Lagrangian fillings for Legendrian links of finite or affine type, and the foldings

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Trends in Cluster Algebras 2022

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Outline

Symplectic / contact geometry | **Cluster algebras**

A Legendrian link λ

If $\lambda =$ positive braid closure $\leftrightarrow$ $\mathcal{A}$: a cluster algebra

a Lagrangian filling $\leftrightarrow$ a seed = (cluster variables, quiver)

$\begin{cases} \text{1-cycles \& intersection} \\ \text{monodromies along cycles} \end{cases}$ $\begin{cases} \text{quiver} \\ \text{cluster variables} \end{cases}$

two fillings are distinct $\Leftrightarrow$ corresponding seeds are different.

§ 1. Cluster structures on double Bott-Samelson cells

§ 2. Combinatorics on exchange graphs of cluster algebras of finite or affine type

§ 3. N-graph realizations of seeds and the foldings

§1. Cluster structures on double Bott-Samelson cells

⊙ WHY Bott-Samelson varieties?

* $G = SL_{n+1}(\mathbb{C})$ (semisimple Lie group of rk n)
* Bott-Samelson varieties are smooth projective varieties parametrized by a word $\mathbb{i} = (i_1, \dots, i_\ell) \in [n]^\ell$

For $\mathbb{i} = (i_1, \dots, i_\ell) \in [n]^\ell$, we denote by $Z_{\mathbb{i}}$ the Bott-Samelson variety

$\dim_{\mathbb{C}} Z_{\mathbb{i}} = \ell$

Bott-Samelson var $Z_{\mathbb{i}}$

- Schubert varieties $\subset G/B$
- Newton-Okounkov bodies of G/B
  string polytopes,
  crystal bases, ..
- Toric degenerations (Bott manifolds)
- Cluster structure on $M_1(\lambda_{\mathbb{i}})$
  rainbow closure of positive braid

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For $\mathbf{i} = (i_1, \ldots, i_\ell) \in [n]^\ell$, we denote by $Z_{\mathbf{i}}$ the Bott-Samelson variety

(1) When $\mathbf{i}$ is reduced, that is, $w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \in \mathfrak{S}_{n+1}$ is of length ℓ

then $Z_{\mathbf{i}}$ is a desingularization of the Schubert variety X_w

(we always have a morphism $\eta : Z_{\mathbf{i}} \longrightarrow G/B$
full flag var)

(2) In such a case, for any line bundle $\mathcal{L} \to G/B$,

$$H^0(X_w, \mathcal{L}) \cong H^0(Z_{\mathbf{i}}, \eta^* \mathcal{L}) \quad \text{(as } B\text{-modules)}$$

$\rightsquigarrow$ used to compute the character of Demazure modules

Newton-Okounkov bodies (string polytopes [Kaveh, 15], generalized string polytopes [Fujita, 18], ..)

⊙ WHAT are Bott-Samelson varieties?

There are several description of BS varieties. The most popular one is "quotient construction". $(P_{i_1} \times P_{i_2} \times \cdots \times P_{i_\ell}) / B^\ell = Z_{\underline{i}}$

Today We consider Magyar's "configuration spaces" description

- $Fl(\mathbb{C}^{n+1}) = \{(\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \subsetneq \mathbb{C}^{n+1}) \mid \dim_{\mathbb{C}} V_i = i \quad \forall i\}$ 

  $\cong G/B \qquad B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \subset G = SL_{n+1}(\mathbb{C})$

  : (full / complete) flag variety (of type A)

We call $(\{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \cdots \subsetneq \langle e_1, \ldots, e_n \rangle \subsetneq \mathbb{C}^{n+1})$: standard flag

$\{e_1, \ldots, e_{n+1}\}$: standard basis vectors

$\dim_{\mathbb{C}}$		
4	$\mathbb{C}^4$ (1)	$\leftarrow \mathbb{C}^4 \supsetneq \langle e_1, e_2, e_3 \rangle$
3	$\langle e_1, e_2, e_3 \rangle$	
2	$\langle e_1, e_2 \rangle$	
1	$\langle e_1 \rangle$	
0	$\{0\}$	: standard flag in $\mathbb{C}^4$

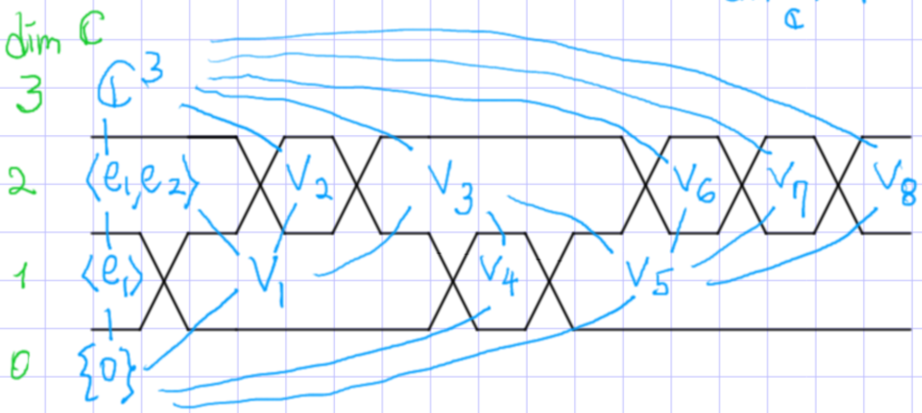
- $\mathbf{i} = (i_1, \ldots, i_\ell) \in [n]^\ell \rightsquigarrow Z_{\mathbf{i}}$ Bott-Samelson variety

  $\rightsquigarrow Z_{\mathbf{i}}$ consists of vector spaces $(V_1, V_2, \ldots, V_\ell)$ s.t. $\dim_{\mathbb{C}} V_k = i_k$

  & satisfies some relations

Eg $\mathbf{i} = (1, 2, 2, 1, 1, 2, 2, 2) \rightsquigarrow (V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8)$

$\dim_{\mathbb{C}}$: 1 2 2 1 1 2 2 2



* We draw | whenever two regions share codim-1 faces

⊙ Cluster structures on double Bott–Samelson cells

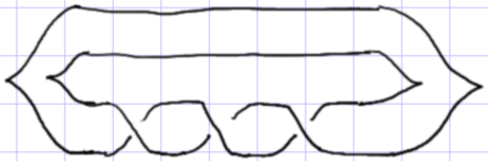
[Shen–Weng] Cluster structures on double Bott–Samelson cells

[Gao–Shen–Weng] Positive braid links with infinitely many fillings

- β : positive braid, $\mathbb{i}$: word of β

  $\rightsquigarrow$ Legendrian link $\lambda_{\mathbb{i}}$

Eg. $\mathbb{i} = (1, 1, 1)$

 $= \lambda_{\mathbb{i}} \sim$ 

undecorated double Bott–Samelson cell [Shen–Weng]

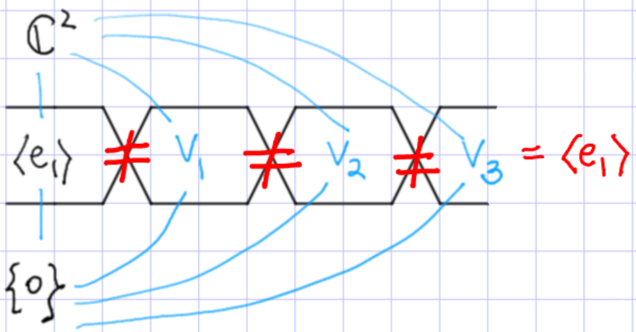
$\boxed{\text{Thm}}$ [Shen–Weng] ① $M_1(\lambda_{\mathbb{i}}) \cong \mathrm{Conf}_{\mathbb{i}}(\mathcal{B})$

② $\mathrm{Conf}_{\mathbb{i}}(\mathcal{B})$ smooth affine variety and its coordinate ring admitting $\mathcal{X}$-cluster structure.

- What is $\mathrm{Conf}_{\mathbb{i}}(\mathcal{B})$?
- What is the cluster structure on $\mathrm{Conf}_{\mathbb{i}}(\mathcal{B})$? (quivers)

- What is $\mathrm{Conf}_{\mathbf{i}}(\mathcal{B})$? double Bott-Samelson cell

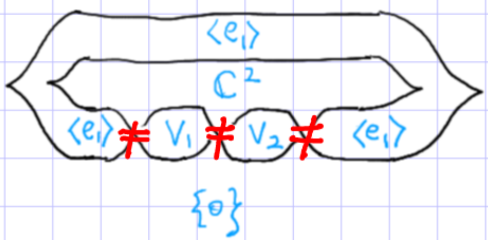
∘ $\mathbf{i} = (1, 1, 1)$ $\quad Z_{\mathbf{i}} = \{ (V_1, V_2, V_3) \mid \{0\} \subsetneq V_i \subsetneq \mathbb{C}^2, \ \dim_{\mathbb{C}} V_i = 1 \}$ of $\dim_{\mathbb{C}} = 3$



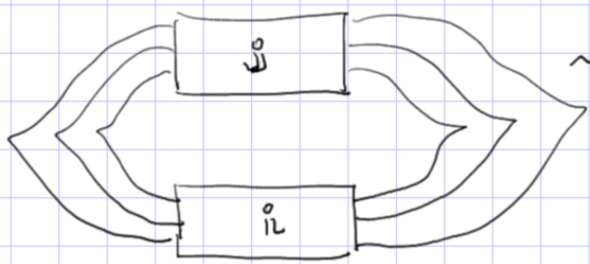
$$\cup$$

$$\mathrm{Conf}_{\mathbf{i}}(\mathcal{B}) = \{ (V_1, V_2, V_3) \in Z_{\mathbf{i}} \mid \langle e_1 \rangle \neq V_1 \neq V_2 \neq V_3 \ \& \ V_3 \cong \langle e_1 \rangle \}$$

of $\dim_{\mathbb{C}} = 2$



<u>Note</u> Why "double"? In fact, Shen-Weng considered



$\rightsquigarrow \mathrm{Conf}_{\mathbf{i}}^{\mathbf{j}}$

- What is the cluster structure on $\mathrm{Conf}_{\underline{i}}^{\circ}(\mathcal{B})$?

