Localizations for quiver Hecke algebras

Myungho Kim (Kyung Hee University, Seoul)

Joint works with M. Kashiwara, S.-j. Oh and E. Park

September 20, 2022,
Trends in Cluster Algebras 2022

Contents

Motivation

- G : group corresp. to a symmetrizable Kac-Moody algebra $\mathfrak{g}$
- $N(w) \subset G$: the unipotent subgroup of G corresponding to w (i.e., $\mathrm{Lie}(N(w)) = \bigoplus_{\beta \in \boldsymbol{\Delta}_+ \cap w\boldsymbol{\Delta}_-} \mathfrak{g}_\beta$)
- $N^w := N_+ \cap \mathring{X}_w \subset G/B_-$: the unipotent cell corresponding to w inside the flag manifold, where $N_+ \hookrightarrow X = G/B_-$ and $\mathring{X}_w$ is the Schubert cell corresponding to w in X.
- There is an isomorphism between coordinate rings

$$\mathbb{C}[N^w] \simeq \mathbb{C}[N(w)][D_{w\lambda,\lambda}^{-1} \,|\, \lambda \in \mathrm{P}_+]$$

- $\mathbb{C}[N(w)]$ has a categorification via a monoidal subcategory $\mathcal{C}_w$ of $R\text{-}\mathrm{gmod}$($=$ the category of f.d. modules over the quiver Hecke algebra R of type $\mathfrak{g}$), which respects the cluster algebra structure on $\mathbb{C}[N(w)]$ i.e., the cluster monomials are simple objects.
- It is desired to extend $\mathcal{C}_w$ to $\widetilde{\mathcal{C}}_w$ in which $M_{w\lambda,\lambda}$ ($=$ simple module corresponding to $D_{w\lambda,\lambda}$) is invertible.

Monoidal categories

A *monoidal category* is a datum consisting of

(a) a category $\mathcal{T}$,

(b) a bifunctor $\cdot\otimes\cdot\colon \mathcal{T}\times\mathcal{T}\to\mathcal{T}$,

(c) an isomorphism $a(X,Y,Z)\colon (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$ which is functorial in $X,Y,Z\in\mathcal{T}$,

(d) an object $\mathbf{1}\in\mathcal{T}$ with an isomorphism $\epsilon\colon \mathbf{1}\otimes\mathbf{1}\xrightarrow{\sim}\mathbf{1}$ such that

(1) the diagram below commutes for all $X,Y,Z,W\in\mathcal{T}$:

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes W \arrow[r, "a(X\otimes Y,Z,W)"] \arrow[d, "a(X,Y,Z)\otimes W"'] & (X\otimes Y)\otimes(Z\otimes W) \arrow[dd, "a(X,Y,Z\otimes W)"] \\
(X\otimes(Y\otimes Z))\otimes W \arrow[d, "a(X,Y\otimes Z,W)"'] & \\
X\otimes((Y\otimes Z)\otimes W) \arrow[r, "X\otimes a(Y,Z,W)"'] & X\otimes(Y\otimes(Z\otimes W)),
\end{tikzcd}$$

(2) the functors $\mathcal{T}\ni X\mapsto\mathbf{1}\otimes X\in\mathcal{T}$ and $\mathcal{T}\ni X\mapsto X\otimes\mathbf{1}\in\mathcal{T}$ are fully faithful. ($\Rightarrow\exists$ canonical isomorphisms $\mathbf{1}\otimes X\simeq X\otimes\mathbf{1}\simeq X$ for any $X\in\mathcal{T}$).

Invertible objects and dual objects

An object $X \in \mathcal{T}$ is invertible if the endofunctors $X \otimes -$ and $- \otimes X$ on $\mathcal{T}$ are equivalence of categories. In the case there exists $Y \in \mathcal{T}$ and isomorphisms $f\colon X \otimes Y \overset{\sim}{\to} \mathbf{1}$ and $g\colon Y \otimes X \overset{\sim}{\to} \mathbf{1}$.

