Kronecker Coefficients via Upper Cluster Algebras

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The Multiple Incarnation of Kronecker Coefficients

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2. Inner plethysm of Schur functions;
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Let S^λ be the irreducible complex representation of $\mathfrak{S}_n$. The Kronecker coefficients $g^\lambda_{\mu,\nu}$ are the tensor product multiplicities:

$$S^\mu \otimes S^\nu \cong \bigoplus_\lambda g^\lambda_{\mu,\nu} S^\lambda.$$

cf. Littlewood-Richardson coefficients

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Inner Plethysm

Let $S_\lambda(V)$ be the irreducible complex representation of $\mathrm{GL}(V)$.

$$S_\lambda(V \otimes W) = \bigoplus_{\mu,\nu} g^{\lambda}_{\mu,\nu} S_\mu(V) \otimes S_\nu(W).$$

This follows from the Schur-Weyl duality because

$$S_\lambda(V) = \mathrm{Hom}_{\mathfrak{S}_n}(S^\lambda, V^{\otimes n}).$$

In terms of Schur functions:

$$s_\lambda(\mathbf{xy}) = \sum_{\mu,\nu} g^{\lambda}_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y}).$$

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Some Algorithms

1. Character table

$$g_{\lambda,\mu,\nu} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma).$$

2. Dvir's recursive algorithm (1993)
3. Counting lattice points (in $\ell(\lambda)!\ell(\mu)!\ell(\nu)!$ polytopes) (2008)

$$\prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} s_\lambda(x) s_\mu(y) s_\nu(z).$$

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The Role in the Geometric Complexity Theory

The Geometric Complexity Theory (GCT) is a research program in computational complexity theory proposed by K. Mulmuley and M. Sohoni to attack the famous open problem in computer science whether $\mathrm{P} = \mathrm{NP}$ by showing that the complexity class P is not equal to the complexity class NP.

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Flagged Kronecker Quivers

Let $K^m_{\ell,\ell}$ be the flagged Kronecker quiver

$$\bar{1} \longrightarrow \bar{2} \longrightarrow \cdots \longrightarrow \bar{\ell} \underset{m \text{ arrows}}{\Longrightarrow} \ell \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

and β_ℓ be the dimension vector defined by $\beta_\ell(i) = |i|$. Consider the product of special linear group SL_{β_ℓ} acting naturally on the quiver representation space $\mathrm{Rep}_{\beta_\ell}(K^m_{\ell,\ell})$.

$$\mathrm{Rep}_{\beta_\ell}(K^m_{\ell,\ell}) := \bigoplus_{i=1}^{l-1} \left(\mathrm{Hom}(V_{-i}, V_{-(i+1)}) \oplus \mathrm{Hom}(V_{i+1}, V_i)\right) \oplus \mathrm{Hom}(V_{-l}, V_l) \otimes W.$$

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Semi-invariants of Flagged Kronecker Quivers

Definition

The semi-invariant ring $\mathsf{SI}_{\beta_\ell}(K^m_{\ell,\ell})$ is by definition equal to $\mathbb{C}[\mathsf{Rep}_{\beta_\ell}(K^m_{\ell,\ell})]^{\mathsf{SL}_{\beta_\ell}}$.

The semi-invariant ring $\mathsf{SI}_{\beta_\ell}(K^m_{\ell,\ell})$ is graded by a weight $\sigma \in \mathbb{Z}^{2\ell}$ and a weight λ of $T \subset \mathsf{GL}(W)$:

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Here

$$\mathsf{SI}_{\beta_\ell}(K^m_{\ell,\ell})_{\sigma,\lambda} = \{f \in \mathbb{C}[\mathsf{Rep}_{\beta_\ell}(K^m_{\ell,\ell})] \mid (g,t)\cdot f = \chi_\sigma(g)t^\lambda f \quad \forall g \in \mathsf{GL}_{\beta_\ell},\ t \in T\}.$$

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Kronecker Coefficients via Semi-invariant Rings

For any pair of partitions μ and ν of *length* no greater than ℓ, we can associate a weight vector $\sigma(\mu,\nu) \in \mathbb{Z}^{K^m_{\ell,\ell}}$.

