?

Namely, $\exists$? continuous & cluster modular group equivariant map

$$E : \mathcal{X}_{\mathbf{s}}(\mathbb{R}_{>0}) \times \mathcal{X}_{\mathbf{s}}(\mathbb{R}^{\mathrm{trop}}) \longrightarrow \mathcal{X}_{\mathbf{s}}(\mathbb{R}_{>0})$$

s.t. for each $g_0 \in \mathcal{X}_{\mathbf{s}}(\mathbb{R}_{>0})$,

$$E(g_0, -) : \mathcal{X}_{\mathbf{s}}(\mathbb{R}^{\mathrm{trop}}) \longrightarrow \mathcal{X}_{\mathbf{s}}(\mathbb{R}_{>0})$$

is a homeomorphism.

Thm [Bonsante-Krasnov-Schlenker '16]

If $\mathbf{s}$ is obtained from a punctured surface (i.e. it has no boundary comp), then the earthquake theorem holds.

Main theorem [Asaka - Ishibashi - K.]

If $\mathfrak{s}$ is of finite, then the earthquake theorem holds.

§1 Earthquake map

Σ : a marked surface.

$\hat{\mathcal{T}}(\Sigma)$: the enhanced Teichmüller sp.

$$\begin{array}{l} \quad\Psi \\ [X, f] : \left\{ \begin{array}{l} X : \text{hyperbolic surf.} \\ \qquad \text{homeomorphic to } \Sigma. \\ f : X \to \Sigma : \text{homeo.} \end{array} \right. \end{array}$$

(+ signing)

$\widehat{\mathcal{ML}}(X)$: the space of the enhanced measured geodesic laminations on X.

$\rotatebox{90}{$\in$}$

$$(G, \mu) \begin{cases} G : \text{unbounded geodesic lamination on } X \\ \mu : \text{transverse measure of } G \end{cases}$$

(+ signing)

$$\begin{tikzcd} \tilde{G} \arrow[d] & \subset & \tilde{X} \arrow[d] & \subset \mathbb{H} \\ G & \subset & X & \end{tikzcd}$$

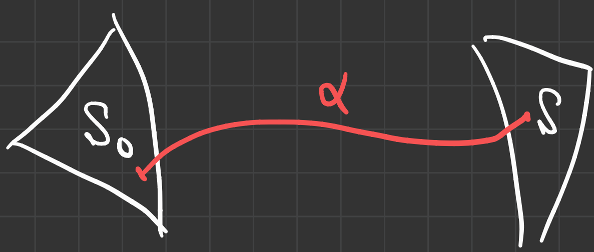
$\nwarrow$ the universal cover

Fix $S_0 \sqsubset \tilde{X} \setminus \tilde{G}$

For $S \subset \tilde{X} \setminus \tilde{G}$,

define $E_S \in PSL(2, \mathbb{R})$ as

- $E_{S_0} := \mathrm{id}$
- E_S is a hyperbolic element w/ the translation length $= \tilde{\mu}(\alpha)$



- the axis of $E_S^{-1} \circ E_{S'}$ weakly separates S & $S' \subset \tilde{X} \setminus \tilde{G}$

$\rightsquigarrow \tilde{E}_{(G,\mu)} := \bigcup_S E_S|_S$

$\partial_\infty \tilde{E}_{(G,\mu)} : \partial_\infty \tilde{X} \longrightarrow \partial_\infty \mathbb{H} \subset \overline{\mathbb{H}}$

↖ contin. ext. of $\tilde{E}_{(G,\mu)}$.

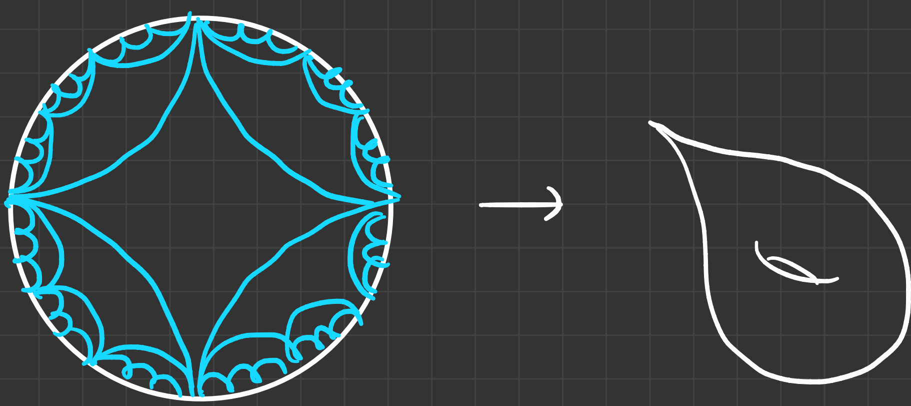
$\hat{T}(\Sigma)$ is parametrized by

$$(\rho, \psi) : \begin{cases} \rho : \pi_1(\Sigma) \longrightarrow PSL(2;\mathbb{R}) \\ \psi : \mathcal{F}_\infty(\Sigma) \longrightarrow \partial_\infty \mathbb{H}. \end{cases}$$

monodromy

: ord. pres.