A pair (X, Y) of objects in $\mathcal{T}$ is called a dual pair if there exists $\varepsilon\colon X \otimes Y \to \mathbf{1}$ and $\eta\colon \mathbf{1} \to Y \otimes X$ such that

$$\begin{tikzcd}
X \arrow[r, "\cong"'] \arrow[rrrr, bend left, "\mathrm{id}_X"] & X \otimes \mathbf{1} \arrow[r, "X \otimes \eta"] & X \otimes Y \otimes X \arrow[r, "\varepsilon \otimes X"] & \mathbf{1} \otimes X \arrow[r, "\cong"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
Y \arrow[r, "\cong"'] \arrow[rrrr, bend left, "\mathrm{id}_Y"] & \mathbf{1} \otimes Y \arrow[r, "\eta \otimes Y"] & Y \otimes X \otimes Y \arrow[r, "Y \otimes \varepsilon"] & Y \otimes \mathbf{1} \arrow[r, "\cong"'] & Y
\end{tikzcd}$$

We say that X is a left dual to Y and Y is a right dual to X in $\mathcal{T}$. A monoidal category $\mathcal{T}$ is left (respectively, right) rigid if every object in $\mathcal{T}$ has a left (respectively, right) dual.

Example : monoidal category R-gmod

Let $\mathbf{k}$ be a base field. The quiver Hecke algebra is a family of associative $\mathbf{k}$-algebras $\{R(\beta)\}_{\beta \in \mathsf{Q}_+}$ which categorify the quantum coordinate ring $A_q(\mathfrak{n})$ of the unipotent subgroup t, where Q_+ is the positive root lattice of $\mathfrak{g}$ (Khovano-Lauda, Rouquier).
For $M \in R(\beta)\text{-}\mathrm{gmod}, N \in R(\gamma)\text{-}\mathrm{gmod}$, the convolution product is given by

$$M \circ N := R(\beta+\gamma) \underset{R(\beta)\otimes_{\mathbf{k}} R(\gamma)}{\otimes} (M \underset{\mathbf{k}}{\otimes} N) \quad \in R(\beta+\gamma)\text{-}\mathrm{gmod}$$

Then the category

$$R\text{-}\mathrm{gmod} := \bigoplus_{\beta \in \mathsf{Q}_+} R(\beta)\text{-}\mathrm{gmod}$$

is a $\mathbf{k}$-linear abelian monoidal category.

The unit object $\mathbf{1}$ is the trivial representation $\mathbf{k}$ of $R(0) \simeq \mathbf{k}$.
If $M \in R(\beta)\text{-}\mathrm{gmod}$ and a morphism $\varepsilon : M \circ N \to \mathbf{1}$ is nonzero, then $\beta = 0$ and $N \in R(0)\text{-}\mathrm{gmod}$.
Hence $\mathbf{1}$ is the only invertible object in R-gmod and there is no dual of M unless $M \in R(0)\text{-}\mathrm{gmod}$.

Real commuting family of braiders in $\mathcal{T}$

We assume that $\mathcal{T}$ is a **k**-linear monoidal category.

Definition

A (left) braider of $\mathcal{T}$ is a pair (C, R_C) of an object C and a morphism

$$R_C(X)\colon C \otimes X \longrightarrow X \otimes C$$

which is functorial in $X \in \mathcal{T}$ such that the followings commute:

$$\begin{tikzcd}
C \otimes X \otimes Y \arrow[r, "R_C(X)\otimes Y"] \arrow[dr, "R_C(X\otimes Y)"'] & X \otimes C \otimes Y \arrow[d, "X\otimes R_C(Y)"] & & C \otimes \mathbf{1} \arrow[r, "R_C(\mathbf{1})"] \arrow[dr, "\simeq"'] & \mathbf{1} \otimes C \arrow[d, "\wr"] \\
& X \otimes Y \otimes C, & & & C.
\end{tikzcd}$$

A braider (C, R_C) is called a central object if $R_C(X)$ is an isomorphism for any $X \in \mathcal{T}$.
The category of braiders in $\mathcal{T}$ will be denoted by $\mathcal{T}_{br}$.

