Bracelets are theta functions

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(joint work with Travis Mandel)

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Overview

- Surface with marked points Σ
- Skein algebra $\overline{\mathrm{Sk}}(\Sigma) = \{\text{unions of curves}\}$
  - {bracelets} is a basis [FG06][MSW13][Thu14].
- Triangulation $\Delta \dashrightarrow$ a seed $t_\Delta \dashrightarrow$ upper cluster algebra $\boldsymbol{U}(t_\Delta)$ [FST08], often:
  - {quantum theta func.} is a basis of $\boldsymbol{U}(t_\Delta)$ [GHKK18][DM21]
  - [Mul16] Unpunctured Σ:

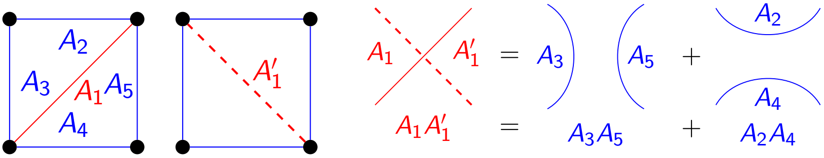
$$\boldsymbol{U}(t_\Delta) = \text{localization of } \overline{\mathrm{Sk}}(\Sigma) \text{ at the boundary arcs}$$

Roughly speaking, for general Σ,
quantum bracelets = quantum theta functions.
(Visualization of theta functions)

- Exception: once-punctured torus
- $\Longrightarrow$ atomicity conjecture: the bracelets basis is the minimal positive basis.

Toy Model

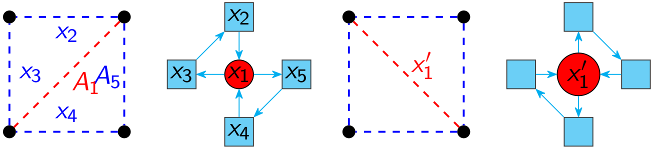
- $\Bbbk = \mathbb{Z}$ (quantum case $\Bbbk = \mathbb{Z}[q^{\pm\frac{1}{2}}]$)
- Surface Σ: 4-gon. Triangulations $\Delta \overset{\text{flip}}{\longleftrightarrow} \Delta'$



- **Skein alg** $\overline{\mathrm{Sk}} := \bigoplus_{D\text{:union of curves}} \Bbbk[D]/(\text{skein relations})$
  - multiplication: $\cup$
  - $\Bbbk$-basis=$\{\text{mono. in } A_1, \ldots, A_5\} \cup \{\text{mono. in } A'_1, A_2, \ldots, A_5\}$
  - $\Bbbk[A_2, \ldots, A_5]$-basis $\{A_1^d\} \cup \{(A'_1)^d\}$
- (Upper) cluster alg $\boldsymbol{U} := \Bbbk[A_1^\pm, \ldots, A_5^\pm] \cap \Bbbk[(A'_1)^\pm, \ldots, A_5^\pm]$
  - triangulation $\Delta = \{A_1, \ldots, A_5\}$: toric local chart of Spec $\boldsymbol{U}$
  - $\boldsymbol{U} = \overline{\mathrm{Sk}}[A_2^{-1}, \ldots, A_5^{-1}]$

Cluster Algebras from Quivers

- Dual of triangulations: quivers



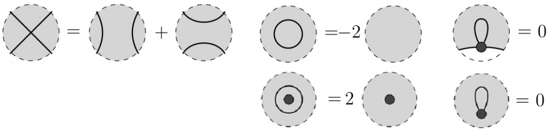
In general, we can construct cluster algebras from quivers:

- **triangulation** $\dashrightarrow$ **seed s** $= (Q, (A_i)_{i \in Q_0})$ ($\sim$local chart)
  - vertices of Q are unfrozen/frozen (internal/boundary)
- **flip** $\dashrightarrow$ **mutation** ($\sim$ change local charts)
- Iterate mutations $\Longrightarrow$ all seeds $\Longrightarrow$ $\boldsymbol{U}$