Farey set of Σ



For $g = (\rho, \psi) \in \hat{\mathcal{T}}(\Sigma)$,

define $E(g, (G, \mu)) = (\rho', \psi')$ as

- $\tilde{E}_{(G,\mu)} \circ \rho(\gamma) = \rho'(\gamma) \circ \tilde{E}_{(G,\mu)}$ for $\forall \gamma \in \pi_1(\Sigma)$
- $\psi' := \partial_\infty \tilde{E}_{(G,\mu)} \circ \psi$.

$$\rightsquigarrow E : \hat{\mathcal{ML}}_\Sigma \longrightarrow \hat{\mathcal{T}}(\Sigma)$$

$$\begin{array}{ccc} \hat{\mathcal{ML}}_\Sigma & \longrightarrow & \hat{\mathcal{T}}(\Sigma) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathcal{X}_\Sigma(\mathbb{R}_{>0}) \times \mathcal{X}_\Sigma(\mathbb{R}^{trop}) & & \mathcal{X}_\Sigma(\mathbb{R}_{>0}) \end{array}$$

<u>Prop 1</u>

$\gamma \subset \Sigma$: an ideal arc, $t \in \mathbb{R}_{>0}$

$(t \cdot \gamma \in \hat{\mathcal{ML}}(X))$

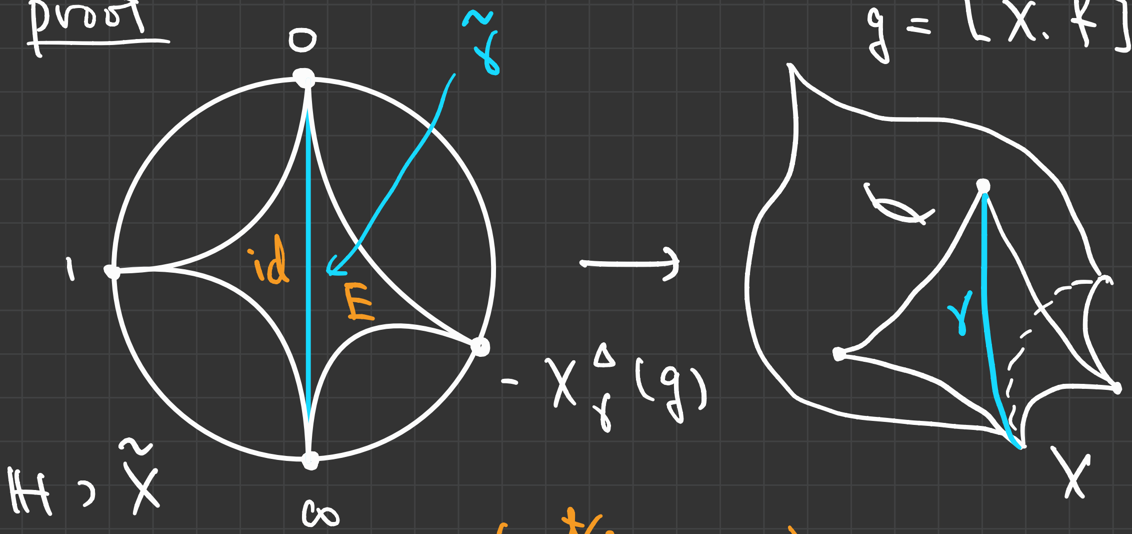
then, for $\forall g \in \hat{\mathcal{T}}(\Sigma)$

$$X_\alpha^\Delta(E(g, t\cdot\gamma)) = \exp(t\,\delta_{\alpha\gamma}) \cdot X_\alpha^\Delta(g)$$

Here Δ is an ideal tri. containing γ.

<u>proof</u>

$g = [X, f]$



$$E = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

$$(-X_\gamma^\Delta \longmapsto -e^t \cdot X_\gamma^\Delta)$$

$$\therefore X_\gamma^\Delta(E(g, t\cdot\gamma)) = [0 : 1 : \infty : -e^t X_\gamma^\Delta]$$

$$= -e^t X_\gamma^\Delta(g) \quad \blacksquare$$

§ 2. Cluster earthquake map.

s : a mutation class

Ψ

$(\varepsilon, \mathbb{X})$ $\begin{cases} \varepsilon : (n \times n) \text{ skew-symmetrizable matrix} \\ \mathbb{X} = (X_1, \cdots, X_n) \end{cases}$

$\rightsquigarrow$ $\mathbb{E}\mathrm{xch}_{\mathsf{s}}$: (labeled) exchange graph

i.e. $(\varepsilon, \mathbb{X}) \overset{k}{\text{———}} (\varepsilon', \mathbb{X}')$ then

