Quantum dilogarithm identities of infinite product and quantum affine algebras

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Plan of this talk

1. Introduction
2. Quantum algebra $U_q(\mathfrak{g})$ and universal R-matrix
3. Convex orders and construction of convex bases for quantum affine algebras
4. Representing root vectors as q-commutator monomial
5. Construction of quantum dilogarithm identities

Quantum dilogarithm

Quantum dilogarithm is the function

$$\mathbb{E}(x) := \prod_{k=0}^{\infty} \frac{1}{1 + q^{2k+1}x} \quad \in \mathbb{Q}(q)[[x]].$$

This is called 'dilogarithm' since $\mathbb{E}(x) = \exp(\mathrm{Li}_{2,q}(-qx))$, where

$$\mathrm{Li}_{2,q}(x) := \sum_{n=1}^{\infty} \frac{x^n}{n(1-q^n)}$$

and $\mathrm{Li}_{2,q}(x)$ degenerates to usual dilogarithm function:

$$\lim_{q \to 1} (1-q)\mathrm{Li}_{2,q}(x) = \mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Quantum dilogarithm identities

It is well known that $\mathbb{E}(x)$ satisfies the following fundamental identity (especially called pentagon identity):

$$\mathbb{E}(x_1)\mathbb{E}(x_2) = \mathbb{E}(x_2)\mathbb{E}(q^{-1}x_1x_2)\mathbb{E}(x_1),$$

where x_1, x_2 are indeterminate satisfying $x_1x_2 = q^2x_2x_1$.

$\mathfrak{g} = sl_3$

$[E_1, E_2]_q \mapsto 0$ $\quad [E_2, E_1]_q \mapsto (1-q^2)e_1e_2 \neq 0$

Relationship with cluster transformations

- The quantum dilogarithm $\mathbb{E}(x)$ appears in the quantization of cluster transformations by Fock-Goncharov (They denote $\mathbb{E}(x)$ as $\Psi_q(x)$) [FG].
- The definitive work of Kashaev-Nakanishi [KN] enabled us to construct (quantum) dilogarithm identities from periods of (quantum) cluster algebras.
- Thus the behavior of quantum dilogarithm identities of finite product are well understood.

Identities of infinite product

On the other hand, Dimofte, Gukov, Soibelman proposed several concrete quantum dilogarithm identities containing infinite product in a context of physics [DGS]:

$$\begin{aligned}\mathbf{U}_{2,-1}\mathbf{U}_{0,1} = (\mathbf{U}_{0,1}\mathbf{U}_{2,1}\mathbf{U}_{4,1}\dots)& \\ \times \mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1}& \\ \times (\dots \mathbf{U}_{6,-1}\mathbf{U}_{4,-1}\mathbf{U}_{2,-1}),& \quad (A_1^{(1)})\end{aligned}$$

where $\mathbf{U}_{m,n} := \mathbb{E}(q^{-mn}x_1^m x_2^n)$.

Identities of infinite product [DGS]

$$\begin{aligned}
\mathbf{U}_{1,-1}\mathbf{U}_{1,0}\mathbf{U}_{0,1} &= (\mathbf{U}_{0,1}\mathbf{U}_{1,1}\mathbf{U}_{2,1}\mathbf{U}_{3,1}\dots) \\
&\quad\times \mathbf{U}_{1,0}^{2}\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\
&\quad\times (\dots\mathbf{U}_{3,-1}\mathbf{U}_{2,-1}\mathbf{U}_{1,-1}), \quad (A_2^{(1)}) \\
\mathbf{U}_{1,-1}^{2}\mathbf{U}_{0,1}^{2} &= (\mathbf{U}_{0,1}^{2}\mathbf{U}_{1,1}^{2}\mathbf{U}_{2,1}^{2}\mathbf{U}_{3,1}^{2}\dots) \\
&\quad\times \mathbf{U}_{1,0}^{4}\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\
&\quad\times (\dots\mathbf{U}_{3,-1}^{2}\mathbf{U}_{2,-1}^{2}\mathbf{U}_{1,-1}^{2}), \quad (A_3^{(1)}) \\
\mathbf{U}_{1,-2}\mathbf{U}_{0,1}^{4} &= (\mathbf{U}_{0,1}^{4}\mathbf{U}_{1,2}\mathbf{U}_{1,1}^{4}\mathbf{U}_{3,2}\mathbf{U}_{2,1}^{4}\dots) \\
&\quad\times \mathbf{U}_{1,0}^{6}\mathbb{E}(-qx_1^2)^{-1}\mathbb{E}(-q^{-1}x_1^2)^{-1} \\
&\quad\times (\dots\mathbf{U}_{2,-1}^{4}\mathbf{U}_{3,-2}\mathbf{U}_{1,-1}^{4}\mathbf{U}_{1,-2}), \quad (D_4^{(1)})
\end{aligned}$$

The aim of this talk

- We introduce another method to construct quantum dilogarithm identities using the product formula for the universal R-matrix of quantum affine algebra.
- Our result is that the four identities can be obtained mathematically by our new method.