Real commuting family of braiders in $\mathcal{T}$

Let I be an index set. We say that a family $\{(C_i, R_{C_i})\}_{i\in I}$ of braiders in $\mathcal{T}$ is a real commuting family if

(a) $R_{C_i}(C_i) \in \mathbf{k}^\times \operatorname{id}_{C_i\otimes C_i}$ for $i \in I$,

(b) $R_{C_j}(C_i) \circ R_{C_i}(C_j) \in \mathbf{k}^\times \operatorname{id}_{C_i\otimes C_j}$ for $i, j \in I$.

Then there exists braider $C^\alpha = (C^\alpha, R_{C^\alpha})$ for each $\alpha \in \mathbb{Z}_{\geq 0}^{\oplus I}$, an isomorphism $\xi_{\alpha,\beta} \colon C^\alpha \otimes C^\beta \xrightarrow{\sim} C^{\alpha+\beta}$ in $\mathcal{T}_{br}$ and $\eta_{\alpha,\beta} \in \mathbf{k}^\times$ for $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{\oplus I}$ such that

(a) $C^0 = \mathbf{1}$ and $C^{e_i} = C_i$ for $i \in I$,

(b) the following diagrams in $\mathcal{T}_{br}$ commute:

$$\begin{tikzcd}
C^\alpha \otimes C^\beta \arrow[r, "R_{C^\alpha}(C^\beta)"] \arrow[d, "\xi_{\alpha,\beta}"'] & C^\beta \otimes C^\alpha \arrow[d, "\xi_{\beta,\alpha}"] \\
C^{\alpha+\beta} \arrow[r, "\eta_{\alpha,\beta}"] & C^{\alpha+\beta}
\end{tikzcd}
\qquad
\begin{tikzcd}
C^\alpha \otimes C^\beta \otimes C^\gamma \arrow[r, "\xi_{\alpha,\beta}\otimes C^\gamma"] \arrow[d, "C^\alpha\otimes\xi_{\beta,\gamma}"'] & C^{\alpha+\beta} \otimes C^\gamma \arrow[d, "\xi_{\alpha+\beta,\gamma}"] \\
C^\alpha \otimes C^{\beta+\gamma} \arrow[r, "\xi_{\alpha,\beta+\gamma}"] & C^{\alpha+\beta+\gamma}
\end{tikzcd}$$

Localization of $\mathcal{T}$ via $\{(C_i, R_{C_i})\}_{i\in I}$

Define a partial order $\preceq$ on $\mathbb{Z}^{\oplus I}$ by

$$\alpha \preceq \beta \quad \text{for } \alpha, \beta \in \mathbb{Z}^{\oplus I} \text{ with } \beta - \alpha \in \mathbb{Z}^{\oplus I}_{\geq 0},$$

and for $\alpha, \beta \in \mathbb{Z}^{\oplus I}$ set

$$\mathcal{D}_{\alpha,\beta} := \{\delta \in \mathbb{Z}^{\oplus I} \mid \alpha + \delta,\ \beta + \delta \succeq 0\}.$$

Define a monoidal category $(\widetilde{\mathcal{T}} = \mathcal{T}[C_i^{\otimes -1} \mid i \in I], \otimes)$ as

$$\mathrm{Ob}(\widetilde{\mathcal{T}}) := \left\{(X, \alpha) \mid X \in \mathrm{Ob}(\mathcal{T}),\ \alpha \in \mathbb{Z}^{\oplus I}\right\},$$

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha),(Y,\beta)) := \varinjlim_{\delta \in \mathcal{D}_{\alpha,\beta}} \mathrm{Hom}_{\mathcal{T}}(C^{\delta+\alpha} \otimes X, Y \otimes C^{\delta+\beta}),$$

$$(X,\alpha) \otimes (Y,\beta) := (X \otimes Y, \alpha + \beta).$$

Localization of $\mathcal{T}$ via $\{(C_i, R_{C_i})\}_{i\in I}$

The structure morphism of the inductive system ($\delta' \succeq \delta$)

