Laurent recurrence formula, positivity and polytope basis on cluster algebras via Newton polytopes

Fang Li
(Zhejiang University)
Joint work with Jie Pan

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Goal

We will discuss the following three issues and some related topics via the method of Newton polytopes.

- For a cluster algebra, it is difficult to give a formula expression of the Laurent expansion of a cluster variable in an initial cluster. So, as an alternative, we want to give a recurrence formula.
- Positivity problem of a cluster variable under Laurent expansion in an initial cluster for a TSSS cluster algebra.
- Give a so-called polytope basis for a (upper) cluster algebra as a generalization of Greedy basis for a cluster algebra of rank 2

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Preliminaries

Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield, and $\mathcal{F}$ be the field of rational functions in n independent variables with coefficients in $\mathbb{QP}$.

Definition

A **seed** in $\mathcal{F}$ is a triple $\Sigma = (X, Y, B)$ such that

- $X = (x_1, x_2, \cdots, x_n)$ is a n-tuple satisfying that the elements form a free generating set of $\mathcal{F}$;
- $Y = (y_1, y_2, \cdots, y_n)$ is a n-tuple of elements of $\mathbb{P}$;
- B is an $n \times n$ totally sign skew-symmetric (TSSS) integer matrix, i.e. every matrix (b'_{ij}) mutation equivalent to B satisfying $sign(b'_{ij}) = -sign(b'_{ji})$.

Preliminaries

Definition

For any seed $\Sigma = (X, Y, B)$ in $\mathcal{F}$ and $k \in [1, n]$, we say that $\Sigma' = (X', Y', B')$ is obtained from Σ by a **mutation** in direction k if

- $$x_j' = \begin{cases} \dfrac{y_k \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+}}{(y_k \bigoplus 1) x_k} & j = k; \\ x_j & \textit{otherwise}. \end{cases}$$
- $$y_j' = \begin{cases} y_k^{-1} & j = k; \\ y_j y_k^{[b_{kj}]_+} (y_k \bigoplus 1)^{-b_{kj}} & \textit{otherwise}. \end{cases}$$
- $$b_{ij}' = \begin{cases} -b_{ij} & i = k \text{ or } j = k; \\ b_{ij} + sgn(b_{ik})[b_{ik} b_{kj}]_+ & \textit{otherwise}. \end{cases}$$

In this case we denote $\Sigma' = \mu_k(\Sigma)$.
Then Σ' is also a seed and the seed mutation μ_k is an involution.

Preliminaries

Let $\mathbb{T}_n$ be an n-regular tree such that n edges emanating from the same vertex are labeled differently by $[1, n]$.

The seed assigned to vertex t is denoted by $\Sigma_t = (X_t, Y_t, B_t)$ with

$$X_t = (x_{1;t}, x_{2;t} \cdots, x_{n;t}), Y_t = (y_{1;t}, y_{2;t} \cdots, y_{n;t})$$

$$and \quad B_t = (b_{ij}^t)_{i,j \in [1,n]}.$$

Definition

Let $\mathcal{S} = \{x_{i;t} \in \mathcal{F} \mid i \in [1, n], t \in \mathbb{T}_n\}$. The **cluster algebra** $\mathcal{A}$ associated with the family of clusters on $\mathbb{T}_n$ is the $\mathbb{ZP}$-algebra generated by $\mathcal{S}$.

Preliminaries

Definition

- A **tropical semifield** $Trop(u_1, u_2, \cdots, u_l)$ is a free abelian multiplicative group generated by $u_1, u_2, \cdots, u_l$ with addition defined by

$$\prod_{j=1}^{l} u_j^{a_j} \bigoplus \prod_{j=1}^{l} u_j^{b_j} = \prod_{j=1}^{l} u_j^{min(a_j, b_j)}.$$

- A cluster algebra is said to have **principal coefficients** at a vertex t_0 if $\mathbb{P} = Trop(y_1, y_2, \cdots, y_n)$ and $Y_{t_0} = (y_1, y_2, \cdots, y_n)$.