Theorem (F)

Let (μ,ν,λ) be a triple of partitions of length no greater than ℓ,ℓ and m respectively, then

$$g^{\lambda}_{\mu,\nu} = \sum_{\omega\in\mathfrak{S}_m} \operatorname{sgn}(\omega) \dim \mathrm{SI}_{\beta_\ell}(K^m_{\ell,\ell})_{\sigma(\mu,\nu),\lambda^\omega},$$

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Upper Cluster Algebra

Let $\mathcal{L}(\mathbf{x})$ be the Laurent polynomial algebra in cluster $\mathbf{x}$ which is polynomial in coefficient variables. The upper cluster algebra $\overline{\mathcal{C}}(\Delta, \mathbf{x})$ is the intersection of all $\mathcal{L}(\mathbf{x}')$ where $\mathbf{x}'$ is a cluster.

$$\overline{\mathcal{C}}(\Delta, \mathbf{x}) := \bigcap_{(\Delta', \mathbf{x}') \sim (\Delta, \mathbf{x})} \mathcal{L}(\mathbf{x}').$$

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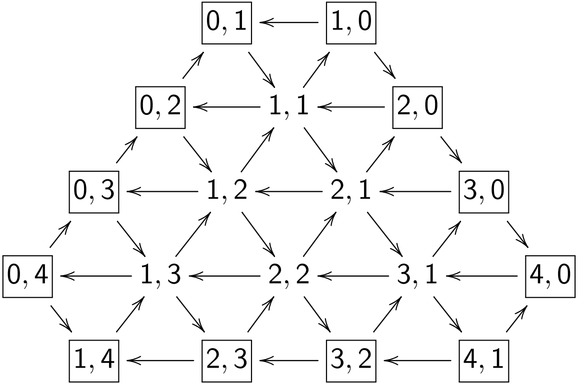
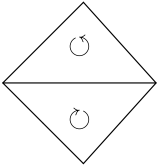
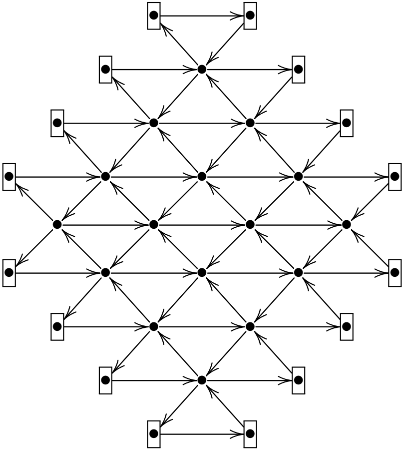
The Hive Quivers

Here is a hive quiver Δ_ℓ ($\ell = 5$).



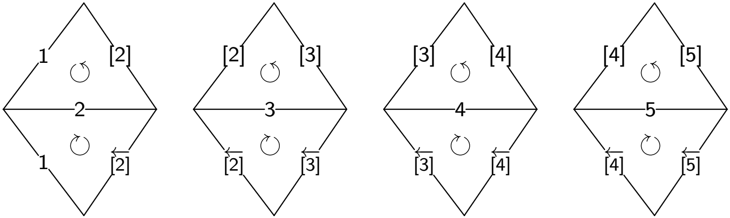
Quivers from Gluing Oriented Triangles





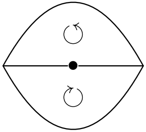
Gluing $m-1$ Diamond Quivers

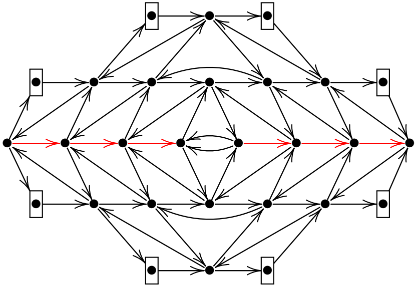
Let $\Diamond_\ell^m$ be quiver obtained by inconsistently gluing the $m-1$ diamond quivers along the edges with the same label. Note that the first one has two edges glued together.



Example: gluing a single diamond quiver

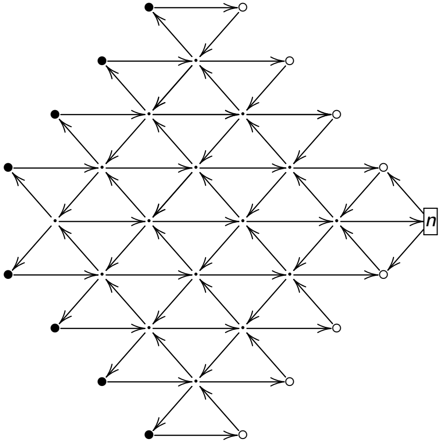
By definition, the quiver of the first diamond after gluing the edge 1 looks like (when $\ell = 5$)





Extending $\diamondsuit_\ell^m$

We insert one frozen vertices n and three arrows to each n-th diamond as shown below, and get a quiver denoted by $\overline{\diamondsuit}_\ell^m$.



