Extended crystals and Hernandez–Leclerc categories

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Motivation and Idea

Let

- $U_q'(\mathfrak{g})$:= a quantum affine algebra (q: indeterminate)
- $\mathscr{C}_\mathfrak{g}$:= the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules (It has a rich structure, e.g., non-semisimplicity, tensor, rigidity, etc.)
- $\mathscr{C}_\mathfrak{g}^0$:= Hernandez-Leclerc category of $\mathscr{C}_\mathfrak{g}$

- ([Hernandez-Leclerc])
  Distinguished subcategories $\mathscr{C}_{\mathfrak{g}}^{-}$, $\mathscr{C}_{\mathfrak{g}}^{\ell}$ of $\mathscr{C}_{\mathfrak{g}}^{0}$,
  Cluster algebra structure of $K(\mathscr{C}_{\mathfrak{g}}^{-})$ and $K(\mathscr{C}_{\mathfrak{g}}^{\ell})$
- ([Kashiwara-Kim-Oh-P.]) (Monoidal categorification)
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- ([Hernandez-Leclerc], [Oh-Suh], [Oh-Scrimshaw], [Fujita-Oh])
  There exist distinguished subcategories $\mathscr{C}_{\mathcal{Q}}$ for a Q-datum $\mathcal{Q}$.
  Moreover,
  - $U_q^{-}(\mathfrak{g}_{\mathrm{fin}})^{\vee} \simeq K_t(\mathscr{C}_{\mathcal{Q}}) \cdots (*)$
    ([Hernandez-Leclerc, Hernandez-Oya, Fujita-Hernandez-Oh-Oya])
- ([Kang-Kashiwara-Kim], [Kashiwara-Kim-Oh-P.])
  $R_{\mathfrak{g}_{\mathrm{fin}}}$ = symmetric quiver Hecke algebra associated with $\mathfrak{g}_{\mathrm{fin}}$.
  For a complete duality datum $\mathcal{D}$,
  $\mathcal{F}_{\mathcal{D}} : \mathrm{R}_{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} \longrightarrow \mathscr{C}_{\mathcal{D}} \subset \mathscr{C}_{\mathfrak{g}}^{0}$ (Generalized Schur-Weyl duality)

Main idea Let $\mathcal{D}$ be a complete duality datum of $\mathscr{C}_{\mathfrak{g}}^0$.

$$\begin{array}{ccccc} R_{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} & \xrightarrow{\quad \mathcal{F}_{\mathcal{D}} \quad} & \mathscr{C}_{\mathcal{D}} & \subset & \mathscr{C}_{\mathfrak{g}}^0 \\ \uparrow & & \uparrow & & \\ B_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \simeq \{\text{simple } R_{\mathfrak{g}_{\mathrm{fin}}}\text{-modules}\} & \xleftrightarrow{1-1} & \{\text{simple modules in } \mathscr{C}_{\mathcal{D}}\} & & \end{array}$$

(a) ([Khovanov-Lauda], [Rouquier]) $U_q^-(\mathfrak{g}_{\mathrm{fin}})^\vee \simeq K(R_{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod})$,

(b) ([Lauda-Vazirani]) $B_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \simeq \{\text{simple } R_{\mathfrak{g}_{\mathrm{fin}}}\text{-modules}\}$ as a crystal,

(c) $\{\text{simple } R_{\mathfrak{g}_{\mathrm{fin}}}\text{-modules}\} \underset{1-1}{\overset{\mathcal{F}_{\mathcal{D}}}{\longleftrightarrow}} \{\text{simple modules in } \mathscr{C}_{\mathcal{D}}\}$

(d) $K(\mathscr{C}_{\mathfrak{g}}^0) \approx$ a product of infinite copies of $K(\mathscr{C}_{\mathcal{D}})$

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$$\prod_{\mathbb{Z}} B_{\mathfrak{g}_{\rm fin}}(\infty) \xleftrightarrow{\quad ??? \quad} \{\text{simple modules in } \mathscr{C}_{\mathfrak{g}}^0\}$$

Goal : Find a crystal-theoretic approach to interpret $\mathscr{C}_{\mathfrak{g}}^0$!