Preliminaries

From vertex t_0 to vertex t, by Laurent phenomenon, we can write as:

$$x_{l;t} = \frac{P_{l;t}^{t_0}}{\prod_{i=1}^{n} x_{i;t_0}^{d_i^{t_0}(x_{l;t})}}$$

where

- The vector $d_{l;t}^{t_0} = (d_1^{t_0}(x_{l;t}), d_2^{t_0}(x_{l;t}), \cdots, d_n^{t_0}(x_{l;t}))^T$ is called the **d-vector** of $x_{l;t}$ with respect to cluster X_{t_0}.
- Moreover, if $\mathcal{A}$ has principal coefficients at t_0, then $P_{l;t}^{t_0} \in \mathbb{Z}[x_{1,t_0}, \cdots, x_{n,t_0}; y_{1,t_0}, \cdots, y_{n,t_0}]$. $F_{l;t}^{t_0} = P_{l;t}^{t_0}|_{x_{i;t_0}=1, \forall i \in [1,n]}$ is a polynomial in $y_{1,t_0}, \cdots, y_{n,t_0}$ called the **F-polynomial** of $x_{l;t}$ with respect to X_{t_0}.

Preliminaries

- Define $\mathbb{Z}^n$-grading on $\mathcal{A}$:

$$deg(x_{i;t_0}) = e_i, \quad deg(y_{j;t_0}) = -b_j^0.$$

Then

$$x_{l;t} = \frac{P_{l;t}^{t_0}}{\prod_{i=1}^{n} x_{i;t_0}^{d_i^{t_0}(x_{l;t})}}.$$

is homogeneous on $\mathbb{Z}^n$-grading.

Define the **g-vector** of $x_{l;t}$:

$$g_{l;t}^{t_0} := deg(x_{l;t})$$

Preliminaries

Theorem [FZ4]

For any cluster algebra $\mathcal{A}$ and any vertices t and t' in $\mathbb{T}_n$, a cluster variable $x_{l;t}$ can be expressed by

$$x_{l;t} = \frac{F_{l;t}^{t'}|_{\mathcal{F}}(\hat{y}_{1;t'}, \cdots, \hat{y}_{n;t'})}{F_{l;t}^{t'}|_{\mathbb{P}}(y_{1;t'}, \cdots, y_{n;t'})} \prod_{i=1}^{n} x_{i;t'}^{g_i},$$

where $\hat{y}_{j;t'} = y_{j;t'} \prod_{i=1}^{n} x_{i;t'}^{b'_{ij}}$, and $g_{l;t}^{t'} = (g_1, \cdots, g_n)^T$.

Denote $\hat{Y}_t = \{\hat{y}_{1;t}, \cdots, \hat{y}_{n;t}\}$ for any $t \in \mathbb{T}_n$

Preliminaries

$$\mathcal{U}_{\geqslant 0}(\Sigma_t) := \mathbb{NP}[X_t^{\pm 1}] \bigcap \mathbb{NP}[X_{t_1}^{\pm 1}] \bigcap \cdots \bigcap \mathbb{NP}[X_{t_n}^{\pm 1}] \subseteq \mathcal{U}(\Sigma_t),$$

$$\mathcal{U}^+(\Sigma_t) := \{f \in \mathcal{U}_{\geqslant 0}(\Sigma_t) \mid L^t(f) \in \mathbb{N}[Y_t][X_t^{\pm 1}] \text{ and } L^{t_i}(f) \in \mathbb{N}[Y_{t_i}][X_{t_i}^{\pm 1}], \forall i \in [1, n]\},$$

where $L^t(-)$ is defined to make $L^t(\rho_h)$ to be the Laurent expression in the cluster algebra over $\mathbb{Z}Trop(Y_t)$.

$$\mathcal{U}^+_{\geqslant 0}(\Sigma_t) := \mathcal{U}^+(\Sigma_t) \bigcap \mathcal{U}_{\geqslant 0}(\Sigma_{t_1}) \bigcap \cdots \bigcap \mathcal{U}_{\geqslant 0}(\Sigma_{t_n}),$$

where $t_i \in \mathbb{T}_n$ is the vertex connected to $t \in \mathbb{T}_n$ by an edge labeled i.