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & i = k \text{ or } j = k \\ \varepsilon_{ij} + \frac{1}{2}\left(|\varepsilon_{ik}| \varepsilon_{kj} + \varepsilon_{ik} |\varepsilon_{kj}| \right) & \text{otherwise} \end{cases}$$

$$X'_i = \begin{cases} X_k^{-1} & i = k \\ X_i \left(1 + X_k^{-\mathrm{sgn}(\varepsilon_{ik})} \right)^{-\varepsilon_{ik}} & i \neq k \end{cases}$$

$\rightsquigarrow$ $\mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0})$: the cluster manifold ass. with $\mathfrak{s}$.

the C^{ω}-mfd homeo. to $\mathbb{R}^n_{>0}$

equipped w/ an atlas consisting of

$$\mathbb{X}^{(v)} : \mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathbb{R}^n_{>0}$$

for $v \in \mathbb{E}\mathrm{xch}_{\mathfrak{s}}$,

s.t. $v \overset{k}{\text{——}} v'$

$\Rightarrow (\mathbb{X}^{(v)})^{-1} \circ \mathbb{X}^{(v')}$: cluster $\mathcal{X}$-transf at k

tropicalize $\rightsquigarrow$ $\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}})$: the tropical cluster manifold the PL mfd homeo. to $\mathbb{R}^n$ equipped w/

an atlas consisting of

$$x^{(v)} : \mathcal{X}_s(\mathbb{R}^{trop}) \xrightarrow{\sim} \mathbb{R}^n$$

for $v \in \mathbb{E}xch_s$

s.t. $v \overset{k}{—} v'$

$\Rightarrow$ $(x^{(v)})^{-1} \circ x^{(v')}$: tropicalized

cluster transf. at k.

$$\left(x'_i = \begin{cases} -x_k & i = k \\ x_i - \varepsilon_{ik} \cdot \min(0, -\mathrm{sgn}(\varepsilon_{ik}) x_k) & i \neq k \end{cases} \right)$$

For $v \in \mathbb{E}\mathrm{xch}_{\mathfrak{s}}$

$$\mathcal{C}^{+}_{(v)} := \{ L \mid x^{(v)}_i(L) \geq 0, \ \forall i \}$$

$$\subset \mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}})$$

The cones $\mathcal{C}^{+}_{(v)}$, $v \in \mathbb{E}\mathrm{xch}_{\mathfrak{s}}$ forms a fan $\mathcal{F}^{+}_{\mathfrak{s}}$ on $\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}})$ called Fock-Goncharov fan.

Prop. 1 is rewritten as follows;

for $\forall L \in \mathcal{C}^{+}_{\Delta} \subset \mathcal{X}_{\Sigma}(\mathbb{R}^{\mathrm{trop}})$,

$$X^{\Delta}_{\alpha}(E(g, L)) = \exp(x^{\Delta}_{\alpha}(L)) \cdot X^{\Delta}_{\alpha}(g)$$

For $g_0 \in \mathcal{X}_*(\mathbb{R}_{>0})$ and $v \in \mathbb{E}\mathrm{xch}_*$

define $\exp_{g_0}^{(v)} : \mathcal{C}_{(v)}^+ \longrightarrow \mathcal{X}_*(\mathbb{R}_{>0})$

by $X_i^{(v)}(\exp_{g_0}^{(v)}(L))$

$$:= \exp(x_i^{(v)}(L)) \cdot X_i^{(v)}(g)$$

<u>Lem</u>

For $v_1 \xrightarrow{k} v_2$ in $\mathbb{E}\mathrm{xch}_*$,

$$\exp_{g_0}^{(v_1)} = \exp_{g_0}^{(v_2)} \text{ on } \mathcal{C}_{(v_1)}^+ \cap \mathcal{C}_{(v_2)}^+$$

$$\rightsquigarrow \exp_{g_0} := \bigcup_v \exp_{g_0}^{(v)}$$

Def

$$E_{pre} : \mathcal{X}_s(\mathbb{R}_{>0}) \times \overset{\bigcup_v \mathcal{C}^+_{(v)}}{\overset{\shortparallel}{|\mathcal{F}^+_s|}} \longrightarrow \mathcal{X}_s(\mathbb{R}_{>0})$$