Given by the brick quiver. cf. [Gao – Shen – Weng]

positive braid $\underline{i}$

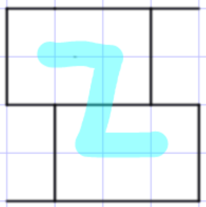
⇝ ① replace each crossing by a vertical bar |

② draw a mutable vertex at each compact brick

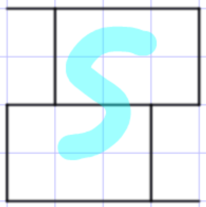
③ for any two adjacent bricks on the same level,

draw a rightward horizontal arrow connecting them

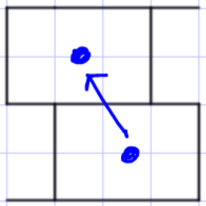
For any two adjacent bricks forming



or



draw a leftward arrow connecting them



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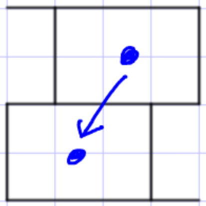
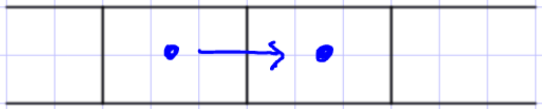
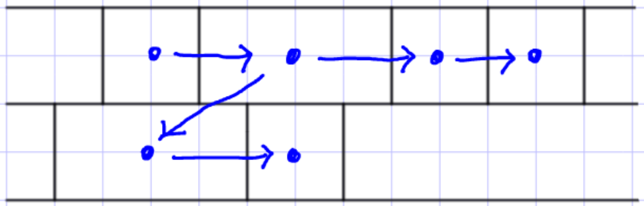


Example $\mathbf{i} = (1,1,1)$



quiver of type A_2

Example $\mathbf{i} = (1,2,2,1,1,2,2,2)$



quiver of type E_6

For each of simply-laced Dynkin diagrams of finite / affine type, except $\widetilde{A}$

[Gao-Shen-Weng] & [Casals-Ng] provide its "standard positive braid"

$$\mathbb{i}_0(a,b,c) := (1, \underbrace{2,\dots,2}_{a}, \underbrace{1,\dots,1}_{b-1}, \underbrace{2,\dots,2}_{c})$$

A_n	D_n	E_n
1^{n+1}	$\mathbb{i}_0(n-2,2,2)$	$\mathbb{i}_0(2,3,n-3)$

finite

$\widetilde{D}_n$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
$(3,2,2,3,\underbrace{2,\dots,2}_{n-4},1,2,2,1)$	$\mathbb{i}_0(3,3,3)$	$\mathbb{i}_0(2,4,4)$	$\mathbb{i}_0(2,3,6)$

affine

§2. Combinatorics on exchange graphs of cluster algebras of finite or affine type

Recall

$\Sigma_{t_0} = (\mathbf{x}, B)$: an initial seed

$\rightsquigarrow \{\Sigma_t = (\mathbf{x}_t, B_t)\}_{t \in \mathbb{T}_n}$ seed pattern (or cluster pattern)

n-regular tree

Two seeds $\Sigma_t = (\mathbf{x}_t, B_t)$, $\Sigma_{t'} = (\mathbf{x}_{t'}, B_{t'})$ are equivalent if

$x_{i;t'} = x_{\sigma(i);t}$ and $b'_{i,j} = b_{\sigma(i),\sigma(j)}$ $\forall i,j \in [m]$

for some permutation $\sigma \in \mathfrak{S}_m$

Def. Let $\mathcal{A}$ be a cluster algebra

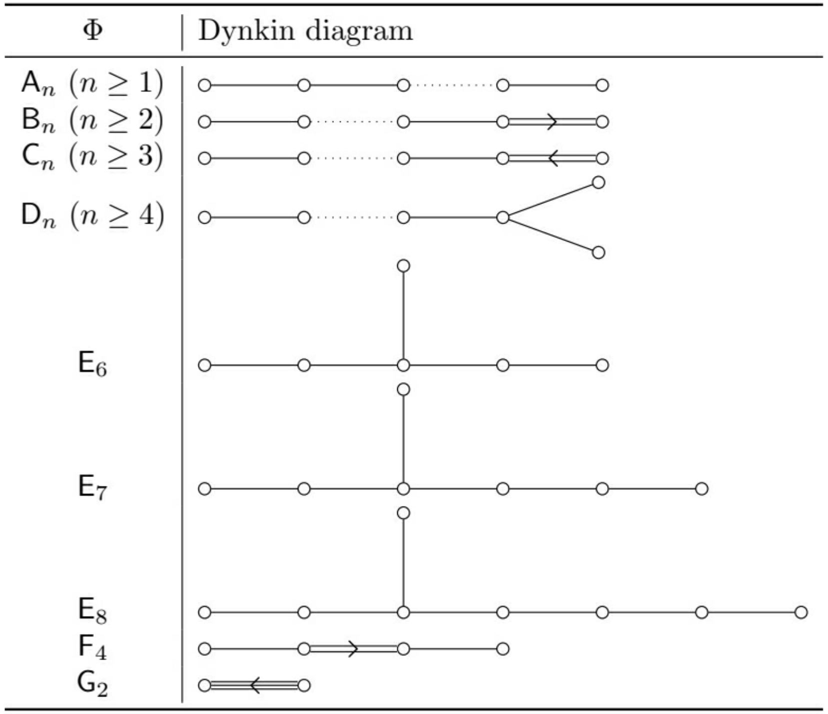
$\mathrm{Ex}(\mathcal{A}) := \mathbb{T}_n / \sim$ where $t \sim t' \Leftrightarrow \Sigma_t \sim \Sigma_{t'}$

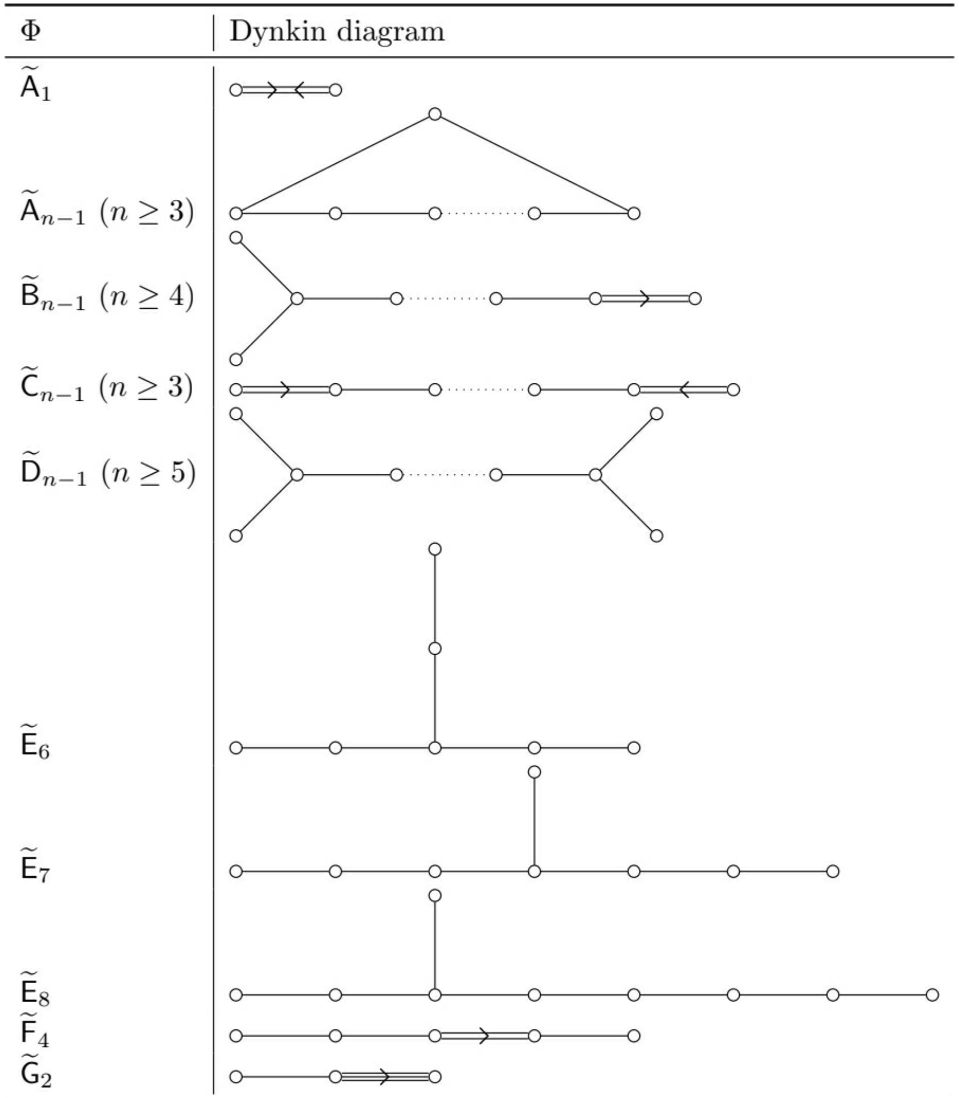
: the exchange graph of $\mathcal{A}$.