- A seed is of *full rank* $\Longleftrightarrow$ it can be *quantized*

Skein Algebra on Surfaces

- $\Sigma = (S, M)$, S: topological surface, M: marked points
    - each connected component of ∂S contains ≥ 1 marked points
    - $M \backslash \partial S$: punctures
- curve C_i: ending at M or a closed loop
- diagram D: a union of curves. $[D]$ isotopy class
    - considered up to isotopy (fixing M and crossings)
    - denote $D = \cup w_j C_j$ where w_j is the weight (multiplicity) of C_j
    - called internal if it does not contain a boundary arc
    - called simple if not reducible by Skein relations
- Skein algebra $\overline{\mathrm{Sk}}(\Sigma) := \oplus \mathbb{k}[D]/(\text{Skein relations})$



- If unpunctured, $\exists$ quantization:
  $[D] * [D'] = [D \text{ put above } D']$, q-Skein relations

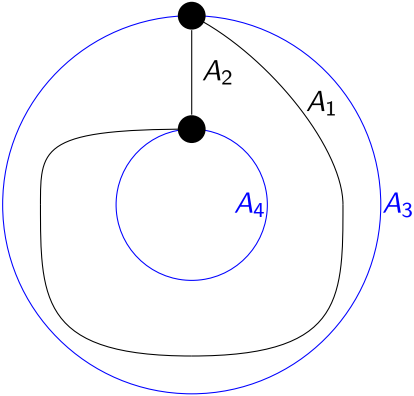
From Skein Algebras to Cluster Algebras

- Arc γ: simple curve ending at M
  - γ_1, γ_2 are compatible if they have no crossing
- Punctured case: tagged arc [FST08]
  - tagged plain or notched at the punctured ending
  - notion of compatibility (no-crossing, compatible tagging)
- Ideal/tagged triangulation Δ: a maximal collection of non-isotopic compatible arcs/tagged arcs.
- $\Delta \dashrightarrow$ seed t_Δ [FST08]
  - cluster variables: $\gamma \in \Delta$ (frozen: boundary)
  - quiver: oriented "dual graph" of Δ
  - if unpunctured $\dashrightarrow$ quantum seed t_Δ [Mul16]
- $\mathrm{Sk}(\Sigma) :=$ localization of $\overline{\mathrm{Sk}}(\Sigma)$ at the boundary arcs, then

$$\mathrm{Sk}(\Sigma) \subset \boldsymbol{U}(t_\Delta).$$

  - [Mul16] For most unpunctured Σ, they coincide.

Example: annulus

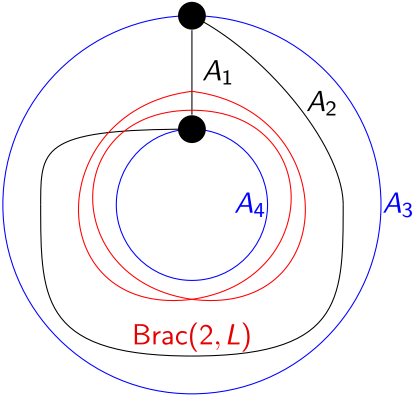


- annulus = area bounded by A_3, A_4.
- initial triangulation $\Delta = \{A_1, A_2\} \cup \{A_3, A_4\}$,
  - t_Δ is associated with a Kronecker quiver
- Infinite many triangulations (by rotating boundary)

Dictionary

Topology (SL_2-local system on Σ)	Cluster theory
tagged triangulation Δ	seed t_Δ
tagged arc	cluster variable
boundary arc	coefficients/frozen variable
$\cup\gamma_i$ for $\gamma_i \in \Delta$	cluster monomial
union	multiplication
flip	mutation

Bracelet Basis (Theta Basis)



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- Brac(w, L) a bracelet loop with $w-1$ self-crossings.
- A bracelet diagram $BracD = \bigcup$ compatible arcs and Brac(w, L)
- $\overline{\mathrm{Sk}}(\Sigma)$ has the $\mathbb{k}[b_1, b_2]$-basis $\{[\text{internal } BracD]\}$
  $= \{[\text{internal } \textit{cluster monomials}]\} \cup \{[\mathrm{Brac}(w, L)] | w > 0\}$