Crystals

Let

- $\mathrm{A} := (a_{ij})_{i,j\in I}$ (symmetrizable generalized Cartan matrix)
- $\Pi := \{\alpha_i \mid i \in I\}$ (simple roots)
- $\Pi^\vee := \{h_i \mid i \in I\}$ (simple coroots) $(\langle h_i, \alpha_j\rangle = a_{i,j}$ for all $i,j)$
- $\mathrm{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ (root lattice)
- P := weight lattice
- $\mathrm{P}^\vee$:= dual weight lattice

Definition (Quantum group)

(i) $U_q(\mathfrak{g})$:= the algebra over $\mathbb{Q}(q)$ generated by e_i, f_i $(i \in I)$ and q^h $(h \in \mathrm{P}^\vee)$ satisfying certain defining relations determined by A.

(ii) $U_q^-(\mathfrak{g})$:= the subalgebra of $U_q(\mathfrak{g})$ generated by f_i $(i \in I)$.

The notion of crystals was introduced by Kashiwara, which is one of the most powerful combinatorial tool to study quantum groups and their representations.

A $U_q(\mathfrak{g})$-crystal is a set B endowed with maps

$$\begin{aligned} \mathrm{wt}\colon B &\to \mathrm{P}, \\ \varphi_i,\ \varepsilon_i\colon B &\to \mathbb{Z} \sqcup \{\infty\} \\ \tilde{e}_i,\ \tilde{f}_i\colon B &\to B \sqcup \{0\} \end{aligned}$$

for all $i \in I$ which satisfy the following axioms:

- $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$,
- $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \in B$, and $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \in B$,
- for $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$,
- for $b \in B$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$,
- if $b \in B$ and $\tilde{e}_i b \in B$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
- if $b \in B$ and $\tilde{f}_i b \in B$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

A crystal B has an I-colored graph structure as follows:

- Vertices $:= B$
- Arrows : $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i(b)$

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Definition $B_{\mathfrak{g}}(\infty) :=$ the crystal of the half $U_q^-(\mathfrak{g})$

Properties

(i) $B_{\mathfrak{g}}(\infty)$ is connected.

(ii) $B_{\mathfrak{g}}(\infty)$ has the highest weight vector $\mathbb{1}$ with weight 0.

(iii) There is an involution $* : B_{\mathfrak{g}}(\infty) \to B_{\mathfrak{g}}(\infty)$, which provides another crystal structure with $\tilde{e}_i^*$, $\tilde{f}_i^*$, ε_i^*, φ_i^*.

(iv) $\tilde{e}_i$ is locally nilpotent on $B_{\mathfrak{g}}(\infty)$.

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Definition

$$\widehat{B}_{\mathfrak{g}}(\infty) := \left\{ (b_k)_{k\in\mathbb{Z}} \in \prod_{k\in\mathbb{Z}} B_{\mathfrak{g}}(\infty) \;\middle|\; b_k = 1 \text{ for all but finitely many } k \right\},$$

and set

$$\mathbf{1} := (1)_{k\in\mathbb{Z}} \in \widehat{B}_{\mathfrak{g}}(\infty).$$

Extended crystal structure

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$$\mathrm{wt}_k(\mathbf{b}) := (-1)^k \,\mathrm{wt}(b_k), \quad \varepsilon_{(i,k)}(\mathbf{b}) := \varepsilon_i(b_k), \quad \varepsilon^*_{(i,k)}(\mathbf{b}) := \varepsilon^*_i(b_k).$$