For $t' \in \mathbb{T}_n$, $h \in \mathbb{Z}^n$ and any homogeneous Laurent polynomial $f \in \mathbb{Z}[Y_{t'}][X_{t'}^{\pm 1}] \subseteq \mathbb{Z}Trop(Y_{t'})[X_{t'}^{\pm 1}]$,
denote by $co_{X_{t'}^h}(f)$ the coefficient of the Laurent monomial $X_{t'}^h$ in f, i.e. if the coefficient is a, then

$$co_{X_{t'}^h}(f) = a.$$

Newton polytope

Any Laurent monomial aY^v, where $a \in \mathbb{Z}$, corresponds to a vector $v \in \mathbb{Z}^n$, called a **point with weight.**

- The **support** of a Laurent polynomial F is a set consisting of $\mathbb{N}$-vectors associated to summand monomials of F.
- The **Newton polytope** N of a Laurent polynomial F is the convex hull of the support of F.
- The support of a Laurent polynomial F is **saturated**(饱和的) if any lattice point in the Newton polytope N is associated to a summand monomial of F.

The relation between Newton polytopes and Laurent polynomials

Given a vector $h \in \mathbb{Z}^n$ and a cluster algebra $\mathcal{A}$ with principal coefficients, there is a bijection:

$\{$homogeneous Laurent polynomials in $\mathbb{Z}[Y][X^{\pm 1}]$ with grade $h\}$ $\overset{\tilde{v}}{\longleftrightarrow}$ $\{$Polytopes with weights$\}$

summand $a\hat{Y}^p X^h$ $\mapsto$ (lattice) point p with weight a

That is, we have

$$\tilde{v}(f(\hat{Y})X^h) = N_h$$

Outline of the idea of the polytope function ρ_h associated to $h \in \mathbb{Z}^n$

As a generalization of cluster monomials, for $h \in \mathbb{Z}^n$, we need to construct a function ρ_h to include X^h as a summand and to be a homogeneous Laurent polynomial in the cluster X_t for any $t \in \mathbb{T}_n$ with coefficients in $\mathbb{N}[Y]$.

Such global conditions are too complicated to deal with directly, so we first try to construct a such function in local conditions for any given vertex $t_0 \in \mathbb{T}_n$ and those vertices around it, and then construct ρ_h in global conditions.

Based on the above idea, ρ_h can be constructed in the following three steps.

(i) In general, X^h can not be expressed as a Laurent polynomial with positive coefficients in any cluster. For example, when $h_k < 0$ for some $k \in [1, n]$, the expression of X^h in X_{t_k} equals to $(\frac{M_{k;t_k}}{x_{k;t_k}})^{h_k} \prod_{i \neq k} x_{i;t_k}^{h_i} = \frac{x_{k;t_k}^{-h_k} \prod_{i \neq k} x_{i;t_k}^{h_i}}{M_{k;t_k}^{-h_k}}$, which is not a Laurent polynomial in X_{t_k}, where $t_k \in \mathbb{T}_n$ is the vertex connected to t_0 by an edge labeled k.

Our method is to add some Laurent polynomial in X to make the summation also a Laurent polynomial in X_{t_k}. Concretely, we find $(\hat{y}_k + 1)^{-h_k} X^h = x_k^{h_k} M_{k;t_k}^{-h_k} \prod_{i \neq k} x_i^{h_i + [-b_{ik}]_+ h_k}$ having X^h as a summand, which can be expressed as a Laurent polynomial in X_{t_k}.

(ii) If there is $k' \in [1, n]$ such that $(\hat{y}_k + 1)^{-h_k} X^h$ can not be expressed as a Laurent polynomial in $X_{t_{k'}}$, then there is a summand $x_{k'}^{-a} p$, where $a \in \mathbb{Z}_{>0}$ and p is some Laurent monomial in $X_{t_{k'}} \setminus \{x_{k'}\}$.

Again we need to find a Laurent polynomial $x_{k'}^{-a} M^a_{k', t_{k'}} q \in \mathbb{N}[Y][X^{\pm 1}]$ which has $x_{k'}^{-a} p$ as a summand, where q is a Laurent monomial in $X_{t_{k'}} \setminus \{x_{k'}\}$.