Chebyshev Polynomials

- Chebyshev polynomial of the first kind $T_w(\ )$, $w \geq 0$:
  $T_w(z+z^{-1}) = z^w + z^{-w}$
  - $T_0(z) = 2,\ T_1(z) = z,\ z \cdot T_w(z) = T_{w+1}(z) + T_{w-1}(z)$
  - $\mathrm{tr}(M^w) = T_w(\mathrm{tr} M)$ for $M \in SL_2$

Theorem 1 ([MSW13])

$[\mathrm{Brac}(w, L)] = T_k([L])$.

Example 2

$[\mathrm{Brac}(2, L)] = [L]^2 - 2$

Atomicity

- An element in $\boldsymbol{U}$ is universally positive if, with respect to any seed, its Laurent coefficients belong to $\Bbbk^+ = \mathbb{N}$ (or $\mathbb{N}[q^{\pm\frac{1}{2}}]$).
- A basis of $\boldsymbol{U}$ is said to be atomic if:
  - it consists of universally positive elements
  - any universally positive element is a $\Bbbk^+$-sum of its elements.
- The atomic basis is unique if it exists.

Conjecture [FG06][MSW13]

Bracelets form the atomic basis (in some settings)

Theorem 3 ([GHKK18][Man17][Yur20])

If no component of Σ is a once-punctured closed surface, then the theta functions form a basis for $\boldsymbol{U}$. Moreover, this theta basis is atomic.

Similar Seeds

Definition 4 ([Qin14, Qin17])

Two seeds t, t' are similar if they share the same unfrozen full subquiver.

- $\Longrightarrow$ $\boldsymbol{U}(t)$ and $\boldsymbol{U}(t')$ share similar structure and properties
- For experts: an element in $\boldsymbol{U}(t)$ is similar to an element in $\boldsymbol{U}(t')$ if they share the same unfrozen g-vector and the same F-polynomial.
  - Examples include the cluster monomials
  - Assume t is of full rank, if we have a good basis for $\boldsymbol{U}(t)$, then the similar elements form a spanning set for $\boldsymbol{U}(t')$.
- Assume t is similar to t_Δ
  - We say $\boldsymbol{U}(t)$ is of type Σ
  - If we have defined a bracelet element in $\boldsymbol{U}(t_\Delta)$, the similar element for $\boldsymbol{U}(t)$ is also called a bracelet element.

Quantization

- For unpunctured Σ:
  - We can $t = t_\Delta$, quantize $\mathrm{Sk}(\Sigma)$, t_Δ as in [Mul16]
- For Σ with punctures: $\mathrm{Sk}(\Sigma)$ and t_Δ cannot be quantized!
  - We can choose a seed t similar to t_Δ (such as the *principal coefficients*) and quantize t instead.

Tagged Bracelet Elements for Upper Cluster Algebras

- A tagged bracelet diagram $D = (\cup k_i \gamma_i) \bigcup (\cup w_j L_j)$ consisting of compatible tagged arcs and bracelet loops.
- Construct (quantum) tagged bracelet $\langle D \rangle_{\mathrm{Brac}}$ in $\boldsymbol{U}(t)$:
  - $\langle \gamma_i \rangle_{\mathrm{Brac}}$: the (quantum) cluster variable corresponding to the tagged arc γ_i
    - Exception: tagged notched arcs on a 1-punctured closed surface are NOT cluster variables. Need definition.
  - $\langle w_j L_j \rangle_{\mathrm{Brac}}$: take the quantum trace of $\mathrm{Brac}(w_j, L_j)$ ([BW11][AK17] define it in the cluster Poisson algebra, consider its image in $\boldsymbol{U}(t)$)
  - $\langle D \rangle_{\mathrm{Brac}} = q^{\alpha} (\prod \langle \gamma_i \rangle_{\mathrm{Brac}}^{k_i}) \prod (\prod \langle w_j L_j \rangle_{\mathrm{Brac}})$, where α is chosen to guarantee the bar-invariance $q \mapsto q^{-1}$.