$$(g, L) \longmapsto \exp_g(L)$$

When s is of finite type,

$\mathcal{F}^+_s$ is complete

i.e. $|\mathcal{F}^+_s| = \mathcal{X}_s(\mathbb{R}^{trop})$

$\rightsquigarrow E := E_{pre}$

: cluster earthquake map

Main thm

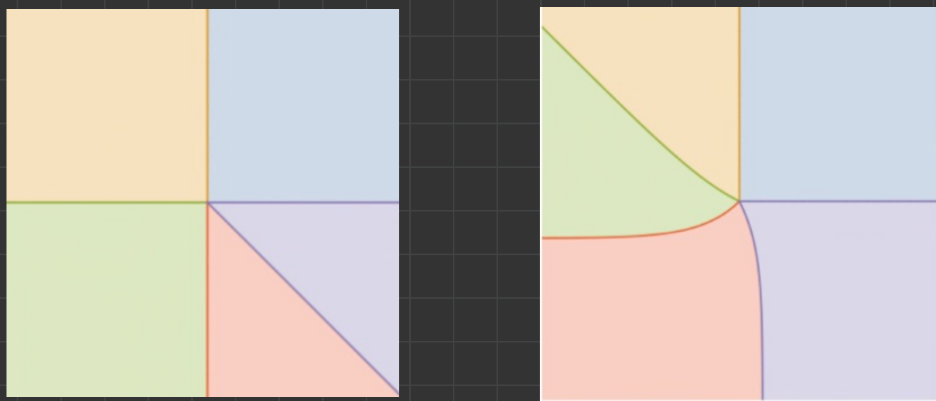
① E is cluster modular group equiv.

② $E(g_0, -) : \mathcal{X}_s(\mathbb{R}^{trop}) \xrightarrow{\sim} \mathcal{X}_s(\mathbb{R}_{>0})$

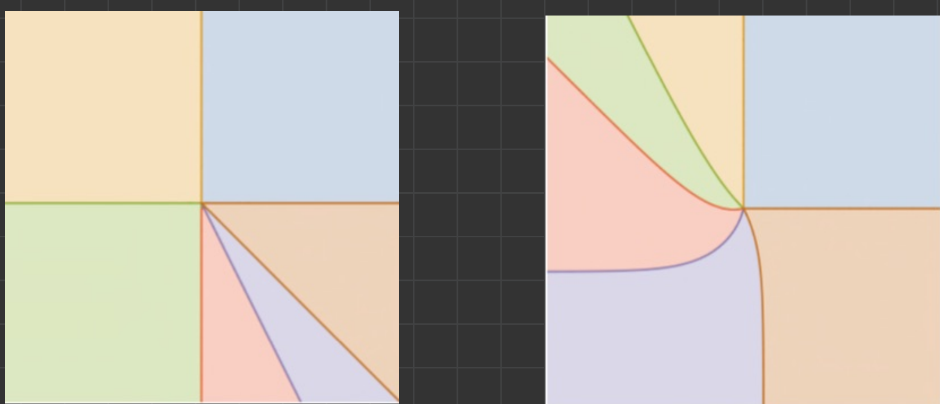
is a homeo.

$\mathfrak{A}^{(v)}(\mathfrak{F}_s^+)$ $\quad\quad$ $\log X^{(v)}(E(g_0 . \mathfrak{F}_s^+))$

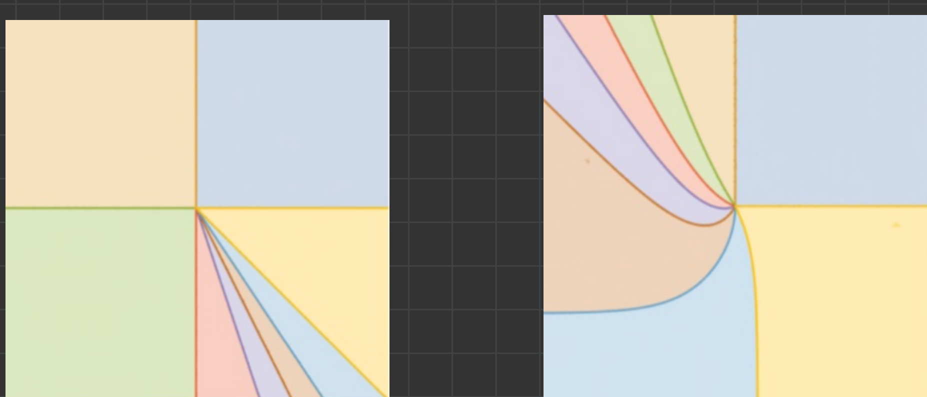
A_2



B_2



G_2



Sketch of proof

① : Easy.

② : By induction on the rank of $\mathfrak{s}$.

- $n = 2$: direct calculation

- $n \geq 3$:

  $E(g_0, -)$ is a local homeo. at each $L \in \operatorname{int} \mathcal{C}^{+}_{(v)}$

We want to prove that it is also a loc. homeo. at $L \in \partial \mathcal{C}^{+}_{(v)}$.