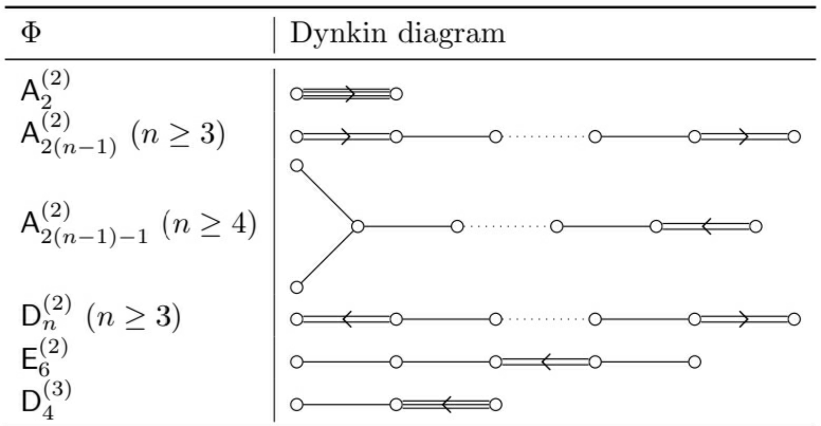
$\boxed{\text{Thm}}$ (cf. [Fomin-Zelevinsky, 03] & [Irelli-Keller-Labardini-Fragoso-Plamondon, 13])

Let $\Sigma_{t_0} = (\mathbf{x}_{t_0}, B_{t_0})$. If $B^{pr}_{t_0}$ is of finite/affine type,

then $\mathrm{Ex}(\mathcal{A}(\Sigma_{t_0}))$ only depends on $B^{pr}_{t_0}$.







finite type

standard affine type

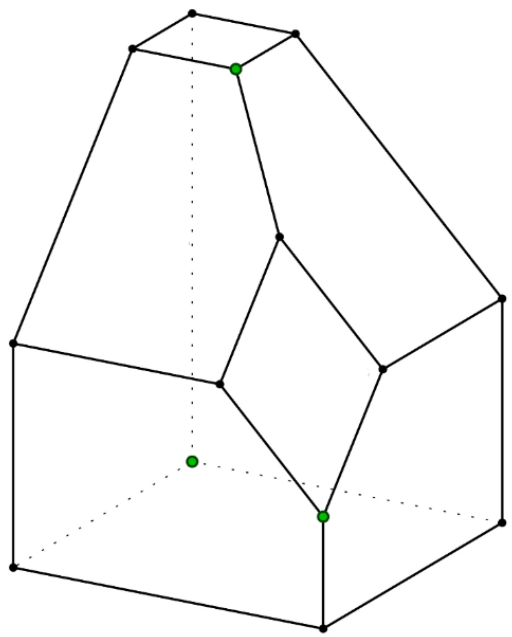
twisted affine type

Assumption B_{t_0} is of finite / affine type & size $n \times n$.

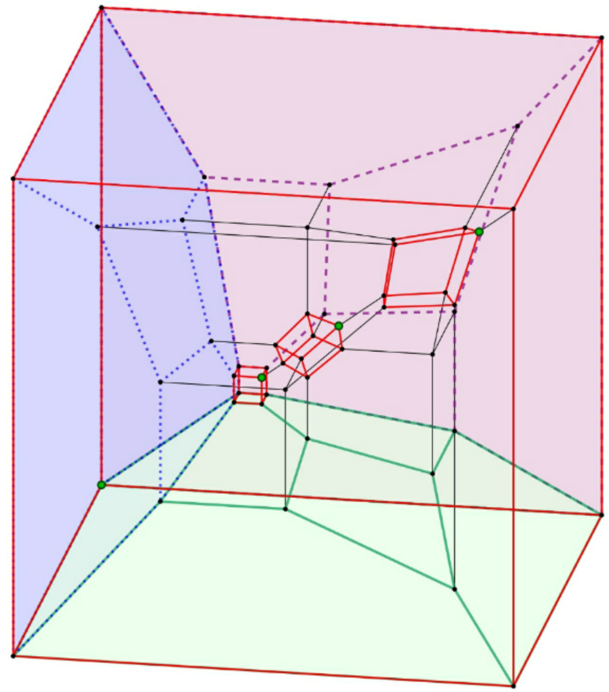
(Cartan counterpart $C(B_{t_0})$ is a Cartan matrix of Dynkin type)

We simply denote by $Ex(B_{t_0})$ the corr. exchange graph

A_3



D_4



Question How to reach all vertices of $Ex(B^{pr})$?

- Coxeter mutation (bipartite coloring)
- induction / Key Lemma

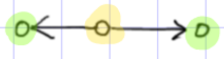
⊙ Coxeter mutation (bipartite coloring)

We call $B=(b_{i,j})$ of size $n\times n$ bipartite if $\exists\ \varepsilon : [m] \to \{+,-\}$, coloring,

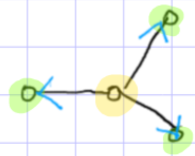
s.t. $\quad b_{i,j} \neq 0 \quad \Rightarrow \quad \varepsilon(i) \neq \varepsilon(j) \qquad \forall\, i\neq j.$

If B is not of type $\widetilde{A}$, then B is bipartite

Ex :



A_3



D_4

: $+$ source

: $-$ sink

$I_+ := \{ i \in [n] \mid \varepsilon(i) = + \}$, $I_- := \{ i \in [n] \mid \varepsilon(i) = - \}$

$\rightsquigarrow [n] = I_+ \sqcup I_-$

$$\mu_\varepsilon := \prod_{i\in I_\varepsilon} \mu_i \qquad \varepsilon = + \text{ or } -$$

$\boxed{\mu_Q} := \mu_- \mu_+$: Coxeter mutation $\quad \boxed{\mu_Q^{-1}} := \mu_+ \mu_-$

cf. a Coxeter element in W is a product of all simple reflections

Note $\quad \mu_-\mu_+(B) = \mu_+\mu_-(B) = B$

eg $\quad \circ \to \circ \leftarrow \circ \overset{\mu_+}{\mapsto} \circ \leftarrow \circ \to \circ \overset{\mu_-}{\mapsto} \circ \to \circ \leftarrow \circ$

- $\Sigma_0 = \Sigma_{t_0}$

$$\Sigma_r = (\mathbf{x}_r, B) := \begin{cases} \mu_Q^r(\Sigma_{t_0}) & \text{if } r>0 \\ (\mu_Q^{-1})^{-r}(\Sigma_{t_0}) & \text{if } r<0 \end{cases}$$

$\{\Sigma_r \mid r \in \mathbb{Z}\}$: bipartite belt. $\quad \mathbf{x}_r = (x_{1;r}, x_{2;r}, \dots, x_{n;r})$

$\boxed{\text{Thm}}$ ([Fomin-Zelevinsky, 03] & [Reading-Stella, 20])

Suppose that B_{t_0} is of finite / affine type

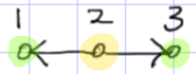
(1) μ_Q acts on $\mathrm{Ex}(B_{t_0})$

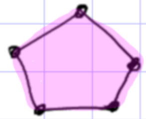
(2) $\mathrm{Ex}(B_{t_0}, x_{\ell;r}) \cong \mathrm{Ex}(B_{t_0}|_{[n]\setminus\{\ell\}})$

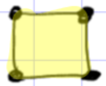
Induced subgraph consisting of seeds having $x_{\ell;r}$

(3) For any seed $\Sigma = (\mathbf{x}, B)$, $\exists\, r \in \mathbb{Z}$ s.t.