Tagged Notched Arcs for Once-Punctured Closed Surfaces

- For $q = 1$. Consider a finite cover $\pi : \widetilde{\Sigma} \to \Sigma$. The lift of a tagged notched arc γ corresponds to a cluster variable x. Define the bracelet element for γ in $\boldsymbol{U}(t)$ as

$$[\gamma] = \pi x.$$

  - Justified by the Teichmüller theory (lambda length).
- Assume Σ is NOT a once-punctured torus.
  - Obstruction: π does NOT work for q-case
  - At $q = 1$, can show $[\gamma]$ is a theta function θ.
  - Donaldson-Thomas transformation = wall-crossing from $\mathbb{R}^n_{>0}$ to $\mathbb{R}^n_{<0}$ in the scattering diagram [GHKK18]
    $\Longrightarrow DT(\ )$ sends $[\gamma]$ to tagged plain arc $[\gamma^{\diamond}]$ at $q = 1$.
  - Define $[\gamma]$ such that we still have

$$DT([\gamma]) = [\gamma^{\diamond}].$$

    - This will force $[\gamma] = \theta$.
  - Better definition?

Result: Visualization of Theta Functions.

- Assume Σ connected for simplicity. Choose a (quantum) seed t of type Σ.

Theorem 5

$\forall$ tagged bracelet diagram D, $\langle D\rangle_{\mathrm{Brac}}$ is a quantum theta function $\vartheta_{g(D)}$ in $\boldsymbol{U}(t)$ iff one the following is satisfied:

- *Σ is NOT a 1-punctured torus, or*
- *Σ is a 1-punctured torus, but D does not contain notched arcs*

If Σ is 1-punctured torus, for $t = t_\Delta$ and $q = 1$,

$$\langle D\rangle_{\mathrm{Brac}} = 4^{\mathrm{wt}(D)}\vartheta_{g(D)}$$

where $\mathrm{wt}(D) := [-\sum g_i]_+$ for the g-vector $g(D) = (g_i)$ of D.

- [Zho20] [notched arc] $\neq$ theta function for 1-punctured torus.

Result: Atomicity

Theorem 6

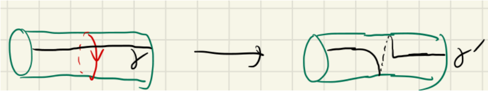
- *For unpunctured Σ, the quantum bracelets is the atomic basis for* $\mathrm{Sk}(\Sigma)$.
- *For any Σ, at $q = 1$, the bracelets form the atomic basis for* $\mathrm{Sk}(\Sigma)$ *with respect to the ideal triangulation atlas (the triangulation only consists of arcs without tagging)*

Proof: Ideas

Ideas for verifying that $\langle D \rangle_{\text{Brac}}$ is a theta function.

1. We can cut the surface and work locally
   - Skein relations are local. Need some work for theta functions.
2. (Key) $[L]$ and, more generally, $\langle wL \rangle_{\text{Brac}}$ is a theta function
   - Use Dehn twists to choose good Δ.
   - Verify the Chebyshev recursion.

Figure: Dehn twist around L

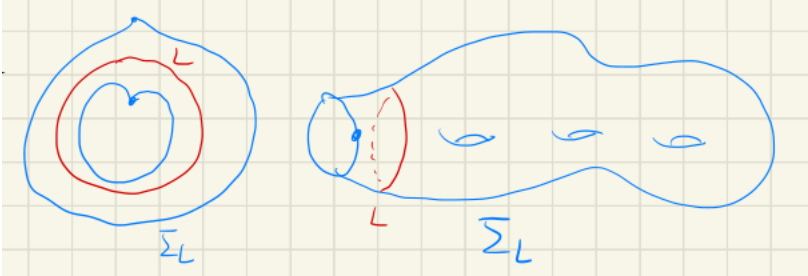


3. In general, $\langle D \rangle_{\text{Brac}}$ is a theta function (not easy)
   - The arguments follow the spirit of step 2

- Detailed arguments are subtle and lengthy.