If $L \in F \subset \mathcal{C}^{+}_{(v)}$: face of dim ≥ 1

$$F = \{ L' \mid x_j^{(v)}(L) \neq 0, \ \forall j \in J \}$$

$\exists J \subset \{1, \cdots, n\}$

$\rightsquigarrow$ s_F : the mutation class

$$\left((\varepsilon_{ij}^{(v)})_{i,j\in J} ,\ (X_j^{(v)})_{j\in J} \right)$$

$$\pi_F^{\mathrm{trop}} : \mathcal{X}_s(\mathbb{R}_{>0}) \longrightarrow \mathcal{X}_{s_F}(\mathbb{R}_{>0})$$

$\mathbb{R}^{\mathrm{trop}}$ $\mathbb{R}^{\mathrm{trop}}$

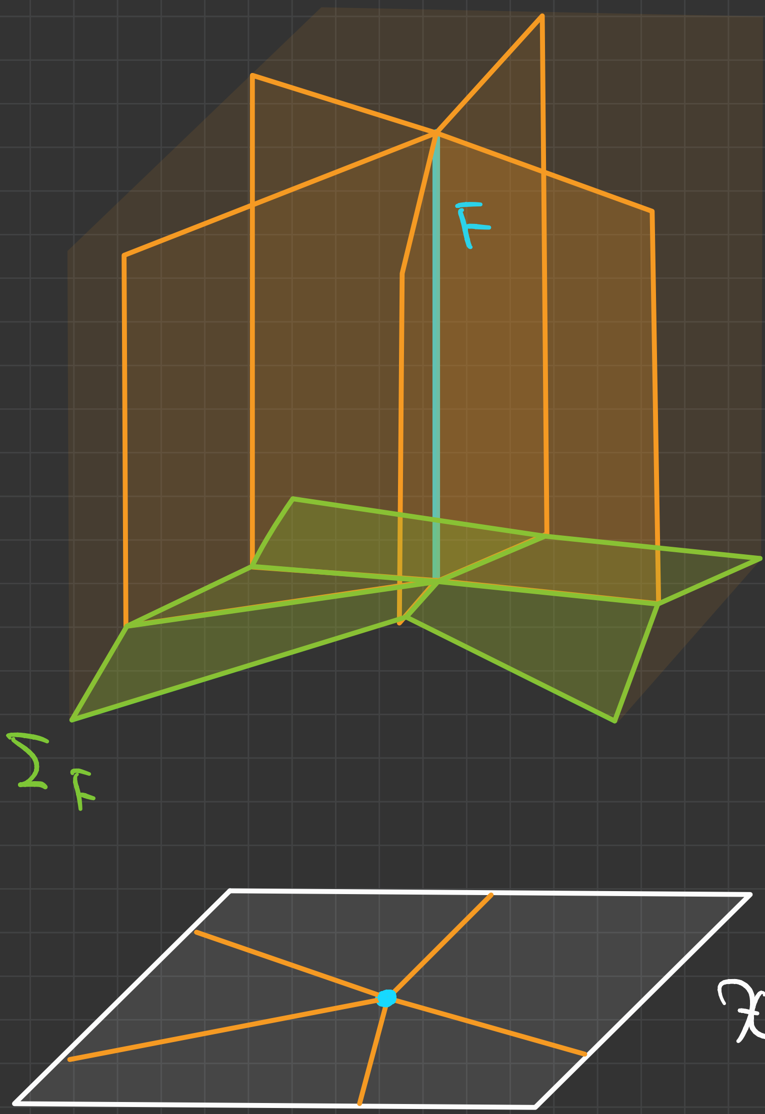
$$(X_i^{(v)})_{i=1}^{n} \longmapsto (X_j^{(v)})_{j\in J}$$

$$D_F := \bigcup_{v'\in \mathbb{E}\mathrm{xch}_{s_F}} \mathcal{C}^+_{(v')}$$

$$= \bigcup \{ \mathcal{C}^+_{(v')} \mid F \subset \mathcal{C}^+_{(v')} \}$$

$$\Sigma_F := \bigcup_{v'\in \mathbb{E}\mathrm{xch}_s} \{ L \in \mathcal{C}^+_{(v')} \mid x_i^{(v')}(L) = 0,\ \forall i \notin J \}$$

$$= \bigcup \{ F' \subset D_F : \text{face} \mid F' \pitchfork F \}$$



$$\begin{tikzcd}
\mathcal{X}_s(\mathbb{R}^{trop}) \\
D_F \arrow[u, phantom, "\cup" description] \\
\Sigma_F \times F \arrow[u, phantom, "\cong" description] \arrow[d, "\pi_F^{trop}"] \\
\mathcal{X}_{s_F}(\mathbb{R}^{trop})
\end{tikzcd}$$

PL bundle with fiber F.

$$
\begin{tikzcd}
{} \arrow[rr, "E(g_0, -)"] \arrow[drr, phantom, "\circlearrowright"] & & \mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0}) \arrow[d, "\pi_F"] \\
{} \arrow[rr, "E(\pi_F(g_0, -))"'] & & \mathcal{X}_{\mathfrak{s}_F}(\mathbb{R}_{>0})
\end{tikzcd}
$$

(homeo. by induction hypothesis)

$$L \neq L' \in \mathcal{X}_{\mathfrak{s}_F}(\mathbb{R}^{trop})$$

$$\Rightarrow \pi_F^{trop}(L) \cap \pi_F^{trop}(L') = \emptyset$$

$\rightsquigarrow$ $E(g_0, -)$ is a local homeo. at each point of F.