The semi-invariant rings are upper cluster algebras

Theorem (F)

For any $\ell, m \geq 2$, the semi-invariant ring $\mathrm{SI}_{\beta_\ell}(K^m_{\ell,\ell})$ is isomorphic to the graded upper cluster algebra $\overline{\mathcal{C}}(\overline{\diamondsuit}^m_\ell, \overline{\mathbf{s}}^m_\ell; \overline{\boldsymbol{\sigma}}^m_\ell)$. Here, each cluster variable in $\overline{\mathbf{s}}^m_\ell$ is a Schofield's semi-invariant.

Category of Quiver Representations

The category of representations of a quiver without oriented cycles is abelian, Krull-Schimdt, having enough projective and injective objects.

The indecomposable projective representations P_i are in bijection with the vertices of Q. The vector space $P_i(j)$ is spanned by all paths from i to j.

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Schofield's Semi-invariants

Let Q be a quiver without oriented cycles. Take a projective presentation f, that is, f is an element in $\mathrm{Hom}_Q(P_1, P_0)$. We apply $\mathrm{Hom}_Q(-, W)$ to f and obtain

$$\mathrm{Hom}_Q(P_0, W) \xrightarrow{f(W)} \mathrm{Hom}_Q(P_1, W).$$

We define a polynomial function s_f on $\mathrm{Rep}_\beta(Q)$ by

$$s_f(W) = \det(f(W)).$$

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How a typical $\bar{\mathbf{s}}_\ell^m$ looks like?

Let $\widetilde{f}_{i,j}^n$ be the following presentation for $n = 2r + 2$

$$P_{i+j} \oplus rP_\ell \xrightarrow{\begin{bmatrix} n & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & n-4 & n-3 \\ 0 & n-1 & 0 & \cdots & 0 & n-2 \end{bmatrix}} P_{-i} \oplus P_{-j} \oplus rP_{-\ell}$$

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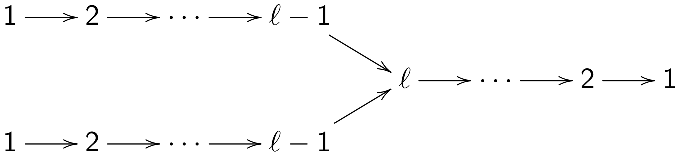
Why?

It is known (F) semi-invariant rings of the triple flag quivers T_ℓ



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There are natural maps from an open subset U of $R^m_{\ell,\ell} /\!\!/ \mathrm{SL}_{\beta_\ell}$ to an open subset of $\mathrm{Rep}_{\beta_\ell'}(T_\ell) /\!\!/ \mathrm{SL}_{\beta_\ell'}$. Our seed is obtained by gluing these seeds from hives according to certain rule.

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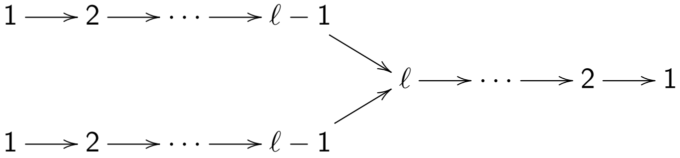

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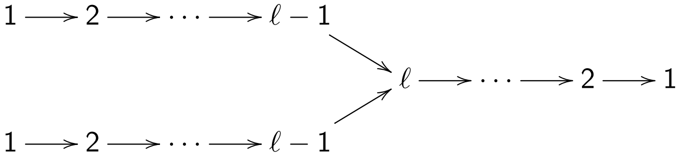

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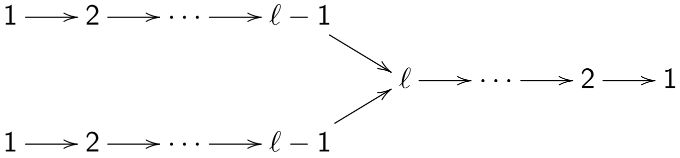

are upper cluster algebras with seeds given by the hive quivers Δ_ℓ.
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$$R^m_{\ell,\ell} = \bigoplus_{i=1}^{l-1} \big(\mathrm{Hom}(V_{-i}, V_{-(i+1)}) \oplus \mathrm{Hom}(V_{i+1}, V_i)\big) \oplus (\mathrm{SL}_\ell \otimes W)\,.$$

There are natural maps from an open subset U of $R^m_{\ell,\ell} /\!\!/ \mathrm{SL}_{\beta_\ell}$ to an open subset of $\mathrm{Rep}_{\beta_\ell{}'}(T_\ell) /\!\!/ \mathrm{SL}_{\beta_\ell{}'}$. Our seed is obtained by gluing these seeds from hives according to certain rule.