Outline of the method

- Use product formula for universal R-matrix $\mathcal{R}$ [Ito] the product presentation depends on convex orders on positive roots, each factor is in fact quantum dilogarithm (except for imaginary roots!).

  $\exp_q(x) = \Psi((q-q^{-1})x)$

  most nontrivial

- By the uniqueness of $\mathcal{R}$, comparing different convex orders we have nontrivial identity.
- Appropriate degeneration process of the identity kills infinitely many factors, and eventually we obtain nontrivial identities of the form finite product $=$ infinite product.

Notation

- $\mathfrak{g}$: symmetrizable Kac-Moody Lie algebra
- $A = (a_{ij}) \in \mathrm{Mat}(n, \mathbb{Z})$: Cartan matrix of $\mathfrak{g}$
- $\mathfrak{h} \subset \mathfrak{g}$: Cartan subalgebra
- $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathfrak{h}^*$: simple roots
- $\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_n \in \mathfrak{h}$: simple coroots
- $(\cdot, \cdot) : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$: invariant bilinear form
- $W := \langle s_i \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$: Weyl group
  s_i: simple reflection w.r.t. α_i
- $\Delta \subset \mathfrak{h}^*$: root system of $\mathfrak{g}$

Kac's textbook

Quantum algebra $U_q(\mathfrak{g})$

Definition

$U_q(\mathfrak{g})$ is the $\mathbb{Q}(q)$-algebra defined by

Generators : $E_i, F_i, K_\lambda \quad (i = 1, 2, \ldots, n,\ \lambda \in P)$.

Relations : $K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_0 = 1 \quad (\lambda, \mu \in P)$;

$$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda,\alpha_i)} E_i,$$

$$K_\lambda F_i K_\lambda^{-1} = q^{-(\lambda,\alpha_i)} F_i \quad (\lambda \in P, i = 1, 2, \ldots, n);$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad (i, j = 1, 2, \ldots, n);$$

Quantum Serre relations;

where $P := \{\lambda \in \mathfrak{h}^* | \lambda(\check{\alpha}_i) \in \mathbb{Z}\,(i = 1, 2, \ldots, n)\}$.

Quantum algebra $U_q(\mathfrak{g})$

Definition

Quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j),$$

where $q_i := q^{\frac{1}{2}(\alpha_i,\alpha_i)}$, $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$,

$[n]_q! := [1]_q [2]_q \dots [n]_q$, $\begin{bmatrix} m \\ k \end{bmatrix}_q := \frac{[m]_q!}{[k]_q! [m-k]_q!}$.

Triangular decomposition

We set the following $\mathbb{Q}(q)$-subalgebras of $U_q(\mathfrak{g})$.

$$U_q^+ := \langle E_i \rangle, \quad U_q^0 := \langle K_\lambda \rangle, \quad U_q^- := \langle F_i \rangle,$$
$$U_q^{\geq 0} := \langle E_i, K_\lambda \rangle, \quad U_q^{\leq 0} := \langle F_i, K_\lambda \rangle.$$

Proposition (Triangular decomposition)

$$U_q^- \otimes U_q^0 \otimes U_q^+ \cong U_q(\mathfrak{g}), \quad x \otimes y \otimes z \mapsto xyz.$$

Hopf algebra

Definition

Hopf algebra is a k-algebra H equipped with coproduct $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$, and antipode $S : H \to H$ satisfying

- $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$
- $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \iota = (\mathrm{id} \otimes \varepsilon) \circ \Delta$
- $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta.$

where $\iota : H \cong k \otimes H \cong H \otimes k$ natural isomorphism,
$\mu : H \otimes H \to H$ is the multiplication,
$u : k \to H$, $1_k \mapsto 1_H$ is the unit map.