$\rightsquigarrow$ $E(g_0, -)|$: $\mathcal{X}_s(\mathbb{R}^{trop}) \setminus \{0\} \to \mathcal{X}_s(\mathbb{R}_{>0}) \setminus \{g_0\}$

is a proper local homeo.

$\rightsquigarrow$ This is a covering map.

- $\mathcal{X}_s(\mathbb{R}^{trop}) \setminus \{0\} \simeq S^{n-1}$ is simply conn. since $n \geq 3$

$\rightsquigarrow$ Filling missing point. $\blacksquare$

Cor.

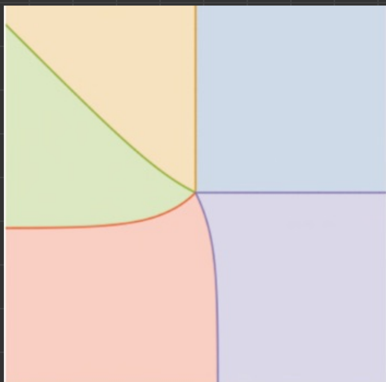
$$dE : \mathcal{X}_s(\mathbb{R}_{>0}) \times \mathcal{X}_s(\mathbb{R}^{trop}) \longrightarrow T\mathcal{X}_s(\mathbb{R}_{>0})$$

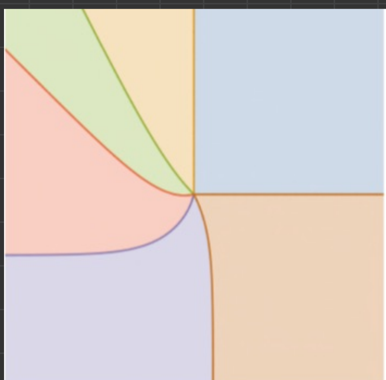
$$(g, L) \longmapsto \left(g, \frac{d}{dt}\Big|_{t=0^+} E(g, t\cdot L)\right)$$

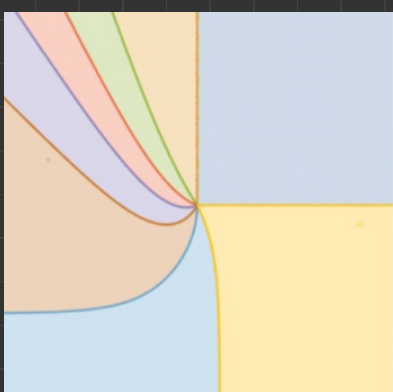
is a cluster modular group equivariant isomorphism of topological fiber bundle

§ 3 Asymptotic behavior

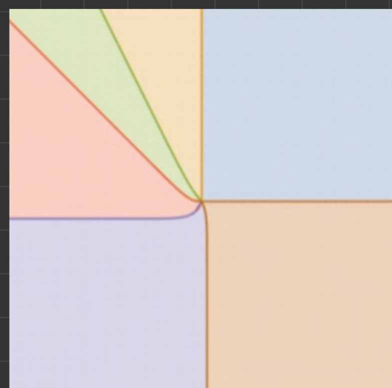
plots in $[-6,6]^2$

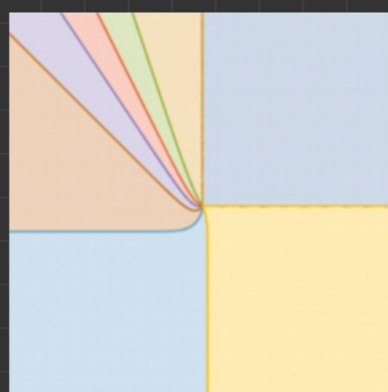






in $[-25,25]^2$





- $\mathfrak{s}$: a mutation class

$\rightsquigarrow$ $-\mathfrak{s}$: the opposite mutation class.

i.e. $(\varepsilon, (X_i)_i) \in \mathfrak{s}$

$$\Rightarrow (-\varepsilon, (X_i^{-1})_i) \in -\mathfrak{s}.$$

$$\rightsquigarrow \quad \mathbb{E}\mathrm{xch}_{\mathfrak{s}} \xrightarrow{\sim} \mathbb{E}\mathrm{xch}_{-\mathfrak{s}}$$
$$v \longmapsto -v$$