In the above process we call $x_k^{-a} M^a_{k, t_k} q$ a **complement** of $x_k^{-a} p$ in direction k.

(iii) Then we focus on the minimal Laurent polynomial having both $(\hat{y}_k + 1)^{-h_k} X^h$ and $x_{k'}^{-a} M^a_{k',t_{k'}} q$ as summands and look for $k'' \in [1, n]$ if it exists such that the minimal Laurent polynomial can not be expressed as a Laurent polynomial in $X_{t_{k''}}$ to repeat step (ii) for k''.

Such construction keeps on until the final (formal) Laurent polynomial can be expressed as a (formal) Laurent polynomial in any X_{t_k} for $k \in [1, n]$, and we denote it by ρ_h.

ρ_h is called a **polytope function**.

Note that ρ_h is a Laurent polynomial if the construction ends in finitely many steps, otherwise it is a formal Laurent polynomial.

Construct ρ_h by induction via Newton polytopes

We find a decomposition

$$x_i \rho_h = \sum_{w,\alpha} c_{w,\alpha} Y^w \rho_\alpha$$

for ρ_h, which simplifies our construction by induction on ρ_α induced by sub-polytopes indexed by α.

During the construction of a polytope function ρ_α, the sub-polytope N_α associated to ρ_α is constructed with the order induced by its sub-polytopes.

Thus, it is more convenient for us to construct and study ρ_h via N_h.

Theorem 1

Let $\mathcal{A}$ be a cluster algebra having principal coefficients and $h \in \mathbb{Z}^n$. Denote N_h as the polytope corresponding to ρ_h and let $H = \{h \in \mathbb{Z}^n : \rho_h \in \mathcal{U}^+_{\geqslant 0}(\Sigma_{t_0})\}$. Then,

(i) There is a unique indecomposable Laurent polynomial $\rho_h := \rho_h^{t_0}$ in $\widehat{\mathcal{U}}^+_{\geqslant 0}(\Sigma_{t_0})$ having X^h as a summand.

(ii) $\rho_h \in \mathbb{N}[Y_t][[X_t^{\pm 1}]]$ for any $t \in \mathbb{T}_n$ and it is indecomposable universally positive. So, $\mathcal{P} = \{\rho_h \in \mathcal{U}(\mathcal{A}) | h \in H\}$ is independent to the choice of the initial seed and contains all monomials in $\{X_t^\alpha \mid \alpha \in \mathbb{N}^n, t \in \mathbb{T}_n\}$.

(iii) $\rho_h|_{x_i \to 1}$ is a polynomial in $\mathbb{Z}[Y]$ having a unique maximal term and constant term, and the coefficients of them are both 1.

Indeed, $\rho_h|_{x_i \to 1}$ can be realized as a generalization of F-polynomial for $h \in \mathbb{Z}^n$. When h is a g-vector, it is just the usual F-polynomial.

Theorem 2

Let $\mathcal{A}$ be a cluster algebra having principal coefficients and $h \in \mathbb{Z}^n$. Then,

(i) For $h \in \mathbb{Z}^n$ such that $\rho_h^{t_0} \in \mathcal{U}^+_{\geqslant 0}(\Sigma)$ and any $k \in [1, n]$, there is

$$h^{t_k} = h - 2h_k e_k + h_k[b_k]_+ + [-h_k]_+ b_k \tag{1}$$

such that $L^{t_k}(\rho_h^{t_0}) = \rho_{h^{t_k}}^{t_k}$, where $t_k \in \mathbb{T}_n$ is the vertex connected to t_0 by an edge labeled k.

(ii) Let S be a r-dimensional face of N_h. Then there are a cluster algebra $\mathcal{A}'_S$ with principal coefficients of rank r, a vector $h' \in \mathbb{Z}_r$ and an isomorphism τ from $N_{h'}$ to S with its induced linear map $\tilde{\tau}$ satisfying $\tilde{\tau}(e_i) \in \mathbb{N}^n$ for any $i \in [1, r]$.