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- We define

$$\widehat{\mathrm{wt}}(\mathbf{b}) := \sum_{k\in\mathbb{Z}} \mathrm{wt}_k(\mathbf{b}), \qquad \widehat{\varepsilon}_{(i,k)}(\mathbf{b}) := \varepsilon_{(i,k)}(\mathbf{b}) - \varepsilon^*_{(i,k+1)}(\mathbf{b}),$$

which gives the maps

$$\widehat{\mathrm{wt}} \colon \widehat{B}_{\mathfrak{g}}(\infty) \longrightarrow \mathsf{P}, \qquad \widehat{\varepsilon}_{(i,k)} \colon \widehat{B}_{\mathfrak{g}}(\infty) \longrightarrow \mathbb{Z}$$

Extended crystal structure (continued)

- For any $(i,k) \in \widehat{I}$, the extended crystal operators

$$\widetilde{F}_{(i,k)} \colon \widehat{B}_{\mathfrak{g}}(\infty) \longrightarrow \widehat{B}_{\mathfrak{g}}(\infty) \quad \text{and} \quad \widetilde{E}_{(i,k)} \colon \widehat{B}_{\mathfrak{g}}(\infty) \longrightarrow \widehat{B}_{\mathfrak{g}}(\infty),$$

are defined by

$$\widetilde{F}_{(i,k)}(\mathbf{b}) := \begin{cases} (\cdots, b_{k+2}, \ b_{k+1}, \ \tilde{f}_i(b_k), \ b_{k-1}, \cdots) & \text{if } \widehat{\varepsilon}_{(i,k)}(\mathbf{b}) \geq 0, \\ (\cdots, b_{k+2}, \ \tilde{e}_i^*(b_{k+1}), \ b_k, \ b_{k-1}, \cdots) & \text{if } \widehat{\varepsilon}_{(i,k)}(\mathbf{b}) < 0, \end{cases}$$

$$\widetilde{E}_{(i,k)}(\mathbf{b}) := \begin{cases} (\cdots, b_{k+2}, \ b_{k+1}, \ \tilde{e}_i(b_k), \ b_{k-1}, \cdots) & \text{if } \widehat{\varepsilon}_{(i,k)}(\mathbf{b}) > 0, \\ (\cdots, b_{k+2}, \ \tilde{f}_i^*(b_{k+1}), \ b_k, \ b_{k-1}, \cdots) & \text{if } \widehat{\varepsilon}_{(i,k)}(\mathbf{b}) \leq 0, \end{cases}$$

for any $(i,k) \in \widehat{I}$ and $\mathbf{b} = (b_k)_{k\in\mathbb{Z}} \in \widehat{B}_{\mathfrak{g}}(\infty)$.

Proposition ([Kashiwara-P.]) Let $\mathbf{b} \in \widehat{B}_{\mathfrak{g}}(\infty)$ and $(i,k) \in \widehat{I}$.

(i) $\widehat{\mathrm{wt}}(\widetilde{F}_{i,k}(\mathbf{b})) = \widehat{\mathrm{wt}}(\mathbf{b}) + (-1)^{k+1}\alpha_i$

(ii) $\widehat{\mathrm{wt}}(\widetilde{E}_{i,k}(\mathbf{b})) = \widehat{\mathrm{wt}}(\mathbf{b}) + (-1)^{k}\alpha_i$.

(iii) $\widehat{\varepsilon}_{i,k}(\widetilde{F}_{i,k}(\mathbf{b})) = \widehat{\varepsilon}_{i,k}(\mathbf{b}) + 1$

(iv) $\widehat{\varepsilon}_{i,k}(\widetilde{E}_{i,k}(\mathbf{b})) = \widehat{\varepsilon}_{i,k}(\mathbf{b}) - 1$.

(v) $\widetilde{F}_{i,k}$ and $\widetilde{E}_{i,k}$ are inverse to each other.

(vi) $\widetilde{E}_{i,k}(\mathbf{b})$ is non-zero for any $\mathbf{b} \in \widehat{B}_{\mathfrak{g}}(\infty)$.