Proof: $[L]$ Is a Theta Function

- Cut out a subsurface Σ_L containing L and work locally.
- We first prove that $[L]$ is the theta function $\vartheta_{g(L)}$, where $g(L)$ is the leading degree (g-vector, incoming direction)
  - Graphical calculus + atomicity of theta functions.



Proof: $\langle wL \rangle_{\mathrm{Brac}}$ Is a Theta Function

A theta function ϑ_g can be computed

- as a Laurent polynomial $\vartheta_{g,\Delta'}$, by choosing any reference seed $t_{\Delta'}$ (choose a generic base point in the chamber for $t_{\Delta'}$).
- $\vartheta_{g,\Delta'}$ can be computed order by order (view the initial cluster Poisson variables $X_k(t_\Delta)$ as infinitesimal)

Chebyshev recursion

For some Δ', we have $\vartheta_{g(wL),\Delta'} = T_w(\vartheta_{g(L),\Delta'})$.

- Start with any Δ. Applying the Dehn twist around L to Δ for $N \gg 1$ times to obtain Δ'. (a base point close to $g(L)$ [Yur20])
- Then $\vartheta_{g(wL),\Delta'} = A^{w\cdot g(L)} + A^{-w\cdot g(L)} +$ higher order terms.
- $(\vartheta_{g(L),\Delta'})^w = \sum v_{s,w} \vartheta_{g(sL),\Delta'}$ has the same coefficients as the Chebyshev polynomials $(z+z^{-1})^w = \sum v'_{s,w} T_s(z+z^{-1})$

References I

[AK17] Dylan GL Allegretti and Hyun Kyu Kim, *A duality map for quantum cluster varieties from surfaces*, Advances in Mathematics **306** (2017), 1164–1208.

[BW11] Francis Bonahon and Helen Wong, *Quantum traces for representations of surface groups in sl2($\mathbb{C}$)*, Geometry & Topology **15** (2011), no. 3, 1569–1615.

[DM21] Ben Davison and Travis Mandel, *Strong positivity for quantum theta bases of quantum cluster algebras*, Inventiones mathematicae (2021), 1–119, `arXiv:1910.12915`.

[FG06] Vladimir Fock and Alexander Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103, 1–211, `arXiv:math/0311149`.

[FST08] Sergey Fomin, Michael Shapiro, and Dylan Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Math. **201** (2008), no. 1, 83–146.

[GHKK18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich, *Canonical bases for cluster algebras*, Journal of the American Mathematical Society **31** (2018), no. 2, 497–608, `arXiv:1411.1394`.

[Man17] Travis Mandel, *Theta bases are atomic*, Compositio Mathematica **153** (2017), no. 6, 1217–1219, `arXiv:1605.03202`.

[MSW13] Gregg Musiker, Ralf Schiffler, and Lauren Williams, *Bases for cluster algebras from surfaces*, Compositio Mathematica **149** (2013), no. 02, 217–263, `arXiv:1110.4364`.

References III

[Mul16] Greg Muller, *Skein and cluster algebras of marked surfaces*, Quantum topology **7** (2016), no. 3, 435–503, arXiv:1204.0020.

[Qin14] Fan Qin, *t-analog of q-characters, bases of quantum cluster algebras, and a correction technique*, International Mathematics Research Notices **2014** (2014), no. 22, 6175–6232, arXiv:1207.6604, doi:10.1093/imrn/rnt115.

[Qin17] ———, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Mathematical Journal **166** (2017), no. 12, 2337–2442, arXiv:1501.04085.

References IV

[Thu14] Dylan Paul Thurston, *Positive basis for surface skein algebras*, Proceedings of the National Academy of Sciences **111** (2014), no. 27, 9725–9732, `arXiv:1310.1959`.

[Yur20] Toshiya Yurikusa, *Density of g-vector cones from triangulated surfaces*, International Mathematics Research Notices **2020** (2020), no. 21, 8081–8119.

[Zho20] Yan Zhou, *Cluster structures and subfans in scattering diagrams*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications **16** (2020), 013.