In the end, we add some coefficients to pass from $R^m_{\ell,\ell}$ to the $\mathrm{Rep}_{\beta_\ell}(K^m_{\ell,\ell})$.

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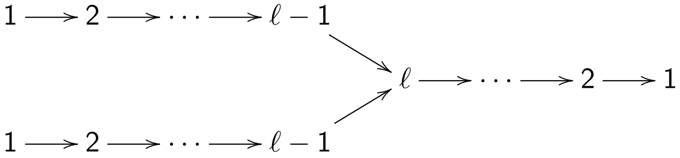

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Quivers with Potentials

The upper cluster algebra can be studied by the representation theory of quivers with potentials (Derksen-Weyman-Zelevinsky).

A *potential* $\mathcal{P}$ on a quiver Δ is a linear combination of oriented cycles of Δ. The Jacobian ideal $\partial\mathcal{P}$ is the two-sided (closed) ideal in $\widehat{\mathbb{C}\Delta}$ generated by all "noncommutative partial derivatives" $\partial_a\mathcal{P}$. The *Jacobian algebra* $J(\Delta, \mathcal{P})$ is the quotient algebra $\widehat{\mathbb{C}\Delta}/\partial\mathcal{P}$.

The quiver mutation can be "lifted" to the mutation of quivers with potentials [DWZ].

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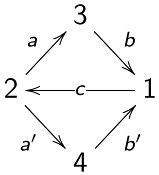
Example

Consider the quiver 

with potential $\mathcal{P} = cba - cb'a'$.

Then the Jacobian ideal is generated by

$$\partial_a \mathcal{P} = cb, \qquad \partial_{a'} \mathcal{P} = cb',$$
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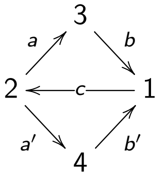
Consider the quiver 

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General Presentations

Similar as the quiver representations, we can consider the projective presentations for any finite-dimensional algebras. We denote $P(\beta) := \bigoplus_{i\in\Delta_0} \beta(i)P_i$. The presentation space of weight $\mathrm{g} \in \mathbb{Z}^{\Delta_0}$ is the space

$$\mathrm{PHom}(\mathrm{g}) := \mathrm{Hom}\left(P([\mathrm{g}]_+), P([-\mathrm{g}]_+)\right),$$

where we denote $[\mathrm{g}]_+ := \max(\mathrm{g}, 0)$.

Definition (F)

A weight vector $\mathrm{g} \in K_0(\mathrm{proj}\text{-}J)$ is called *μ-supported* if the cokernel of a general presentation in $\mathrm{PHom}(\mathrm{g})$ is supported only on mutable vertices. Let $G(\Delta, W)$ be the set of all μ-supported vectors in $K_0(\mathrm{proj}\text{-}J)$.

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The Cluster Model from QP

Definition (Dupont etc)

We define the *generic character* $C_{\mathrm{gen}} : G(\Delta, W) \to \mathbb{Z}(\mathbf{x})$ by

$$C_{\mathrm{gen}}(\mathrm{g}) = \mathbf{x}^{\mathrm{g}} \sum_{\mathrm{e}} \chi\big(\,\mathrm{Gr}^{\mathrm{e}}(\mathrm{Coker}(\mathrm{g}))\big)\mathbf{y}^{\mathrm{e}},$$

where $\mathrm{Gr}^{\mathrm{e}}(M)$ is the variety parameterizing e-dimensional quotient representations of M, and $\chi(-)$ is the topological Euler-characteristic.

It is known that $C_W(\mathrm{g})$ is an element in $\overline{\mathcal{C}}(\Delta)$.

Definition

We say that an IQP (Δ, W) is a *cluster model* if C_{gen} maps $G(\Delta, W)$ onto a basis of $\overline{\mathcal{C}}(\Delta)$.

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We say that a (frozen or mutable) vertex e can be *optimized* in Δ if there is a sequence of mutations away from e making e into a sink or source of Δ (possibly after deleting arrows between frozen vertices).