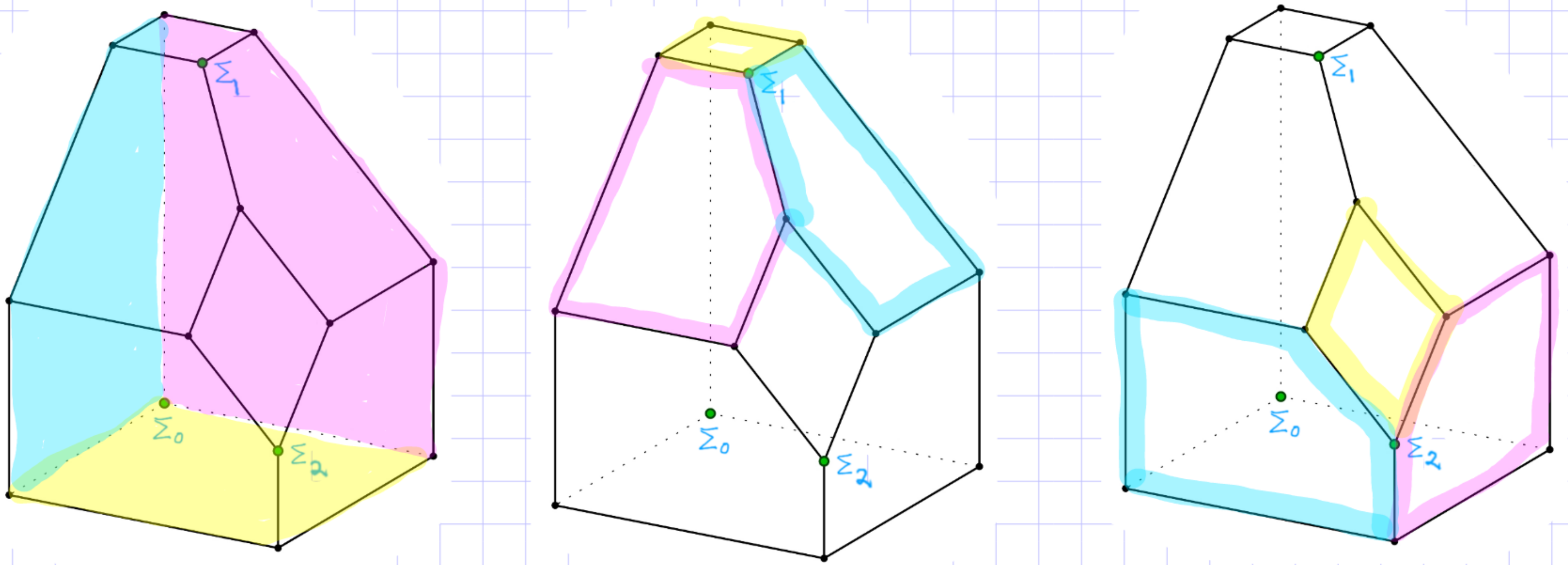
$$|\{x_{1;r}, \dots, x_{n;r}\} \cap \{x_1, \dots, x_n\}| \geq 2.$$

A_3 

$A_3|_{\{2,3\}}$ $\circ \rightarrow \circ$: A_2 

$A_3|_{\{1,3\}}$ $\circ \quad \circ$: $A_1 \times A_1$ 

$A_3|_{\{1,2\}}$ $\circ \leftarrow \circ$: A_2 



Key Lemma 1

Let Σ be a seed in a cluster pattern of finite/affine type (of rank n).

Then $\exists\, r \in \mathbb{Z}$ and $\ell \in [n]$ s.t. Σ & $\Sigma_r \in \mathrm{Ex}(B_{t_0}, x_{\ell;r})$.

Indeed, $\exists\, j_1, \ldots, j_L \in [n] \setminus \{\ell\}$ s.t.

① $\mu_Q^r(\Sigma_{t_0}),\ \mu_{j_1} \cdot \mu_Q^r(\Sigma_{t_0}),\ \mu_{j_2}\mu_{j_1}\mu_Q^r(\Sigma_{t_0}), \ldots,$

$$\mu_{j_L}\cdots\mu_{j_2}\mu_{j_1}\mu_Q^r(\Sigma_{t_0}) \in \mathrm{Ex}(B_{t_0}, x_{\ell;r})$$

② $\Sigma = \mu_{j_L}\cdots\mu_{j_2}\mu_{j_1}\mu_Q^r(\Sigma_{t_0})$.

§3. N-graph realization of seeds

Main tool : N-graph !

a Lagrangian filling $\leftrightarrow$ a seed = (cluster variables, quiver)

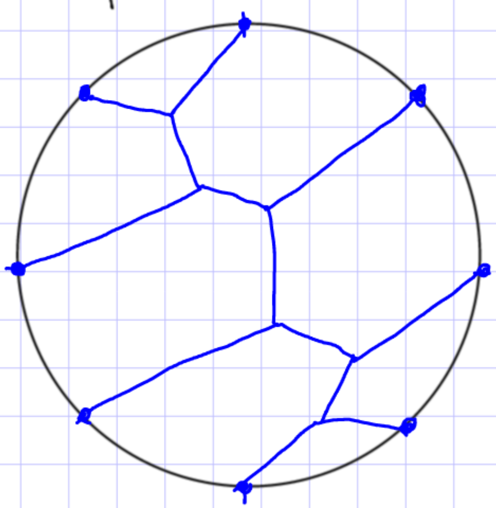
$\left\{ \begin{array}{l} \text{1-cycles \& intersection} \\ \text{monodromies along cycles} \end{array} \right.$ $\left\{ \begin{array}{l} \text{quiver} \\ \text{cluster variables} \end{array} \right.$

Def [Casals - Zaslow] An N-graph $\mathcal{G}$ on a smooth surface C is a set

$\mathcal{G} = \{ G_1, G_2, \dots, G_{N-1} \}$ of embedded graphs s.t.

- each G_i is trivalent
- each pair G_i, G_{i+1} intersect at hexagonal point

Example



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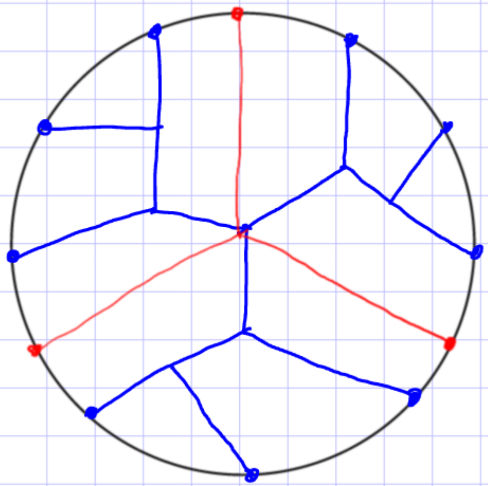

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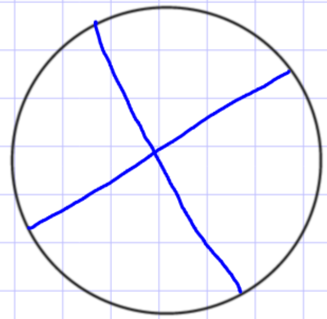

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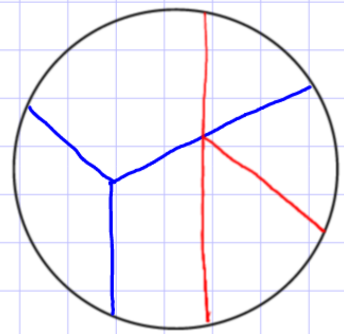

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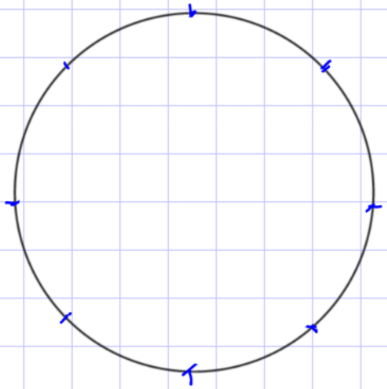
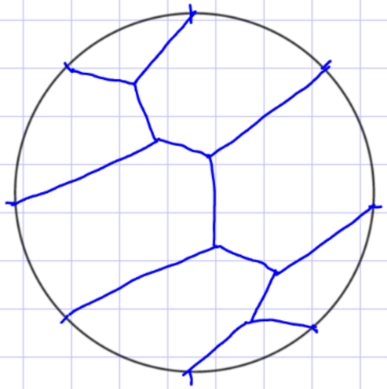
Def [Casals - Zaslow] A Legendrian weave $\Lambda(\mathcal{G})$ is a Legendrian surface obtained by weaving the local Legendrian sheets.



$$\begin{array}{c} J^1 S^1 \\ \cup \\ \lambda \end{array}$$

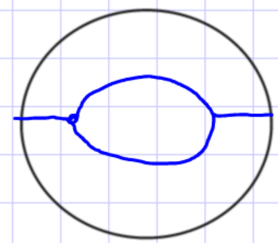


$$\begin{array}{ccc} T^*\mathbb{D}^2 \times \mathbb{R}_w & & \\ \wr\Vert & & \\ J^1\mathbb{D}^2 & \xrightarrow{\pi_w} & T^*\mathbb{D}^2 \\ \cup & & \cup \\ \Lambda(\mathcal{G}) & & \pi_*(\Lambda(\mathcal{G})) \end{array}$$

$\pi_*(\Lambda(\mathcal{G}))$ Lagrangian surface with boundary λ.