Hopf algebra structure

$U_q(\mathfrak{g})$ has a structure of Hopf algebra defined by

$$\begin{aligned}
\Delta(E_i) &:= E_i \otimes 1 + K_i \otimes E_i, \\
\Delta(F_i) &:= F_i \otimes K_i^{-1} + 1 \otimes F_i, \\
\Delta(K_\lambda) &:= K_\lambda \otimes K_\lambda, \\
\varepsilon(E_i) &:= 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_\lambda) := 1, \\
S(E_i) &:= -K_i^{-1} E_i, \quad S(F_i) := -F_i K_i, \quad S(K_\lambda) := K_\lambda^{-1}
\end{aligned}$$

$$(i = 1, 2, \ldots, n;\ \lambda \in P).$$

Universal R-matrix

Definition

For Hopf algebra$H$, an invertible element $\mathcal{R} \in H \otimes H$ is called universal R-matrix *if it satisfies*

$$\Delta^{\mathrm{op}}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \quad (x \in H),$$
$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \quad \in H \otimes H \otimes H,$$
$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12} \quad \in H \otimes H \otimes H,$$

where $\mathcal{R}_{12} := \sum_i a_i \otimes b_i \otimes 1$ *and so on when* $\mathcal{R} = \sum_i a_i \otimes b_i$.

Hopf pairing

There exists unique non-degenerate bilinear form $(\cdot|\cdot) : U_q^{\geq 0} \otimes U_q^{\leq 0} \to \mathbb{Q}(q)$ having the following properties, which is called Hopf pairing.

$$
\begin{aligned}
(x|y_1y_2) &= (\Delta(x)|y_1 \otimes y_2) \quad (x \in U_q^{\geq 0},\ y_1, y_2 \in U_q^{\leq 0}),\\
(x_1x_2|y) &= (x_2 \otimes x_1|\Delta(y)) \quad (x_1, x_2 \in U_q^{\geq 0},\ y \in U_q^{\leq 0}),\\
(K_\mu|K_\nu) &= q^{-(\mu,\nu)} \quad (\mu, \nu \in P),\\
(E_i|K_\mu) &= (K_\mu|F_i) = 0 \quad (\mu \in P,\ i = 1, 2, \ldots, n),\\
(E_i|F_j) &= \frac{\delta_{ij}}{q_i^{-1} - q_i} \quad (i, j = 1, 2, \ldots, n),
\end{aligned}
$$

where $(x_1 \otimes x_2|y_1 \otimes y_2) := (x_1|y_1)\,(x_2|y_2)$.

Construction of universal R-matrix

Theorem (Drinfel'd, Tanisaki)

$$\mathcal{R} = \Theta q^{-T} \quad \in U_q(\mathfrak{g})\widehat{\otimes} U_q(\mathfrak{g})$$

Φ doesnot convergent

is the universal R-matrix of $U_q(\mathfrak{g})$, where $T \in \mathfrak{h} \otimes \mathfrak{h}$ is the canonical element of invariant bilinear form $(\cdot,\cdot)$, and $\Theta \in U_q^+ \widehat{\otimes} U_q^-$ is the canonical element of Hopf paring $(\cdot|\cdot)|_{U_q^+ \otimes U_q^-}$.

quasi R-matrix

Convex bases

It is known that Θ can be more explicitly described using convex basis.

Definition (Convex bases)

Let $U = U_q(\mathfrak{g})$, $\Lambda \subset U$, $\leq$: *total order on* Λ.
For $\Sigma \subset \Lambda$, *let*
$\mathscr{E}_<(\Sigma) :=$ {*increasing monomial consists of*Σ}.
$\mathscr{E}_<(\Lambda)$ *is called* *convex basis* *of* U *when*

- $\mathscr{E}_<(\Lambda)$ *is a* $\mathbb{Q}(q)$*-basis of* U
- *For every interval* $I \subset \Lambda$, $\mathscr{E}_<(I)$ *forms a* $\mathbb{Q}(q)$*-basis of* $U_I := \langle I \rangle \subset U$.
  Inverval is a subset of the form Λ, $(x, *)$, $[x, *)$, $(*, y)$, $(*, y]$, (x, y), $[x, y)$, $(x, y]$, $[x, y]$ $(x, y \in \Lambda)$.