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Let $\widetilde{W}$ be any potential on $\widetilde{\Delta}$ such that its restriction on Δ is W. Suppose that $B(\Delta)$ has full rank, and each vertex in $\mathbf{e}$ can be optimized in $(\widetilde{\Delta}, \widetilde{W})$. If (Δ, W) is a (polyhedral) cluster model, then so is $(\widetilde{\Delta}, \widetilde{W})$.

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$(\overline{\Diamond}_\ell^m, \overline{W}_\ell^m)$ is a Polyhedral Cluster Model

Using the above theorem, we can reduce the QP $(\overline{\Diamond}_\ell^m, \overline{W}_\ell^m)$ to a polyhedral cluster model.

Theorem (F)

There is a rigid potential $\overline{W}_\ell^m$ on $\overline{\Diamond}_\ell^m$ such that $\overline{\mathcal{C}}(\overline{\Diamond}_\ell^m, \bar{\mathbf{s}}_\ell^m)$ has a basis parametrized by μ-supported g-vectors, which lie in a polyhedral cone G_ℓ^m.

So to compute each $g_{\mu,\nu}^\lambda$ we only need to count lattice points in at most $\ell(\lambda)!$ fibre polytopes inside the g-vector cone.

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Computing Kronecker Coefficients

Theorem (F)

Let μ, ν (resp. λ) be partitions of length $\leq \ell$ (resp. $\leq m$). Then

$$g_{\mu,\nu}^{\lambda} = \sum_{\omega \in \mathfrak{S}_m(\lambda)} \mathrm{sgn}(\omega) \left| \mathsf{G}_\ell^m(\mu, \nu, \lambda^\omega) \cap \mathbb{Z}^{(\overline{\Diamond}_\ell^m)_0} \right|.$$

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The Tropical F-polynomials

The μ-supported condition is given by

$$\operatorname{Hom}(\mathrm{g}, T_v) = 0, \ \ (v \text{ frozen})$$

where T_v is the *boundary* representation associated to v.

Theorem (F)

If M is negative reachable, then for any $\delta \in \mathbb{Z}^{\Delta_0}$ we have that

$$f_M(-\mathrm{g}) = \dim \operatorname{Hom}(\mathrm{g}, M),$$

where f_M is the tropical polynomial *of M:*

$$\mathrm{g} \mapsto \max_{L \hookrightarrow M} \mathrm{g}(\underline{\dim} L).$$

The Boundary Representations of $(\overline{\diamondsuit}_\ell^m, \overline{W}_\ell^m)$

The polytope G_ℓ^m has a hyperplane presentation:

$$H\mathrm{g} \geq 0, \quad \mathrm{g} \in \mathbb{Z}^{(\overline{\diamondsuit}_l^m)_0}$$

where rows of H are exactly the dimension vectors of subrepresentations of T_v's for all frozen v.

For each frozen vertex v, we define a boundary representation T_v by injective presentations. Here, instead, we give a concrete description using paths. For any path p, we can associate the (uniserial) path module. If v is a frozen vertex on the last diamond, then T_v is the path module associated to a path p_v. If $v = n$ is an extended frozen vertex, then T_n is the path module associated to the n-th diamond diagonal.

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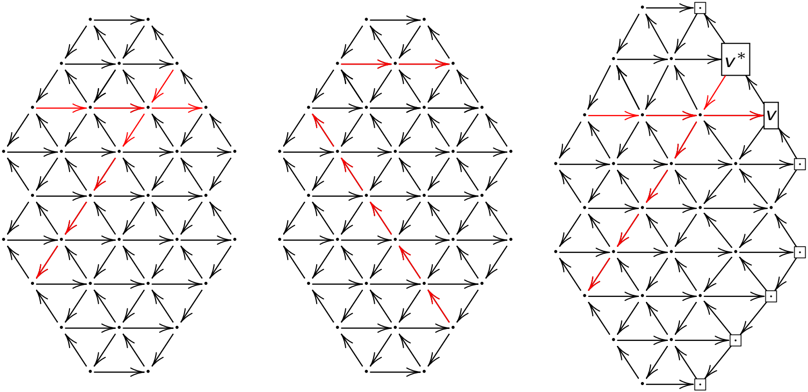
How a typical p_v looks like?

Here is a picture for such a path p_v (when $\ell = 5$ and $m = 4$).



The explicit H-matrix for $(\ell = m = 3, 4, 5)$ can be downloaded from my web page:
https://sites.google.com/a/umich.edu/jiarui/research/tensor-product-multiplicities/symmetric-groups
I hope that the full implementation can be available on SAGE soon.

Thank you!

Time for
questions and comments