In skew-symmetrizable case

Theorem

Let $\mathcal{A}$ be a skew-symmetrizable cluster algebra with principal coefficients, $h \in \mathbb{Z}^n$ such that E_h is finite and S be a r-dimensional face of N_h. Let B' be the initial exchange matrix of the cluster algebra $\mathcal{A}'_S$ related to S in the above theorem.

Then $B' = \overline{W}^\top BW$, where $W = (\tilde{\tau}(e_1), \cdots, \tilde{\tau}(e_r))$, $\overline{W} = (\overline{\tau(e_1)}, \cdots, \overline{\tau(e_r)})$ are $n \times r$ integer matrices, $\tilde{\tau}(e_i) = \sum_{j=1}^{r} w_{ji} e_j$, $\overline{\tau(e_i)} = \sum_{j=1}^{r} \frac{d_j}{d_s} w_{ji} e_j$ as column vectors with $s \neq \emptyset$ the label of the edge in S parallel to $\tilde{\tau}(e_i)$ and $\overline{\tau(e_i)} = \sum_{j=1}^{r} w_{ji} e_j$ when the label is $\emptyset$.

Construction of ρ_h (N_h)

When $\mathcal{A}$ is a cluster algebra with principal coefficients of rank n and $h \in \mathbb{Z}^n$. We can decompose N_h as a summation of smaller polytopes:

$$N_h = \sum_{N_{\alpha_j}[w_j] \in \bigcup_l U_h^l} N_{\alpha_j}[w_j],$$

where $U_h^0 = \{N_{h+e_i}\} \cup \bigcup_j \{N_{\iota_{k;h_k+\iota_{k;0}(w_j)}(b_k^\top)\alpha_j}[\iota_{k;0}(w_j)]\}$ and U_h^l is iteratively determined to make ρ_h an indecomposable Laurent polynomial ρ_h in $\mathcal{U}^+_{\geqslant 0}(\Sigma_{t_0})$.

Moreover in a cluster algebra over arbitrary semifield $\mathbb{ZP}$, let $F_h = \rho_h^{pr}|_{x_i \to 1, \forall i \in [1,n]}$. Then,

$$\rho_h = \frac{F_h|_{\mathcal{F}}(\hat{Y})}{F_h|_{\mathbb{P}}(Y)} X^h \in \mathbb{NP}[[X^{\pm 1}]].$$

Example of Construction of ρ_h (N_h) for rank 2

Denote by $l(\overline{pq})$ the length of the segment connecting two points p and q.

$$\tilde{C}_i^j \triangleq \begin{cases} \binom{i}{j} & \text{if } i \geqslant 0; \\ 0 & \text{if } i < 0. \end{cases}$$

When $\mathcal{A}$ is a cluster algebra with principal coefficients of rank 2,

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix},$$

where $b, c \in \mathbb{Z}_{>0}$. For $h = (h_1, h_2) \in \mathbb{Z}^2$, $V_h = \{(0,0), ([-h_1]_+, 0]), (0, [-h_2]_+), ([-h_1]_+, [-h_2 + c[-h_1]_+]_+), ([-h_1]_+ - [[-h_1]_+ - b[c[-h_1]_+ - h_2]_+]_+, [-h_2 + c[-h_1]_+]_+)\}$ while E_h is the set consisting of edges connecting points in V_h and parallel to e_1 or e_2.

Example of Construction of $\rho_h\ (N_h)$ for rank 2

For any point $p_0 = (u_0, v_0)$ in an arbitrary edge p_1p_2 in E_h with $p_1, p_2 \in V_h$, define the weight $co_{p_0} = \tilde{C}_{l(\overline{p_1p_2})}^{l(\overline{p_0p_2})}$, and denote

$$m_1(p_0) = \begin{cases} co_{p_0}, & \text{if } u_0 = -h_1; \\ 0, & \textit{otherwise}. \end{cases}$$

and

$$m_2(p_0) = \begin{cases} co_{p_0}, & \text{if } v_0 = 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Example of Construction of $\rho_h\ (N_h)$ for rank 2