Braid group action on $U_q(\mathfrak{g})$

- Associated with Weyl group W, braid group $\mathcal{B}$ is the group whose defining relations are same with W's except for $s_i^2 = 1$.
  $\rightsquigarrow$ canonical projection $p : \mathcal{B} \twoheadrightarrow W$
- $s_{i_1} s_{i_2} \dots s_{i_m} = s'_{j_1} s'_{j_2} \dots s'_{j_m} \in W$ are both reduced expressions in W
  $\Rightarrow$ both side coincide in $\mathcal{B}$.
  $\rightsquigarrow$ Taking reduced expression defines natural map $W \hookrightarrow \mathcal{B}$ (*not* group hom)

Braid group action on $U_q(\mathfrak{g})$

Theorem (Lusztig)

There exists unique $T_i \in \operatorname{Aut} U_q(\mathfrak{g})$ *satisfying*

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i, \quad T_i(K_\lambda) = K_{s_i(\lambda)}$$

$$T_i(E_j) = \frac{1}{[-a_{ij}]_{q_i}!} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} \begin{bmatrix} -a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{-a_{ij}-k} E_j E_i^k,$$

$$T_i(F_j) = \frac{1}{[-a_{ij}]_{q_i}!} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} \begin{bmatrix} -a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{-a_{ij}-k} \ (j \neq i)$$

There exists unique group homomorphism $T : \mathcal{B} \to \operatorname{Aut}(U_q(\mathfrak{g}))$ *such that* $T(s_i) = T_i$.

Construction of convex bases ($\dim \mathfrak{g} < \infty$)

- When $\mathfrak{g}$ is simple Lie algebra, $|W| < \infty$.
  The longest element $w_\circ \in W$ exists.
- Choose a reduced expression $w_\circ = s_{i_1} s_{i_2} \dots s_{i_N}$, and set $\beta_k := s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k})$.
  Then $\Delta_+ = \{\beta_k | k = 1, 2, \dots, N\}$.
- $\beta_i \leq \beta_j \Leftrightarrow i \leq j$ defines a total order on Δ_+.
- Set root vectors $E_{\leq,\beta_k} := T_{i_1} T_{i_2} \dots T_{i_{k-1}}(E_{i_k})$, $\Lambda := \{E_{\leq,\beta_k} | k = 1, 2, \dots, N\}$, then $\mathscr{E}_<(\Lambda)$ is a convex basis of U_q^+.
- Using Chevalley involution $\Omega : U_q^+ \to U_q^-$, we can construct convex basis for U_q^-, and eventually for whole $U_q(\mathfrak{g})$.

Convex bases for quantum affine algebra

When $\mathfrak{g}$ is of affine type, several problems appear!

- $\dim \mathfrak{g} = |\Delta| = |W| = \infty$. No longest element
  How to choose presentation $\alpha = w(\alpha_i)$ for $\alpha \in \Delta_+^{\mathrm{re}}$?
- Existence of imaginary roots
  How to construct root vectors for imaginary roots?

These problems has been solved by Beck [B] and Ito [Ito2].

Convex orders

Definition ([Ito])

A total order $\leq$ on $B \subset \Delta_+$ is called convex if

- *For* $\beta, \gamma \in \Delta_+^{\mathrm{re}} \cap B$,
  $\beta < \gamma,\ \beta + \gamma \in B \Rightarrow \beta < \beta + \gamma < \gamma$.
- *For* $\beta \in B,\ \gamma \in \Delta_+ \setminus B,\ \beta + \gamma \in B \Rightarrow \beta < \beta + \gamma$.

Ito classified convex orders on Δ_+ when $\mathfrak{g}$ is untwisted affine algebra [Ito].

Construction of general convex order

- When $\mathfrak{g}$ is of type $X_\ell^{(1)}$, $X = A \sim G$
  choose $w \in \mathring{W}$ (finite Weyl group of type X_ℓ).
  $\rightsquigarrow \Delta_+ = \Delta(w,-) \amalg \Delta_+^{\mathrm{im}} \amalg \Delta(w,+)$
  $\Delta(w,\pm) := \{m\delta + w\varepsilon \mid m \in \mathbb{Z}_{\geq 0}, \varepsilon \in \mathring{\Delta}_\pm\} \cap \Delta_+$
- Convex orders on $\Delta(w,\pm)$ are classified, whose ordinal number is $n\omega$ $(n \leq \ell)$ (n-row type).
- Choose any total order on Δ_+^{im}.
- Concatenating these orders yields a convex order on whole Δ_+.

Theorem (Ito, 2001)

Any convex order on Δ_+ can be constructed by above procedure.