- $\overline{\mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0})} := \mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0}) \sqcup \mathbb{S}\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}})$

tropical compactification

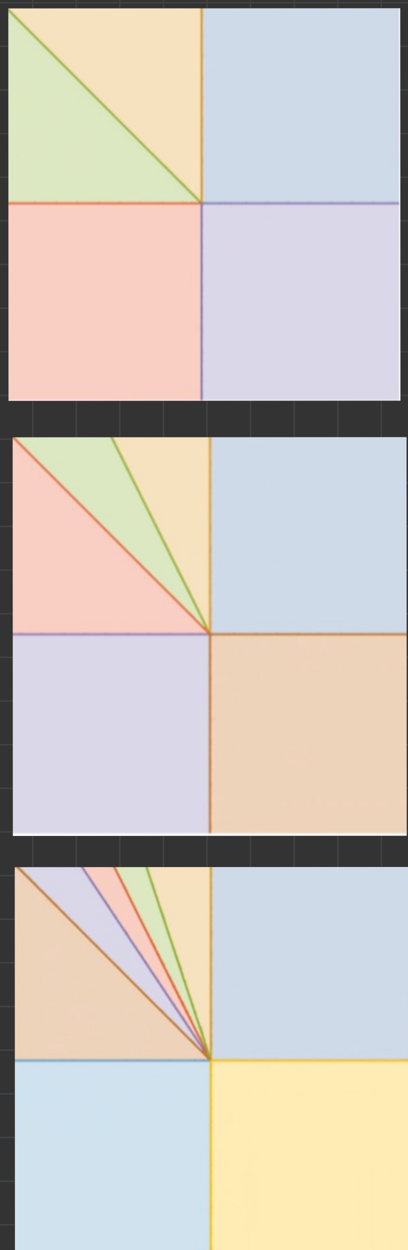
$$\left(\mathbb{S}\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}}) := (\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}}) \setminus \{0\}) / \mathbb{R}_{>0} \right)$$

i.e. $\mathcal{X}_{\mathfrak{s}}(\mathbb{R}_{>0}) \ni g_n \xrightarrow[n\to\infty]{} [L] \in \mathbb{S}\mathcal{X}_{\mathfrak{s}}(\mathbb{R}^{\mathrm{trop}})$

$$\Longleftrightarrow [\log \mathbb{X}^{(v)}(g_n)] \xrightarrow[n\to\infty]{} [x^{(v)}(L)]$$

in $\mathbb{S}\mathbb{R}^n = (\mathbb{R}^n \setminus \{0\}) / \mathbb{R}_{>0}$

$\chi^{(-n)}(\mathcal{F}^{+}_{-\lambda})$



For $g_0 \in \mathcal{X}_s(\mathbb{R}_{>0})$, $v \in \mathbb{E}\mathrm{xch}_s$,

$$\mathcal{E}^+_{(v)}(g_0) := \left\{ \lim_{t\to\infty} E(g_0, t\cdot L) \;\middle|\; L \in \mathcal{C}^+_{(v)} \right\}$$

$$\subset \$\mathcal{X}_s(\mathbb{R}^{\mathrm{trop}})$$

<u>Thm</u> [Asaka-Ishibashi-K.]