For point $p = (u, v)$ not in E_h, define co_p inductively as follows:

$$co_p = max\{ \sum_{i=1}^{[-h_2+c[-h_1]_+]_+-u} m_1((u+i, v))\tilde{C}^i_{-h_1-bv}, \sum_{i=1}^{v} m_2((u, v-i))\tilde{C}^i_{-h_2+cu}\}$$

$$m_1(p) = co_p - \sum_{i=1}^{[-h_2+c[-h_1]_+]_+-u} m_1((u+i, v))\tilde{C}^i_{-h_1-bv},$$

$$m_2(p) = co_p - \sum_{i=1}^{v} m_2((u, v-i))\tilde{C}^i_{-h_2+cu}.$$

Then, we denote by N_h the convex hull of the set $\{p \in \mathbb{N}^2 \mid co_p \neq 0\}$.

Cluster variables and polytopes

In particular, since $\rho_{e_i} = x_i$, we can see that $\rho_{g_{l;t}} = x_{l;t} \in \mathcal{P}$. So cluster variables and their associated polytopes satisfy the above results.

Theorem (Recurrence formula)

Let $\mathcal{A}$ be a TSSS cluster algebra having principal coefficients, then $x_{l;t} = \rho_{g_{l;t}}$ and $N_{l;t} = N_{g_{l;t}}$. Following this, we have that

$$co_p(N_{l;t}) = co_p(N_{g_{l;t}}) = \sum_{N_{h_{j'}}[w_{j'}] \in \bigcup_r U^r_{g_{l;t}}} co_p(N_{h_{j'}}[w_{j'}]) \qquad (2)$$

and

$$x_{l;t} = X_{t_0}^{g_{l;t}} \Big(\sum_{p \in N_{g_{l;t}}} co_p(N_{l;t}) \hat{Y}^p \Big). \qquad (3)$$

Positivity conjecture

In [FZ1], the positivity conjecture for cluster variables is suggested, that is,

Conjecture [FZ1]

Every cluster variable of a cluster algebra $\mathcal{A}$ is a Laurent polynomial in cluster variables from an initial cluster X with positive coefficients.

So far, the recent advance on the positivity conjecture is the proof in skew-symmetrizable case given in [GHKK]. For totally sign-skew-symmetric cluster algebras, it was only proved in acyclic case in [HL].

Positivity conjecture

As a harvest of this polytope method, a natural conclusion of the above Theorem is the following corollary, which actually completely confirms the positivity conjecture in the most general case:

Corollary (Positivity for TSSS cluster algebras)

Let $\mathcal{A}$ be a TSSS cluster algebra with principal coefficients and (X, Y, B) be its initial seed. Then every cluster variable in $\mathcal{A}$ is a Laurent polynomial over $\mathbb{N}[Y]$ in X.

Proof.

It follows from the fact that in the right-hand side of recurrence formula (2), the coefficients of $x_{l;t}$ in (3) are always positive due to our construction of N_h. □

From g-vectors to F-polynomials

Let $\mathcal{A}$ be a TSSS cluster algebra having principal coefficients, there is a bijective map

$$\begin{array}{ccc} \{\text{non-initial g-vectors of } \mathcal{A}\} & \xrightarrow{\cong} & \{\text{non-initial F-polynomials of } \mathcal{A}\} \\ g_{l;t} & \mapsto & \rho_{g_{l;t}}|_{x_i \to 1} \end{array}$$

From F-polynomials to d-vectors

$\forall k \in [1, n]$, we can write $P_{l;t}$ as a sum of x_k-homogeneous polynomials $P_{l;t} = \sum_{s=d_k(x_{l;t})}^{deg_k(P_{l;t})} x_k^s P_s(k) + \sum_{s=0}^{d_k(x_{l;t})-1} x_k^s M_k^{d_k(x_{l;t})-s} P_s(k)$, where $M_k = x_k \mu_k(x_k)$.

Define $\widetilde{deg}_k(P_{l;t}) = max\{r \mid M_k^r \text{ divides } (P_{l;t}\mid_{x_k \to M_k})\}$.