Example of convex order, type $A_1^{(1)}$

$\mathfrak{g} = \widehat{\mathfrak{sl}_2}$

$$\Delta_+ = \{m\delta - \alpha_1 | m \in \mathbb{Z}_{\geq 1}\} \amalg \{m\delta | m \in \mathbb{Z}_{\geq 1}\}$$
$$\amalg \{m\delta + \alpha_1 | m \in \mathbb{Z}_{\geq 0}\}.$$

$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_2$

$\downarrow$

$\alpha_2 < \alpha_1 + \alpha_2 < \alpha_1$

$$\delta - \alpha_1 < 2\delta - \alpha_1 < 3\delta - \alpha_1 < \dots$$
$$< \delta < 2\delta < 3\delta < 4\delta < \dots$$
$$\dots < 3\delta + \alpha_1 < 2\delta + \alpha_1 < \delta + \alpha_1 < \alpha_1,$$

where $\delta := \alpha_0 + \alpha_1$ is the null root.

Example of convex order, type $A_2^{(1)}$

$$\Delta_+ = \{m\delta - \varepsilon | m \in \mathbb{Z}_{\geq 1},\ \varepsilon = \alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$$
$$\amalg\ \Delta_+^{\mathrm{im}} \amalg \{m\delta + \varepsilon | m \in \mathbb{Z}_{\geq 0},\ \varepsilon = \alpha_1, \alpha_1 + \alpha_2, \alpha_2\}.$$

Set $w = s_1$. This is a convex order of 2-row type.

$$\delta - \alpha_1 - \alpha_2 < \delta - \alpha_2 < 2\delta - \alpha_1 - \alpha_2 < 2\delta - \alpha_2 < \dots$$
$$< \alpha_1 < \delta + \alpha_1 < 2\delta + \alpha_1 < 3\delta + \alpha_1 < \dots$$
$$< \delta < 2\delta < 3\delta < 4\delta < \dots$$
$$\dots < 3\delta - \alpha_1 < 2\delta - \alpha_1 < \delta - \alpha_1$$
$$\dots < \delta + \alpha_1 + \alpha_2 < \delta + \alpha_2 < \alpha_1 + \alpha_2 < \alpha_2,$$

where $\delta := \alpha_0 + \alpha_1 + \alpha_2$.

Construction of real root vectors (outline)

Convex order determines a presentation $\alpha = w(\alpha_i)$ for each positive real root $\alpha \in \Delta_+^{\mathrm{re}}$
$\Rightarrow$ Define $E_{\leq,\alpha} := T_w(E_i)$

Example

When $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$: type $A_2^{(1)}$ and using the convex order of previous page,
the root vector for $m\delta - \alpha_1 - \alpha_2$ $(m{\geq}1)$ is

$$E_{\leq, m\delta-\alpha_1-\alpha_2} = \overbrace{T_0T_1T_2T_0T_1T_2\dots}^{2(m-1)}(E_k)\,(k{=}1{-}m \bmod 3).$$

q-bracket

- There is natural weight space decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\mu \in \mathfrak{h}^*} U_\mu,$$

where for each $\mu \in \mathfrak{h}^*$,

$$U_\mu := \{x \in U_q(\mathfrak{g}) | K_\lambda x K_\lambda^{-1} = q^{(\lambda,\mu)} x \quad (\forall \lambda \in P)\}.$$

- For $x \in U_\mu, y \in U_\nu$ $(\mu, \nu \in \mathfrak{h}^*)$, we define q-bracket

$$[x, y]_q := xy - q^{(\mu,\nu)} yx.$$

Construction of imaginary root vectors

Imaginary root vectors $I_{i,m}$ associated with $m\delta \in \Delta_+^{\mathrm{im}}$ is constructed as following procedure. $(1 \le i \le \ell,\ m \in \mathbb{Z}_{\ge 1})$.