Def $\mathcal{G}$ is free if $\Lambda(\mathcal{G})$ has no interior Reeb chords

Note $\mathcal{G}$ is free if each $\mathcal{G}_i$ is a tree

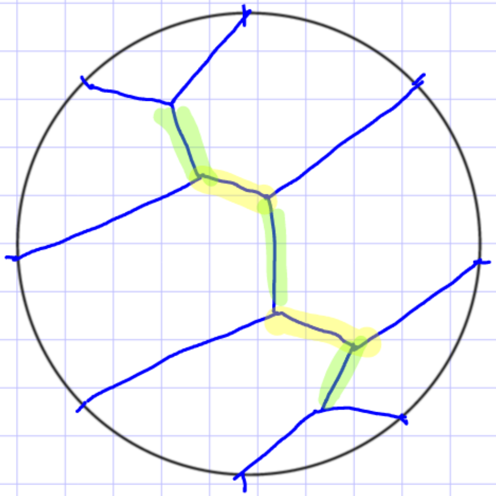


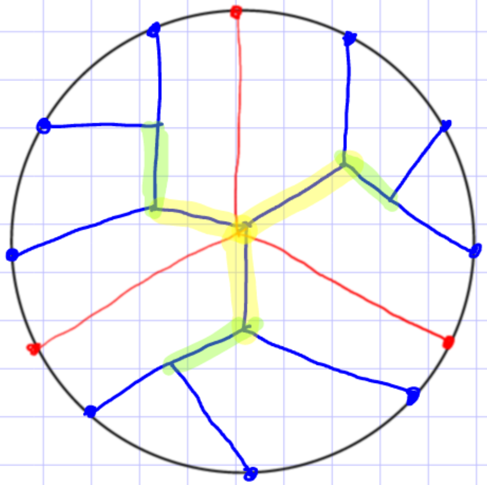
$\boxed{\text{Prop}}$ [Casals - Zaslow] If $\mathcal{G}$ is free, then $\pi_*(\Lambda(\mathcal{G}))$ is an embedded exact Lagrangian filling for λ.

$\odot$ $\mathcal{G}$: N-graph, $\mathcal{B} = (\gamma_1, \dots, \gamma_n)$ $\gamma_i \in H_1(\Lambda(\mathcal{G}))$

$\Rightarrow \Psi(\mathcal{G}, \mathcal{B}) = (\underbrace{\mathbf{y}(\mathcal{G}, \mathcal{B})}_{\text{variables}}, \underbrace{\mathcal{Q}(\mathcal{G}, \mathcal{B})}_{\text{quiver}})$

$\rightsquigarrow$ ① quiver : algebraic intersection number (γ_i, γ_j)





$1 \to 2 \leftarrow 3 \to 4 \leftarrow 5$

: A_5 type

$$\begin{tikzcd} 3 \arrow[dr] & & 2 \arrow[dl] \\ & 1 & \\ & 4 \arrow[u] & \end{tikzcd}$$

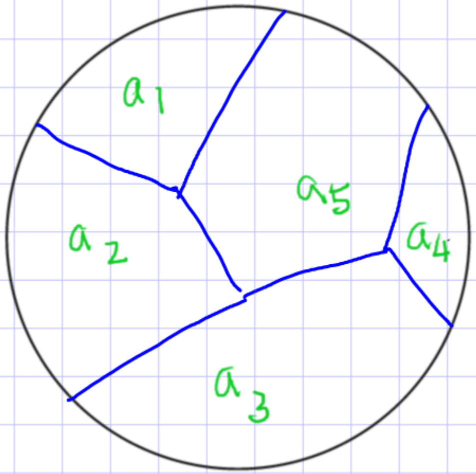
: D_4 type

Note : $M_1(\lambda\ \overset{\circ}{\square})$ is of the same type.

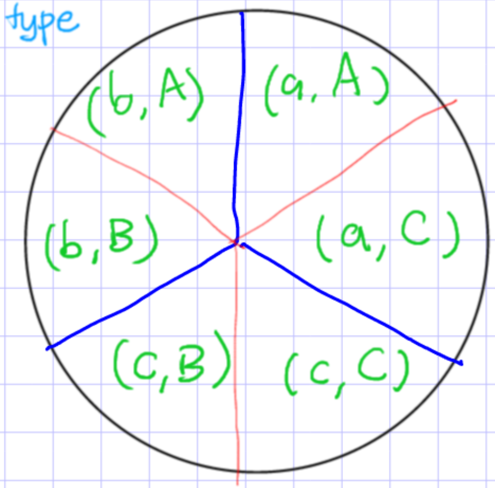
② variables $\subset \mathbb{C}[\mathcal{M}_1(\lambda_i)]$

: defined via microlocal monodromy

- $\mathcal{M}(G) \subset \mathcal{M}_1(\lambda_i)$ when λ_i is of finite or affine type



$A = a \oplus b$

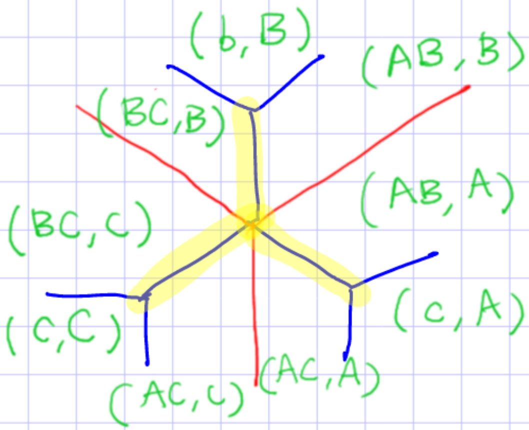


$a \neq b,\ b \neq c,\ c \neq a$

$A \neq B,\ B \neq C,\ C \neq A$

$a_i \neq a_{i+1}$ & $a_2 \neq a_5,\ a_3 \neq a_5$

$$\frac{a_1 \wedge a_2}{a_2 \wedge a_3} \cdot \frac{a_3 \wedge a_4}{a_4 \wedge a_1}$$

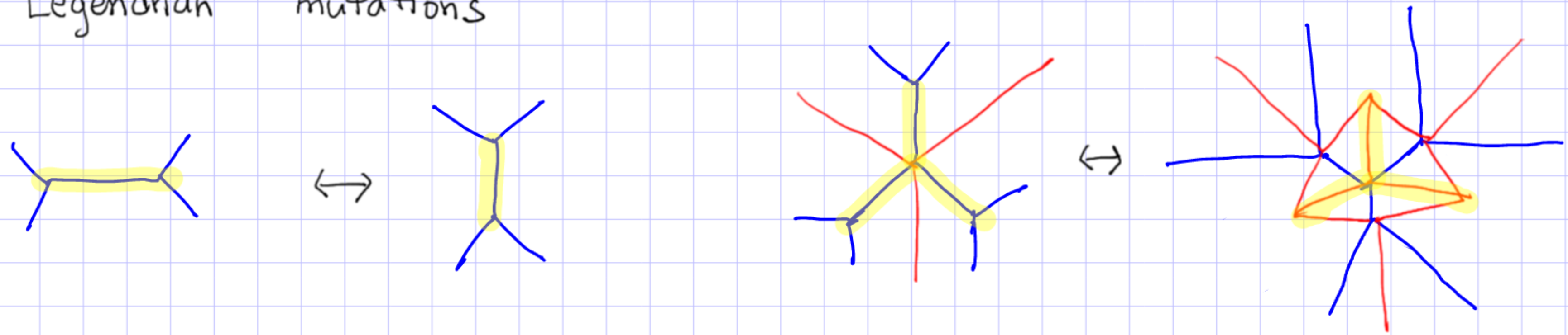


$$\frac{C(a)\,B(c)\,A(b)}{C(b)\,B(a)\,A(c)}$$

Thm [Casals–Zaslow] $\Psi(G, B) = \Psi(G', B')$

$\Leftrightarrow \quad \Lambda(G) \simeq \Lambda(G')$ exact Lagrangian isotopic

◎ Legendrian mutations



$\boxed{\text{Thm}}$ [Casals - Zaslow] Legendrian mutation induces X-cluster mutation.

$\boxed{\text{Questions}}$

$\boxed{\text{Q 1}}$ Can we mutate N-graphs as many times as we want?

A 1 · For N-graphs of type A or D, YES.
[Y. Pan] [J. Hughes]

- Not known yet in general

$\boxed{\text{Q 2}}$ Can any seed in the cluster pattern be realized as an N-graph?

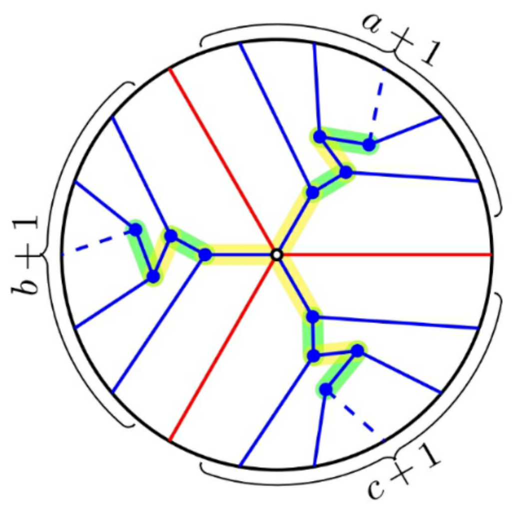
- $Q_1 \Rightarrow Q_2$
- If Q_2 is true, then we have at least as many Lagrangian fillings as there are seeds for Legendrian links of type X.