Theorem

Let $\mathcal{A}$ be a TSSS cluster algebra having principal coefficients. $d_k(x_{l;t}) = \widetilde{deg}_k(P_{l;t}) = \widetilde{deg}_k(P_{l;t}\mid_{x_k=0}) \in \mathbb{N}$.
This means the positivity conjecture of d-vectors of a cluster variable is always true.

From F-polynomials to d-vectors

From [FZ4], we have

$$x_{l;t} = \frac{F_{l;t}^{t'}|_{\mathcal{F}}(\hat{y}_{1;t'}, \cdots, \hat{y}_{n;t'})}{F_{l;t}^{t'}|_{\mathbb{P}}(y_{1;t'}, \cdots, y_{n;t'})} \prod_{i=1}^{n} x_{i;t'}^{g_i},$$

Then, $P_{l;t}$ is the factor of $F_{l;t}|_{\mathcal{F}}(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n)$ which is coprime with $x_i, i \in [1, n]$ $x_{i,t'}$.

Let $\mathcal{A}$ be a cluster algebra having principal coefficients, there is a surjective map

$$\{\text{non-initial F-polynomials }\} \quad \twoheadrightarrow \quad \{\text{positive d-vectors}\}$$

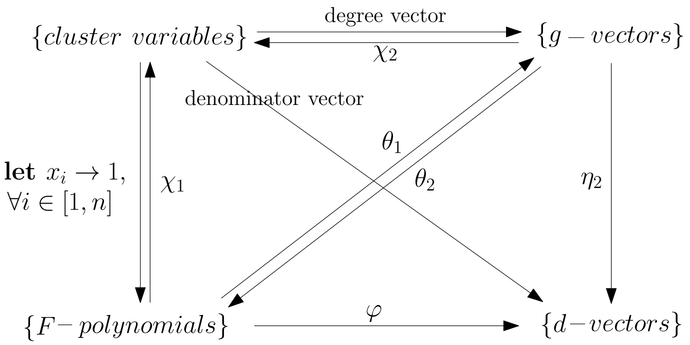
$$F_{l;t} \mapsto P_{l;t} \mapsto (\widetilde{deg}_1(P_{l;t}\ |_{x_1=0}), \cdots, \widetilde{deg}_n(P_{l;t}\ |_{x_n=0}))$$

From F-polynomials to d-vectors

Corollary

Let $\mathcal{A}$ be a cluster algebra with principal coefficients, $x_{l;t}$, $x_{l';t'}$ be two non-initial cluster variables and $F_{l;t}$, $F_{l';t'}$ the F-polynomials associated to $x_{l;t}$, $x_{l';t'}$ respectively. If $F_{l;t} = F_{l';t'}$, then $x_{l;t} = x_{l';t'}$.

Relation diagram



Polytope basis

$$\chi_2 : \quad \{g - vectors\} \longrightarrow \{cluster\ variables\}$$

$$g_{l;t} \mapsto \rho_{g_{l;t}} = x_{l;t}$$

The map χ_2 can be generalized to $\mathbb{Z}^n$. Denote

$$\mathcal{P} = \{\rho_h \in \mathbb{NP}[X^{\pm 1}] \mid h \in \mathbb{Z}^n\}$$

Proposition

Let $\mathcal{A}$ be a cluster algebra.
(i) $\mathcal{P}$ is independent of the choice of the initial seed. Hence $\mathcal{P} \subseteq \mathcal{U}(\mathcal{A})$ and all elements in $\mathcal{P}$ are universally positive and indecomposable.
(ii) $\mathcal{P}$ contains all cluster monomials with coefficient 1.

Polytope basis

Theorem

Let $\mathcal{A}$ be a cluster algebra with principal coefficients. Then $\mathcal{P}$ is a strongly positive $\mathbb{Z}Trop(Y)$-basis for the upper cluster algebra $\mathcal{U}(\mathcal{A})$.

We would like to call $\mathcal{P}$ the **polytope basis** for $\mathcal{U}(\mathcal{A})$.

In rank 2 case, it is coincident to the greedy basis introduced in [LLZ].

Thanks for your attention!