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{T}}(C^{\delta+\alpha}\otimes X, Y\otimes C^{\delta+\beta}) & \xrightarrow{\zeta_{\delta',\delta}} & \mathrm{Hom}_{\mathcal{T}}(C^{\delta'+\alpha}\otimes X, Y\otimes C^{\delta'+\beta}) \\ f & \mapsto & \zeta_{\delta',\delta}(f) \end{array}$$

is given by

$$\begin{tikzcd}
C^{\delta'-\delta}\otimes C^{\delta+\alpha}\otimes X \arrow[r, "C^{\delta'-\delta}\otimes f"] \arrow[dd, "\xi_{\delta'-\delta,\delta+\alpha}"', "\wr"] & C^{\delta'-\delta}\otimes Y\otimes C^{\delta+\beta} \arrow[d, "R_{C^{\delta'-\delta}}(Y)"] \\
 & Y\otimes C^{\delta'-\delta}\otimes C^{\delta+\beta} \arrow[d, "\xi_{\delta'-\delta,\delta+\beta}", "\wr"'] \\
C^{\delta'+\alpha}\otimes X \arrow[r, "\zeta_{\delta',\delta}(f)"] & Y\otimes C^{\delta'+\beta}.
\end{tikzcd}$$

Then $\zeta_{\delta'',\delta} = \zeta_{\delta'',\delta'}\circ\zeta_{\delta',\delta}$ for $\delta'' \succ \delta' \succ \delta$.

There is a natural way to define the composition of morphisms and the tensor product of morphisms with careful manipulation of coefficients.

Localization of $\mathcal{T}$ via $\{(C_i, R_{C_i})\}_{i\in I}$

Recall $(X,\alpha)\,\widetilde{\otimes}\,(Y,\beta) := (X\otimes Y, \alpha+\beta)$ in $\widetilde{\mathcal{T}}$.

Proposition

For $\alpha \in \mathbb{Z}^{\oplus I}$, set $\widetilde{C}^{\alpha} := (\mathbf{1}, \alpha)$. Then

1. *$\widetilde{C}^{\alpha}$ is invertible in $\widetilde{\mathcal{T}}$ with an inverse $\widetilde{C}^{-\alpha} = (\mathbf{1}, -\alpha)$.*
2. *$(C^{\alpha}, 0) \simeq \widetilde{C}^{\alpha} = (\mathbf{1}, \alpha)$ if $\alpha \succeq 0$ ($\Leftrightarrow \alpha \in \mathbb{Z}^{\oplus I}_{\geq 0}$).*

proof of ②: For any $\delta \succ 0$, we have $\zeta_{\delta,0}(\mathrm{id}_{C^{\alpha}}) = R_{C^{\delta}}(C^{\alpha})$, which is an isomorphism, since $\{(C_i, R_{C_i})\}_{i\in I}$ is a real commuting family. Hence it defines an isomorphism in $\mathrm{Hom}_{\widetilde{\mathcal{T}}}((C^{\alpha}, 0), (\mathbf{1}, \alpha))$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(C^{\alpha}, C^{\alpha}) \arrow[d, equal] & \\
\mathrm{Hom}_{\mathcal{T}}(C^{0}\otimes C^{\alpha}, \mathbf{1}\otimes C^{0+\alpha}) \arrow[r, "\zeta_{\delta,0}"] & \mathrm{Hom}_{\mathcal{T}}(C^{\delta}\otimes C^{\alpha}, \mathbf{1}\otimes C^{\delta+\alpha}) \\
\mathrm{id}_{C^{\alpha}} \arrow[r, mapsto] & R_{C^{\delta}}(C^{\alpha}).
\end{tikzcd}$$

Universal property of $\widetilde{\mathcal{T}}$

Let $\Psi : \mathcal{T} \to \widetilde{\mathcal{T}}$ the canonical functor such that $X \mapsto (X, 0)$ (it is a monoidal functor).