- Let $\mathcal{E}_{m\delta-\alpha_i} := T_{\varepsilon_i}^m T_i^{-1}(E_i)$.
  T_{ε_i}: Action of translation w.r.t. fundamental coweight ε_i.
  (Constructed by extended braid group action [B])
- Let $\varphi_{i,m} := [\mathcal{E}_{m\delta-\alpha_i}, E_i]_q$, and Beck
  $\varphi_i(z) := (q_i - q_i^{-1}) \sum_{m\ge1} \varphi_{i,m} z^m \in U_q^+[[z]]$.
- Set $I_i(z) := \log(1+\varphi_i(z))$, Then $I_{i,m} \in U_q^+$ are determined by $I_i(z) = (q_i - q_i^{-1}) \sum_{m\ge1} I_{i,m} z^m$.
- Each $I_{i,m}$ is a polynomial of $\varphi_{i,m}$

Convex bases for quantum affine algebra

Theorem ([Ito2])

Choose a convex order on Δ_+, and set

$$\Lambda := \{E_{\leq,\alpha} | \alpha \in \Delta_+^{\mathrm{re}}\} \amalg \{T_w(I_{i,m}) | m \in \mathbb{Z}_{\geq 1}, i = 1, 2, \ldots, \ell\}.$$

Using given convex order, we set a total order on Λ. The ordering between $I_{i,m}$ is defined by

$$I_{i,m} \leq I_{j,m'} \Leftrightarrow (m \leq m') \text{ or } (m = m', i \leq j).$$

$\Rightarrow \mathscr{E}_<(\Lambda)$ *is a convex basis of U_q^+, where $w \in \mathring{W}$ was determined by given convex order.*

Explicit product presentation of Θ

Theorem ([Ito2])

For any convex order $\leq$ *on* Δ_+,

$$\Theta = \prod_{\alpha\in\Delta_+}^{>} \Theta_{\leq,\alpha} \quad \in U_q^+ \widehat{\otimes} U_q^-,$$

where $\prod^>$ *means* $\alpha < \beta \Rightarrow \Theta_{\leq,\beta}\Theta_{\leq,\alpha}$. *Each factor* $\Theta_{\leq,\alpha}$ *is written by root vectors:*

$$\Theta_{\leq,\alpha} = \mathbb{E}_{q_\alpha}(-(q_\alpha - q_\alpha^{-1})^2 E_{\leq,\alpha} \otimes F_{\leq,\alpha}) \quad (\alpha \in \Delta_+^{\mathrm{re}}),$$

where $q_\alpha := q^{\frac{1}{2}(\alpha,\alpha)}$, $F_{\leq,\alpha} := \Omega(E_{\leq,\alpha})$,
and $\mathbb{E}_{q_\alpha}(x)$ *means replacing* $q \mapsto q_\alpha$

Explicit product presentation of Θ

$$b_{i,j;n} := \mathrm{sgn}(a_{ij})^n \frac{[a_{ij} n]_{q_i}}{n(q_j^{-1} - q_j)}, \quad \mathrm{sgn}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

Let $(c_{i,j;n})_{i,j=1}^{\ell} \in \mathrm{Mat}(\mathbb{Q}(q), \ell)$ denote the inverse matrix of $(b_{i,j;n})_{i,j=1}^{\ell}$ and $J_{i,n} := \Omega(I_{i,n})$.

$$S_n := \sum_{i,j \in \mathring{I}} c_{j,i;n} I_{i,n} \otimes J_{j,n} \quad \in U_q^+ \otimes U_q^-,$$

$$\Theta_{\leq, n\delta} := \exp\{T_w \otimes T_w(S_n)\}$$

Remark: $\Theta_{\leq, n\delta}$ itself is *not* quantum dilogarithm!

q-commutator monomials

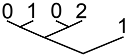
Definition ([S])

The elements in U_q^+ which can be represented by (nonzero scalar multiple of) finitely many applications of q-bracket among the positive generators E_i are called q-commutator monomial.

We illustrate the manipulation of taking q-bracket as

$$:= [X, Y]_q \quad (X, Y \in U_q(\mathfrak{g})),$$

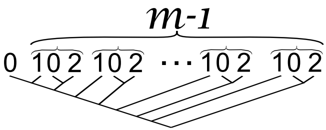
and abbreviate E_i as just i.

Example: $\left[\left[[E_0, E_1]_q, [E_0, E_2]_q\right]_q, E_1\right]_q =$  .

Root vectors are q-commutator monomial

Example of presentation of real root vectors:

$$E_{\leq, m\delta-\alpha_1-\alpha_2} =$$



$$(m \geq 1).$$

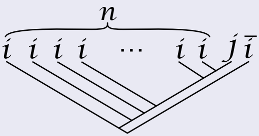
Theorem ([S])

Let $\mathfrak{g}$ be untwisted affine Lie algebra of type $X_\ell^{(1)}$. Then for any convex order $\leq$ on Δ_+, the real root vectors $E_{\leq,\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}})$ and $T_w(\varphi_{i,m})$ $(w \in \mathring{W}, 1 \leq i \leq \ell, m \in \mathbb{Z}_{\geq 1})$ are q-commutator monomial.