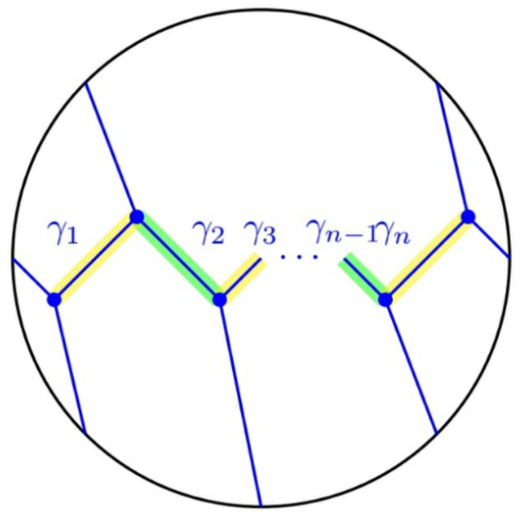
⓪ Tripod N-graphs & quivers.

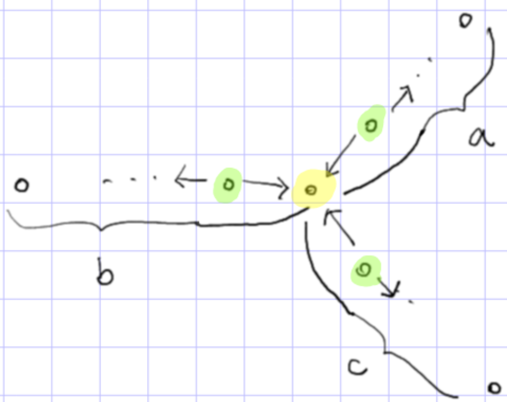
- For $a, b, c \geq 1$, $\lambda(a,b,c) :=$ closure of $\sigma_2 \sigma_1^{a+1} \sigma_2 \sigma_1^{b+1} \sigma_2 \sigma_1^{c+1}$

  For $n \geq 1$, $\lambda(n) :=$ closure of σ_1^{n+3}

- Define N-graphs $\mathcal{G}(a,b,c)$ & $\mathcal{G}(n)$:







$=: Q(a,b,c)$

\# of vertices $= a+b+c-2$

$=: Q(n)$

(of type A_n)

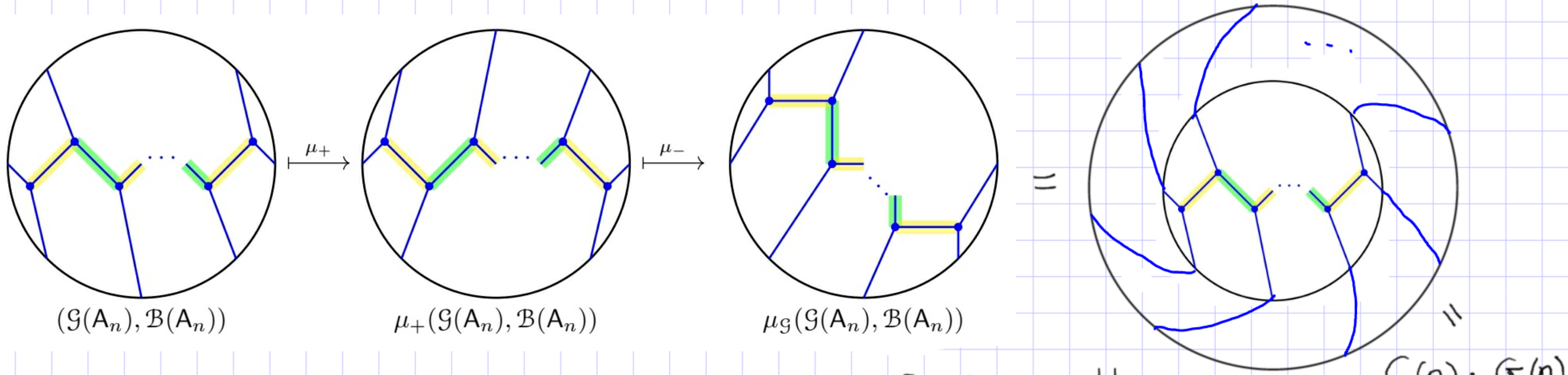
STRATEGY

① Well-definedness of Legendrian Coxeter mutations so that

$$\Psi(\mu_Q(\mathcal{G},\mathcal{B})) = \mu_Q(\Psi(\mathcal{G},\mathcal{B}))$$

② use the KEY lemma & induction on n

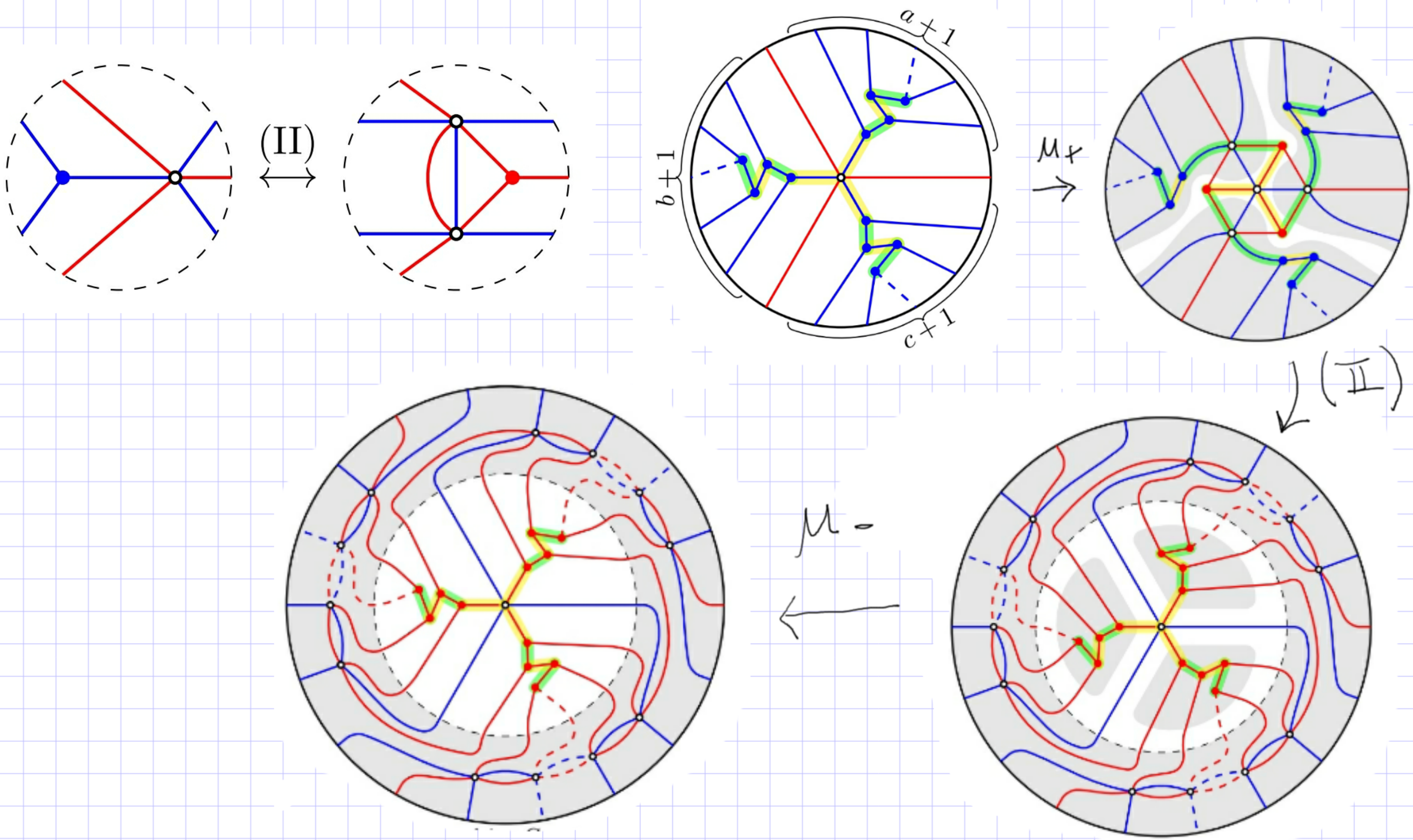
①-1 For $\mathcal{G}(n)$, μ_Q is $\frac{2\pi}{n+3}$-rotation.



adding a Coxeter padding.