$$\mathcal{E}^+_{(v)}(g_0) = \iota\left(\$\mathcal{C}^+_{(-v)}\right)$$

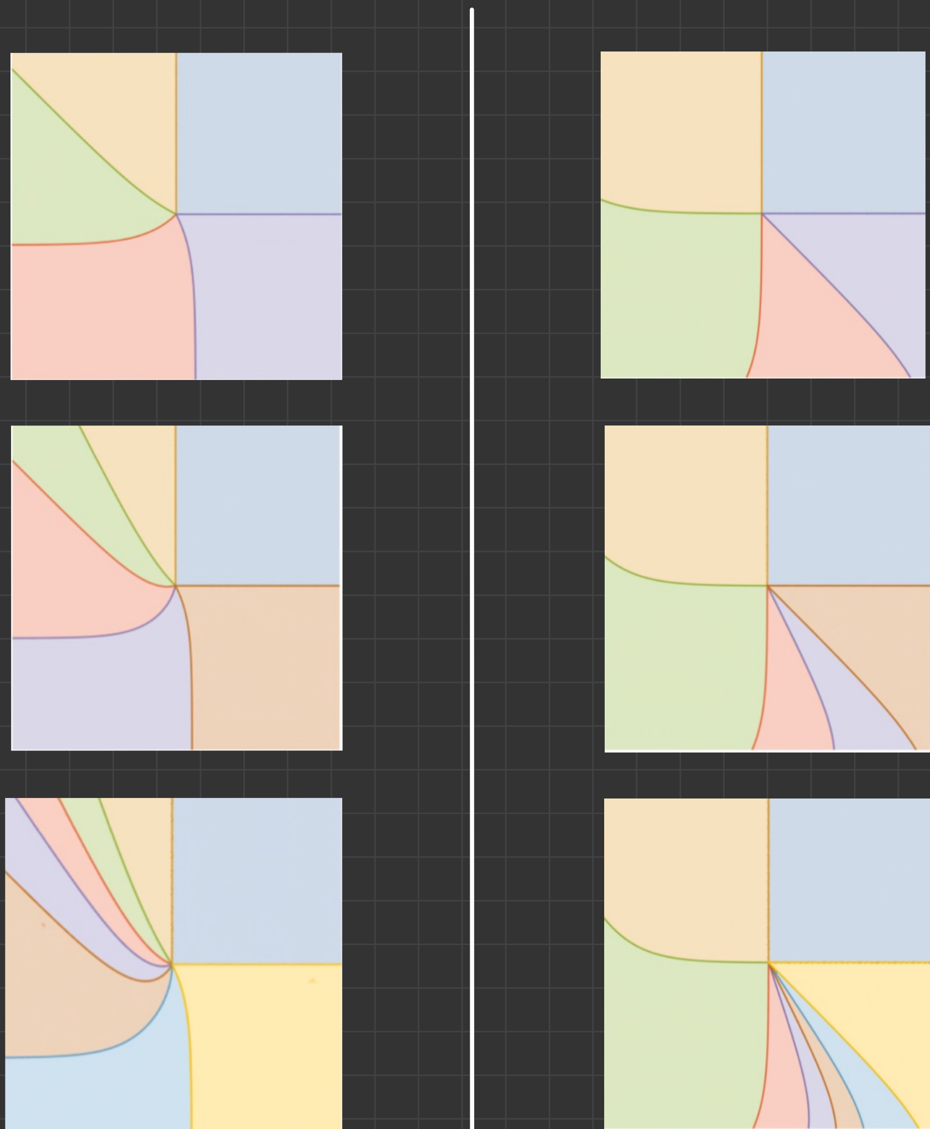
where

$$\begin{array}{ccc} \mathcal{X}_{-s}(\mathbb{R}^{\mathrm{trop}}) & \xrightarrow{\iota} & \mathcal{X}_s(\mathbb{R}^{\mathrm{trop}}) \\ {\scriptstyle x^{(-v)}}\downarrow & & \downarrow{\scriptstyle x^{(v)}} \\ \mathbb{R}^n & \xrightarrow{(-1)} & \mathbb{R}^n \end{array}$$

This theorem describe the asymp. behavior of $E(g_0.L)$ as $L \to \infty$.

Next, we consider the asymp. behavior of $E(g_0, L)$ as $g_0 \to \infty$.

$\chi^{(v)}(g_0) = (1,1)$ $\qquad\qquad$ $(500, 500)$



For $g \in \mathcal{X}_s(\mathbb{R}_{>0})$, $v \in \mathbb{E}\mathrm{xch}_s$,

$$\mathcal{U}_g^{(v)} : \mathcal{X}_s(\mathbb{R}^{\mathrm{trop}}) \longrightarrow \mathbb{R}^n$$

$$L \longmapsto \log \frac{\mathbb{X}^{(v)}(E(g,L))}{\mathbb{X}^{(v)}(g)}$$

$$(0 \longmapsto 0)$$

<u>Thm</u> [Asaka-Ishibashi-K]

$$\mathcal{U}_g^{(v)}(\mathcal{F}_s^+) \longrightarrow x^{(v)}(\mathcal{F}_s^+)$$

as g diverges toward $\mathrm{int}\, \mathcal{S}\mathcal{C}_{(v)}^+$.

$\rightsquigarrow$ $E(g, \mathcal{F}_s^+)$ is a continuous deformation of $\mathcal{F}_s^+$.

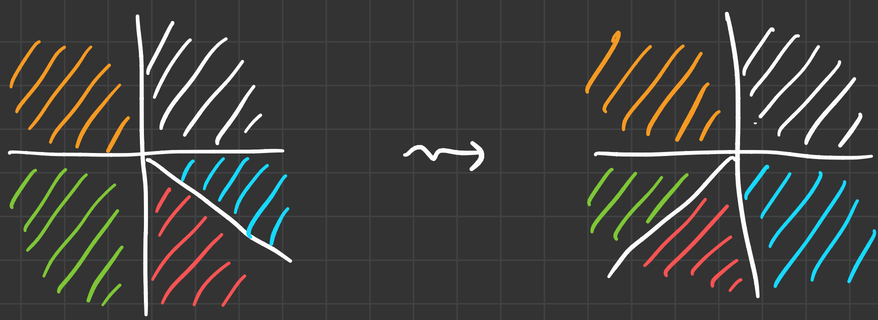
Prob.

How behave $\mathcal{U}_g^{(v)}(\mathcal{F}_s^+)$ when

g diverges to int $\mathcal{C}^+_{(v')}$ for $v' \neq v$.

We observe that it is obtained

by mutation for fan from v to v'.



§4 Future work

Thm [Yurikusa '20]

s is obtained from a marked surf.

$|\mathcal{F}_s^+|$ is dense in $\mathcal{X}_s(\mathbb{R}^{trop})$.

"g-tame"

We hope that the earthquake thm. holds for the g-tame cluster alg's.

It is known that the cluster alg's corresp. to affine Dynkin types are g-tame.

We now trying to define the earthquake map and to prove the earthquake thm for them.

Thank you for your attention!