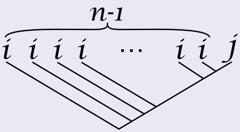
The key formula

Lemma ([S])

In general $U_q(\mathfrak{g})$*, for any* $n \geq 0$ *and any two indices* i, j $(i \neq j)$*, we have the following reduction formula:*



$$= [n]_{q_i}[1 - a_{ij} - n]_{q_i}$$



,

where $\bar{i} := T_i(E_i) = -F_iK_i$*,*
and let RHS $:= 0$ *when* $n = 0$*.*

Algorithm to compute root vectors

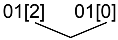
To construct concrete identities, we have to compute root vectors explicitly. Since root vectors are described by braid group action,

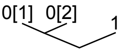
- For $\alpha = s_{i_1} s_{i_2} \dots s_{i_{m-1}}(\alpha_{i_m})$ ($s_{i_1} \dots s_{i_m}$: reduced), represent $T_{i_1} T_{i_2} \dots T_{i_{m-1}}(E_{i_m})$ as a q-commutator monomial.
- We manipulate a binary tree, each of whose leaf holds a pair of a reduced expression and an index.
- At the beginning we have a binary tree consists of only one leaf, which has $(s_{i_1} s_{i_2} \dots s_{i_{m-1}}, i_m)$.

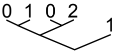
Algorithm to compute root vectors

- For each leaf of the binary tree, the following manipulations are applied recursively.
  The process terminates when the length of reduced expression $m = 0$.
  - $s_{i_1} s_{i_2} \dots s_{i_{m-2}}(\alpha_{i_m}) \in \Delta_+ \Rightarrow$ make branch:
    generate new 2 leaves: (s', i_{m-1}), (s', i_m),
    where $s' := s_{i_1} s_{i_2} \dots s_{i_{m-2}}$
  - $s_{i_1} s_{i_2} \dots s_{i_{m-2}}(\alpha_{i_m}) \in \Delta_- \Rightarrow$ reduction:
    $(\dots s_k s_i s_j, i) \mapsto (\dots s_k, j)$ (ADE).

Example: $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$, type $A_2^{(1)}$

$T_0 T_1 T_2(E_0) = 012[0] =$ 

$=$ 

$=$ 

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Degeneration of Θ

Choose $\sigma_{ij} \in \{\pm 1\}$ for $i < j$ s.t. $a_{ij} \neq 0$, and set

$$b_{ij} := \begin{cases} \sigma_{ij}(\alpha_i, \alpha_j) & i < j \\ 0 & i = j \\ -\sigma_{ij}(\alpha_i, \alpha_j) & i > j \end{cases}.$$

There is natural 1:1 correspondence between sign data $(\sigma_{ij})_{ij}$ and orientation of Dynkin quiver.
Let $\mathcal{P}_B^+$ be a $\mathbb{Q}(q)$-algebra defined by the generators and relations below.

$$\text{Generators} : e_1, e_2, \ldots, e_n.$$

$$\text{Relations} : e_i e_j = q^{b_{ij}} e_j e_i \quad (i, j = 1, 2, \ldots, n).$$

If $i < j$ and $\sigma_{ij} = +1$, then $[e_i, e_j]_q = 0$.

Degeneration of Θ

Proposition

There exists unique Q-graded algebra surjection

$$\pi_B^+ : U_q^+ \to \mathcal{P}_B^+$$

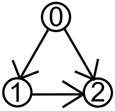
such that $\pi_B^+(E_i) = e_i$ for all $i = 1, \ldots, n$.

In the same way, we have algebra surjection $\pi_B^- : U_q^- \to \mathcal{P}_B^- = \langle f_i \rangle$ such that $[f_i, f_j]_q = 0$ if $\sigma_{ij} = +1$.
Extending continuously these maps, we have

$$\pi_B^+ \widehat{\otimes} \pi_B^- : U_q^+ \widehat{\otimes} U_q^- \to \mathcal{P}_B^+ \widehat{\otimes} \mathcal{P}_B^-,$$

and thus the image $\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta)$ makes sense.
We can degenerate Θ using this map.