Theorem

1. *$\Psi(C_i)$ is invertible in $\widetilde{\mathcal{T}}$ for $i \in I$.*
2. *For $i \in I$, $X \in \mathcal{T}$,*

$$\Psi(R_{C_i}(X)) : \Psi(C_i) \otimes \Psi(X) \to \Psi(X) \otimes \Psi(C_i)$$

is an isomorphsm.

3. *If $\mathcal{F} : \mathcal{T} \to \mathcal{T}'$ is a monoidal functor such that ① and ② hold for $\mathcal{F}$, then there exists a monoidal functor $\widetilde{\mathcal{F}}$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\Psi"] \arrow[dr, "\mathcal{F}"'] & \widetilde{\mathcal{T}} \arrow[d, dotted, "\widetilde{\mathcal{F}}"] \\
 & \mathcal{T}'
\end{tikzcd}$$

Category $\mathcal{C}_w$ and its localization

R : quiver Hecke algebra of type $\mathfrak{g}$ (symmetrizable KM algebra)
$R\text{-gmod} := \bigoplus_{\beta \in \mathsf{Q}_+} R(\beta)\text{-gmod}$
For each $w \in \mathsf{W}$, there exist a full subcategory $\mathcal{C}_w \hookrightarrow R\text{-gmod}$ which is stable under taking subquotients and convolution products. One has

$$A_q(\mathfrak{n}(w)) \simeq K(\mathcal{C}_w) \hookrightarrow K(R\text{-gmod}) \simeq A_q(\mathfrak{n}),$$

where $A_q(\mathfrak{n}(w))$ denotes the quantum coordinate ring of the unipotent subgroup associated with w.

Proposition (Nondegenerate (graded) braiders in R-gmod)

For any simple R-module M, there exists a braider (M, R_M) in $R\text{-gmod}$ such that $R_M(L(i))$ is nonzero for $i \in I$, where $L(i)$ is the 1-dim'l simple module in $R(\alpha_i)\text{-gmod}$. Moreover such (M, R_M) is unique up to a scalar.

Category $\mathcal{C}_w$ and its localization

For $i \in I$ (the index set of simple roots of $\mathfrak{g}$) set

$$C_i = C_{i,w} := M(w\Lambda_i, \Lambda_i) \in \mathcal{C}_w$$

where $M(\lambda, \mu)$ denotes the simple $R(\mu - \lambda)$-module corresponding to $D_{w\Lambda_i,\Lambda_i} := \Delta_{w\Lambda_i,\Lambda_i}|_{A_q(\mathfrak{n})}$ (unipotent quatum minor).

Recall that $\{D_{w\Lambda_i,\Lambda_i}\}_{i\in I}$ is the set of frozen variables of the quantum cluster algebra $A_q(\mathfrak{n})$.

Proposition

The family $\{(C_i, R_{C_i})\}_{i\in I}$ forms a real commuting family of graded braiders in R-gmod (and hence in $\mathcal{C}_w$).

Category $\mathcal{C}_w$ and its localization

Theorem

Let $\iota_w : \mathcal{C}_w \hookrightarrow R\text{-gmod}$ be the embedding functor. Then the functor $\widetilde{\iota}_w : \mathcal{C}_w[C_i^{\circ -1} \mid i \in I] \to R\text{-gmod}[C_i^{\circ -1} \mid i \in I]$ induced from ι_w is an euqivalence of categories.

$$\begin{tikzcd}
R\text{-gmod} \arrow[r, "Q_w"] & R\text{-gmod}[C_i^{\circ -1} \mid i \in I] \\
\mathcal{C}_w \arrow[u, hook, "\iota_w"'] \arrow[r, "\Phi_w"] & \mathcal{C}_w[C_i^{\circ -1} \mid i \in I]. \arrow[u, dotted, "\simeq", "\widetilde{\iota}_w"']
\end{tikzcd}$$

Corollary

$X \in R\text{-gmod}$ belongs to $\mathcal{C}_w$ if and only if $R_{C_i}(X)$ is an isomorphism for each $i \in I$.