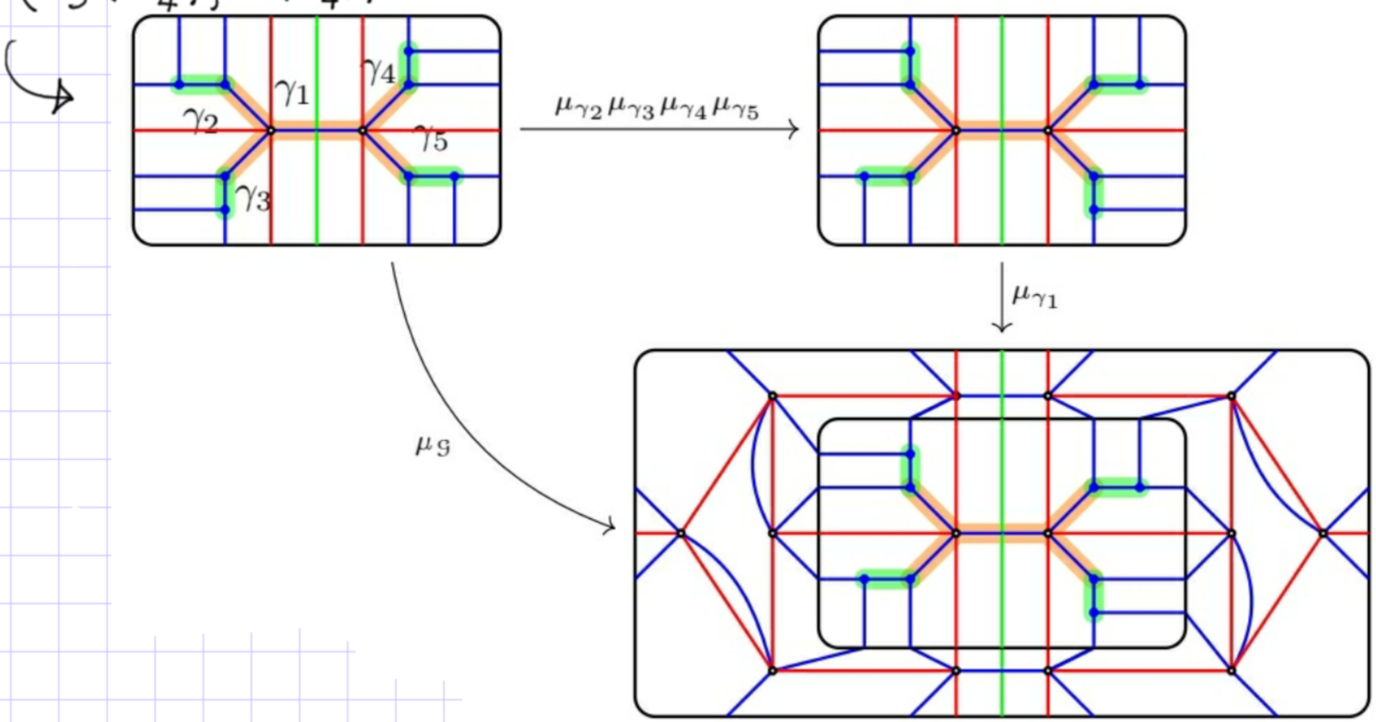
$$\mu_Q^r(\mathcal{G}(n),\mathcal{B}(n)) = \underbrace{C(n)\cdot\cdots\cdot C(n)}_{r}\cdot(\mathcal{G}(n),\mathcal{B}(n))$$

①-2 For $\mathcal{G}(a,b,c)$, we use move (II) several times.



① -3 For $\tilde{D}_n$,

$(\mathcal{G}(\tilde{D}_4), \mathcal{B}(\tilde{D}_4))$



$\boxed{\text{Prop}}$ For $(\mathcal{G}, \mathcal{B}) = (\mathcal{G}(a,b,c), \mathcal{B}(a,b,c))$, $(\mathcal{G}(n), \mathcal{B}(n))$, or $(\mathcal{G}(\tilde{D}_n), \mathcal{B}(\tilde{D}_n))$,

$\forall r \in \mathbb{Z}$ $\mu_Q^r(\mathcal{G}, \mathcal{B})$ is well-defined.

$\boxed{\text{Cor}}$ Seeds in the bipartite belt are realizable.

Thm Let Σ_t be a seed in the cluster pattern finite or affine acyclic simply-laced type X given by $\Sigma_{t_0} = \Psi(\mathcal{G}_0, \mathcal{B}_0)$

Then $\exists$ free N-graph $(\mathcal{G}, \mathcal{B})$ s.t. $\partial\mathcal{G} = \lambda(X)$ &

$\Psi(\mathcal{G}, \mathcal{B}) = \Sigma_t$.

Indeed, we have at least as many Lagrangian fillings as there are seeds for Legendrian links of type X.

(proof)

X	A_n	D_n	$E_n (n=6,7,8)$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$	$\widetilde{D}_n$
$\mathcal{G}$	$\mathcal{G}(n)$	$\mathcal{G}(n-2,2,2)$	$\mathcal{G}(2,3,n-3)$	$\mathcal{G}(3,3,3)$	$\mathcal{G}(2,4,4)$	$\mathcal{G}(2,3,6)$	$\mathcal{G}(\widetilde{D}_n)$

① Seeds in the bipartite belt are realizable.

② Use the KEY lemma 1 & induction. □

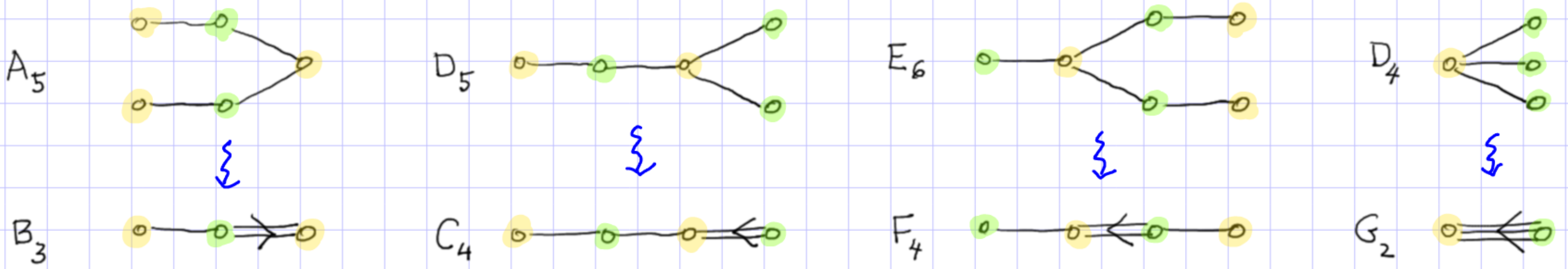
Question. How to get a similar result for non-simply-laced types?

⓪ Foldings

For some finite group G, one can fold

seeds in the cluster pattern of simply-laced type having G-symmetry

$\overset{\text{fold}}{\rightsquigarrow}$ seeds in the cluster pattern of non-simply-laced type



$$C(B_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \qquad C(C_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

$$C(F_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad C(G_2) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Note $\mathcal{B}$: exchange matrix of non-simply-laced Dynkin type.

① $\mathcal{B}$ is NOT skew-symmetric but skew-symmetrizable.

② $\exists$ skew-symmetric $\tilde{\mathcal{B}}$ of Dynkin type s.t. $\mathcal{B} = \tilde{\mathcal{B}}^G$

- Q : quiver on $[n] = \{1, 2, \dots, n\}$ $(n = m)$

  G : a finite group acting on $[n]$. $i \sim i' \Leftrightarrow G \cdot i = G \cdot i'$

Def Q is G-invariant if $\forall g \in G$, $i, j \in [n]$

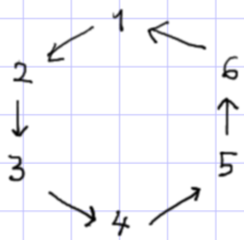
$$i \to j \Leftrightarrow g(i) \to g(j).$$

A G-invariant quiver Q is admissible if

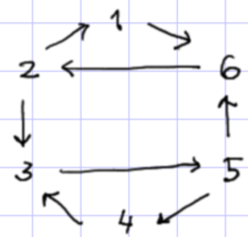
(1) $\forall i \sim i'$, $\nexists$ arrow connecting i & i'.

(2) $\forall i \sim i'$, $i \to j \Leftrightarrow i' \to j$ $\quad i \leftarrow j \Leftrightarrow i' \leftarrow j$

Example $n = 6$. $G = \mathbb{Z}/2\mathbb{Z}$. $i \mapsto i+3 \pmod 6$



$\xrightarrow{\mu_4 \mu_1}$



: G-admissible

: G-invariant but not G-admissible

Def Q is globally foldable w.r.t. G-action if $\forall I_1, \dots, I_\ell$ G-orbits,

$(\mu_{I_\ell} \cdots \mu_{I_1})(Q)$ is G-admissible.

A seed $(\mathbf{x} = (x_1, \dots, x_n), Q)$ is (G, ψ)-admissible if Q is G-admissible

& $\psi(x_i) = \psi(x_{i'})$ $\forall i \sim i'$ under the identification ψ

$\boxed{\text{Thm}}$ (cf. [Fomin-Williams-Zelevinsky])

Let Q be a globally foldable quiver

$\Sigma_{t_0} = (\mathbf{x}, Q)$ an initial seed

Then $\forall\, I_1, \dots, I_\ell$ G-orbits, $(\mu_{I_\ell} \cdots \mu_{I_1})(\Sigma_{t_0})$ is (G,ψ)-admissible, and moreover, the folded seeds $((\mu_{I_\ell} \cdots \mu_{I_2})(\Sigma_{t_0}))^G$ form the cluster pattern given by $\Sigma_{t_0}^G$.

Z	A_{2n-1}	D_{n+1}	E_6	D_4
G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
Z^G	B_n	C_n	F_4	G_2

Z	$\widetilde{A}_{2,2}$	$\widetilde{A}_{n,n}$	$\widetilde{D}_4$		$\widetilde{D}_n$		$\widetilde{D}_{2n}$		$\widetilde{E}_6$		$\widetilde{E}_7$
G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
Z^G	$\widetilde{A}_1$	$D^{(2)}_{n+1}$	$A^{(2)}_2$	$D^{(3)}_4$	$\widetilde{C}_{n-2}$	$A^{(2)}_{2(n-1)-1}$	$\widetilde{B}_n$	$A^{(2)}_{2n-2}$	$\widetilde{G}_2$	$E^{(2)}_6$	$\widetilde{F}_4$

$\boxed{\text{Key Lemma 2}}$ For (X, G, Y) as above,

Q : quiver of simply-laced Dynkin type X, $G \circlearrowright Q$ as above.