Example: $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$, type $A_2^{(1)}$

- Let $\leq$ be the convex order of 2-row type presented earlier.
- Write down two product presentations of Θ determined by $\leq$ and the reversed order $\leq^{\mathrm{op}}$.
- Set $B =$ 

 $(\sigma_{01} = \sigma_{02} = \sigma_{12} = +1)$ and explicitly compute $\pi_B^+ \widehat{\otimes} \pi_B^-(\Theta)$ using the algorithm.
- Equating them, we eventually obtain the quantum dilogarithm identity of next page.
- By change of variables, this identity coincide with the first one of page 8, labeled $A_2^{(1)}$.
- In the same way, other 3 identities can also be derived.

Example: $\mathfrak{g} = \widehat{\mathfrak{sl}_3}$, type $A_2^{(1)}$

Let $y_i := -(q-q^{-1})^2 e_i \otimes f_i$ $(i = 0, 1, 2)$, then

$$
\begin{aligned}
&\mathbb{E}(:y_2:)\mathbb{E}(:y_1:)\mathbb{E}(:y_0:) \\
&= \left\{ \prod_{m\geq 0}^{\rightarrow} \mathbb{E}(:y_0^{m+1}y_1^m y_2^m:)\mathbb{E}(:y_0^{m+1}y_1^{m+1}y_2^m:) \right\} \mathbb{E}(:y_0y_2:) \\
&\times \mathbb{E}(-q:y_0y_1y_2:)^{-1}\mathbb{E}(-q^{-1}:y_0y_1y_2:)^{-1} \\
&\times \mathbb{E}(:y_1:) \left\{ \prod_{m\geq 0}^{\leftarrow} \mathbb{E}(:y_0^m y_1^{m+1} y_2^{m+1}:)\mathbb{E}(:y_0^m y_1^m y_2^{m+1}:) \right\},
\end{aligned}
$$

where $\prod_{m\geq 0}^{\rightarrow} a_m := a_0 a_1 a_2 \ldots$, $\prod_{m\geq 0}^{\leftarrow} a_m := \ldots a_2 a_1 a_0$,
$:y_0^l y_1^m y_2^n: \; := q^{lm+mn+nl} y_0^l y_1^m y_2^n$.

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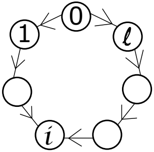
Some conjecture

Let $\mathfrak{g} = \widehat{\mathfrak{sl}_N}$ $(N \geq 2)$, $\ell := N - 1$,

$Q_i :=$ 

be the Dynkin quiver whose vertex 0 is unique source and vertex i is unique sink,

$$A_i := \{ \alpha_j + \alpha_{j+1} + \cdots + \alpha_k \mid 1 \leq j \leq i \leq k \leq \ell \} \subset \mathring{\Delta}_+$$

be the set of positive roots containing α_i component, and for $w \in \mathring{W}$,

$$\Phi(w) := w\mathring{\Delta}_- \cap \mathring{\Delta}_+.$$

Some conjecture

Conjecture

If $A_i \not\subseteq \Phi(w)$*, then* $\pi^+_{Q_i}(T_w(\varphi_{j,m})) = 0$ *for all* j *and* m*.*
When $A_i \subset \Phi(w)$*, then there exists a permutation of indices* σ *such that*

$$(q-q^{-1})\pi^+_{Q_i}(T_w(\varphi_{j,m})) = \begin{cases} D & m=1, |j-\check{i}|=1 \\ [m+1]_q D^m & j=\check{i} \\ 0 & \textit{otherwise} \end{cases},$$

where $\check{i} := N - i$ *and* $D := (q-q^{-1})^N e_{\sigma(0)} e_{\sigma(1)} \dots e_{\sigma(\ell)}$*.*

Some conjecture

Corollary

If the conjecture is true, then for arbitrary convex order

$$\pi^+_{Q_i} \widehat{\otimes} \pi^-_{Q_i}(\Theta_{\mathrm{im}}) = \begin{cases} \mathbb{E}(-q{:}y_0y_1\dots y_\ell{:})^{-1}\mathbb{E}(-q^{-1}{:}y_0y_1\dots y_\ell{:})^{-1} \\ 1 \end{cases}$$

depending on w. Thus using the quiver Q_i, we can obtain quantum dilogarithm identities using any convex order.

Any acyclic Dynkin quiver of type $A_\ell^{(1)}$ can be obtained by applying mutation at source or sink finitely many times on the quiver Q_i.