Rigidity

Theorem

The category $R\text{-}\mathrm{gmod}[C_i^{\circ -1} \,|\, i \in I]$ *is left-rigid, and hence* $\mathcal{C}_w[C_i^{\circ -1} \,|\, i \in I]$ *is left-rigid.*

Theorem

There is a equivalence of monoidal categories

$$\mathcal{C}_w[C_{i,w}^{\circ -1} \,|\, i \in I] \simeq \left(\mathcal{C}_{w^{-1}}[C_{i,w^{-1}}{}^{\circ -1} \,|\, i \in I]\right)^{\mathrm{rev}}.$$

Corollary

The category $\mathcal{C}_w[C_{i,w}^{\circ -1} \,|\, i \in I]$ *is right-rigid and hence rigid.*

Remark: The functor taking left-dual, $X \mapsto \mathscr{D}^{-1}(X)$, corresponds to the *twist automorphism* on $A_q(N^w)$(=quantum analogue of $\mathbb{C}[N^w]$) up to an easy antiautomorphism.

Kernel of $Q_w : R\text{-gmod} \to R\text{-gmod}[C_i^{\circ -1} \mid i \in I]$

Recall $Q_w : R\text{-gmod} \to R\text{-gmod}[C_i^{\circ -1} \mid i \in I] (\simeq \mathcal{C}_w[C_i^{\circ -1} \mid i \in I])$. The kernel $\mathrm{Ker}(Q_w) := \{M \in R\text{-gmod} \mid Q_w(M) \simeq 0\}$ is a Serre subcategory and an $\circ$-ideal of $R\text{-gmod}$, since Q_w is an exact monoidal functor.

Theorem

$$\{\textit{self-dual simples in } \mathrm{Ker}(Q_w)\} = \{S(b) \mid b \in B(\infty) \setminus B_w(\infty)\},$$

where

$$B_w(\infty) := \left\{ \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_\ell}^{a_\ell} b_\infty \mid (a_1, \ldots, a_\ell) \in \mathbb{Z}_{\geq 0}^{\ell} \setminus \{0\} \right\} \subset B(\infty),$$

$w = s_{i_1} \ldots_{i_\ell}$ *is a reduced expression of* w*, and*

$$S : B(\infty) \xrightarrow{\simeq} \{\textit{self-dual simples in } R\text{-gmod}\}.$$

Kernel of $Q_w : R\text{-gmod} \to R\text{-gmod}[C_i^{\circ -1} \mid i \in I]$

We have a commutative diagram of functors:

$$\begin{tikzcd}
\mathcal{C}_w \arrow[rr, "\Phi_w"] \arrow[d, hook] & & \mathcal{C}_w[C_i^{\circ -1} \mid i \in I] \arrow[d, "\widetilde{\iota}_w", "\simeq"'] \\
R\text{-gmod} \arrow[r] \arrow[rr, bend left, "Q_w"] & R\text{-gmod}/\mathrm{Ker}\, Q_w \arrow[r] & R\text{-gmod}[C_i^{\circ -1} \mid i \in I].
\end{tikzcd}$$

Taking their Grothendieck groups, we have

$$\begin{tikzcd}
A_q(\mathfrak{n}(w)) \arrow[rr] \arrow[d, hook] & & A_q(\mathfrak{n}(w)) D(w\Lambda, \Lambda_i)^{-1}; i \in I] \arrow[d, "{[\widetilde{\iota}_w]}", "\simeq"'] \\
A_q(\mathfrak{n}) \arrow[r] & A_q(\mathfrak{n})/I_w \arrow[r] & \big(A_q(\mathfrak{n})/I_w\big)[[D(w\Lambda, \Lambda_i)]^{-1}; i \in I],
\end{tikzcd}$$

where I_w is the ideal corresponding to $\mathrm{Ker}\, Q_w$.

If $\mathfrak{g}$ is symmetric and $\mathbf{k}$ is of characteristic zero, then the ideal I_w coincides with the ideal $(U_{w,q}^-)^\perp$ of Kimura(-Oya). And the above diagram recovers the theorem of Kimura-Oya which asserts that $[\iota_w]$ is an isomorphism.

Thank You!