Then, any $Q' \sim Q$, if Q' is G-invariant, then Q' is G-admissible.

$\{ \Sigma^G \mid \Sigma : (G,\psi)\text{-invariant} \}$ forms a cluster pattern of type Y.

$$\boxed{\{\text{seeds of type } X\} \supset \{G\text{-invariant seeds}\} = \{\text{seeds of type } Y\}}$$

- Consider two actions on $S^3 \times \mathbb{R}_u$

$$R_{\theta_0}(z_1, z_2, u) = (z_1 \cos\theta_0 - z_2 \sin\theta_0,\ z_1 \sin\theta_0 + z_2 \cos\theta_0,\ u) \quad \text{rotation}$$

$$\eta(z_1, z_2, u) = (\bar{z}_1, \bar{z}_2, u) \quad \text{conjugation}$$

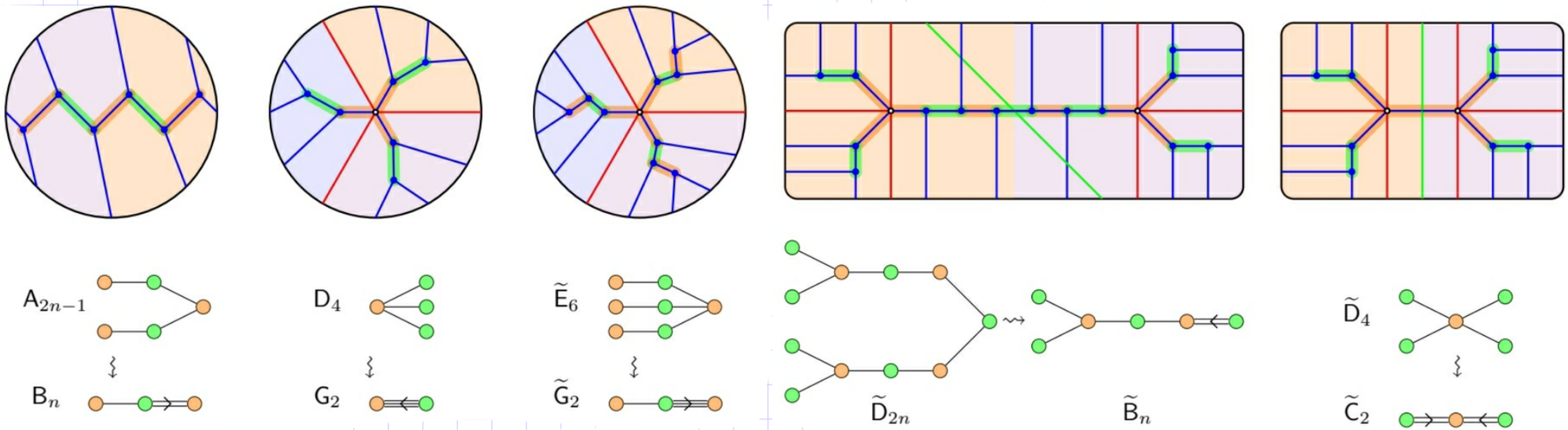
$\rightsquigarrow$ restriction on J^1S^1

$$R_{\theta_0}|_{J^1S^1}(\theta, p_\theta, z) = (\theta + \theta_0, p_\theta, z) \rightsquigarrow \text{rotating the disk } D^2$$

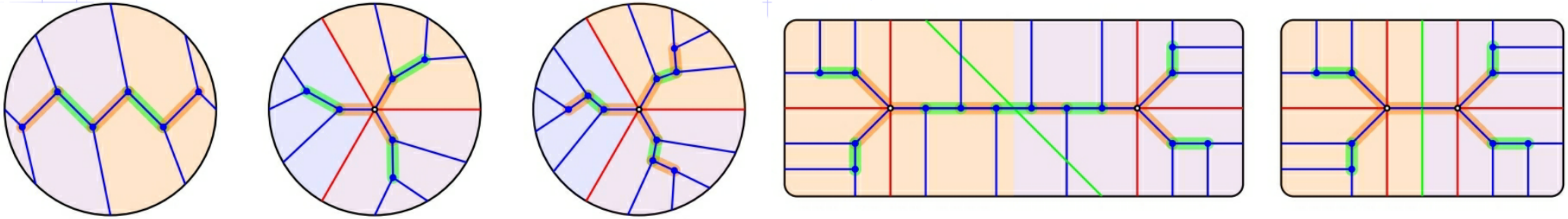
$$\eta|_{J^1S^1}(\theta, p_\theta, z) = (\theta, -p_\theta, -z) \rightsquigarrow \text{flipping } z\text{-coordinates, i.e.,}$$

$$S_1 \leftrightarrow S_3$$

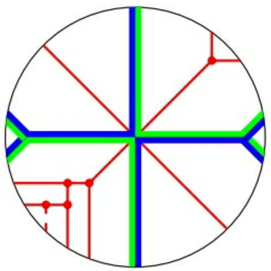
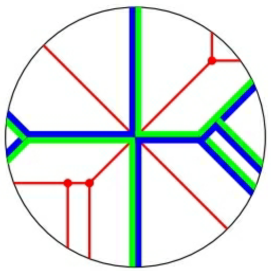
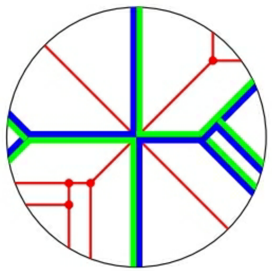
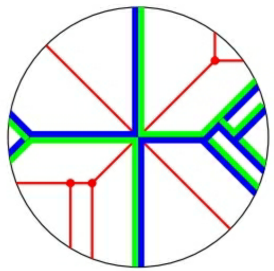
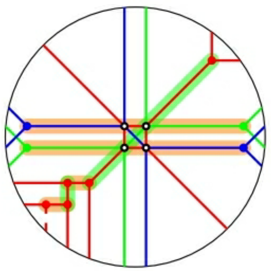
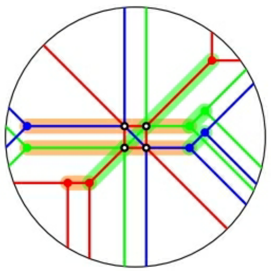
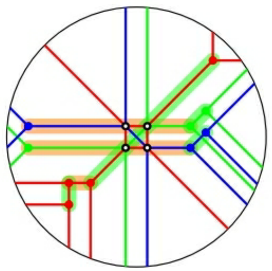
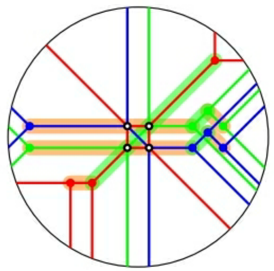
$R_{\theta_0}|_{J'S'}(\theta, p_\theta, z) = (\theta+\theta_0, p_\theta, z) \rightsquigarrow$ rotating the disk D^2



$R_{\theta_0}|_{J'S'}(\theta, p_\theta, z) = (\theta+\theta_0, p_\theta, z) \rightsquigarrow$ rotating the disk D^2



$\eta|_{J'S'}(\theta, p_\theta, z) = (\theta, -p_\theta, -z) \rightsquigarrow$ flipping z-coordinates, i.e., $S_1 \leftrightarrow S_3$

Z	D_{n+1}	E_6	$\widetilde{E}_6$	$\widetilde{E}_7$
$\tilde{\mathcal{G}}(Z)$				
Perturb.				

Thm Let Σ_t be a seed in the cluster pattern finite or affine acyclic non-simply-laced type X given by $\Sigma_{t_0} = \Psi(\mathcal{G}_0, \mathcal{B}_0)$.
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$$\Psi(\mathcal{G},\mathcal{B}) = \Sigma_t .$$

Indeed, we have at least as many Lagrangian fillings as there are seeds for Legendrian links of type X.

(proof)

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Z^G	B_n	C_n	F_4	G_2

Z	$\widetilde{A}_{2,2}$	$\widetilde{A}_{n,n}$	$\widetilde{D}_4$		$\widetilde{D}_n$		$\widetilde{D}_{2n}$		$\widetilde{E}_6$		$\widetilde{E}_7$
G	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
Z^G	$\widetilde{A}_1$	$D^{(2)}_{n+1}$	$A^{(2)}_2$	$D^{(3)}_4$	$\widetilde{C}_{n-2}$	$A^{(2)}_{2(n-1)-1}$	$\widetilde{B}_n$	$A^{(2)}_{2n-2}$	$\widetilde{G}_2$	$E^{(2)}_6$	$\widetilde{F}_4$

Use the first main theorem for X: simply-laced & symmetries

KEY Lemma 2 $\square$

Future works

Q Can we extend the result to other Dynkin type ?

a) $\widetilde{A}_{p,q}$ is remained among finite / affine type

i) [Casals - Ng] provides a candidate Legendrian λ of type $\widetilde{A}_{1,1}$.

However, λ is NOT a rainbow closure of a positive braid.

ii) We don't know a candidate for other type $\widetilde{A}_{p,q}$

b) Hyperbolic Dynkin type.

A big obstruction is the KEY lemmas 1 & 2.

Thank you !