Extended crystal graph

As a usual crystal, the set $\widehat{B}_{\mathfrak{g}}(\infty)$ has the $\widehat{I}$-colored graph structure induced by the operators $\widetilde{F}_{i,k}$ for $(i,k) \in \widehat{I}$.

(a) vertices $:= \widehat{B}_{\mathfrak{g}}(\infty)$

(b) arrows: $\mathbf{b} \xrightarrow{(i,k)} \mathbf{b}'$ if and only if $\mathbf{b}' = \widetilde{F}_{i,k}\mathbf{b}$ $\quad ((i,k) \in \widehat{I})$

We call $\widehat{B}_{\mathfrak{g}}(\infty)$ the extended crystal of $B_{\mathfrak{g}}(\infty)$.

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Proposition ([Kashiwara-P.])

(i) For $k \in \mathbb{Z}$, $\exists$ injection $\iota_k : B_{\mathfrak{g}}(\infty) \rightarrowtail \widehat{B}_{\mathfrak{g}}(\infty)$ s. t., for $b \in B_{\mathfrak{g}}(\infty)$,

$$\widetilde{F}_{i,k}(\iota_k(b)) = \iota_k(\tilde{f}_i(b)),$$
$$\widetilde{E}_{i,k}(\iota_k(b)) = \iota_k(\tilde{e}_i(b)) \quad \text{if } \tilde{e}_i(b) \neq 0.$$

(ii) As an $\widehat{I}$-colored graph, $\widehat{B}_{\mathfrak{g}}(\infty)$ is connected.

Note $\quad$ **1** plays the role of a highest weight vector of $\widehat{B}_{\mathfrak{g}}(\infty)$.

- **Bijection** D^t

  For $t \in \mathbb{Z}$ and $\mathbf{b} = (b_k)_{k\in\mathbb{Z}} \in \widehat{B}_{\mathfrak{g}}(\infty)$, define $\mathrm{D}^t(\mathbf{b}) = (b'_k)_{k\in\mathbb{Z}}$ by

$$b'_k = b_{k-t} \qquad \text{for any } k \in \mathbb{Z}.$$

  This gives a bijection

$$\mathrm{D}^t \colon \widehat{B}_{\mathfrak{g}}(\infty) \longrightarrow \widehat{B}_{\mathfrak{g}}(\infty).$$

Proposition For any $(i,k) \in \widehat{I}$,

$$\mathrm{D}^t(\widetilde{F}_{i,k}(\mathbf{b})) = \widetilde{F}_{i,k+t}(\mathrm{D}^t(\mathbf{b})).$$

Example ($\mathfrak{sl}_2$ case)

Let $I = \{1\}$ and let $B(\infty)$ be the crystal of $U_q^-(\mathfrak{sl}_2)$.
We identify $B(\infty)$ with $\mathbb{Z}_{\geq 0}$, and simply write $\widetilde{F}_k$ instead of $\widetilde{F}_{1,k}$.

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Then we have the following.

- $\widehat{B}(\infty)$ is equal to $(\mathbb{Z}_{\geq 0})^{\oplus \mathbb{Z}}$,
- For any $k \in \mathbb{Z}$ and $\mathbf{b} = (b_k)_{k\in\mathbb{Z}} \in \widehat{B}(\infty)$, we have

$$\widetilde{F}_k(\mathbf{b}) = \begin{cases} (\cdots, b_{k+2},\ b_{k+1},\ b_k + 1,\ b_{k-1}, \cdots) & \text{if } b_{k+1} \leq b_k, \\ (\cdots, b_{k+2},\ b_{k+1} - 1,\ b_k,\ b_{k-1}, \cdots) & \text{if } b_{k+1} > b_k. \end{cases}$$

- If $\mathbf{b} = (\cdots, 0, 0, 0, 1, 2, \underline{3}, 4, 5, 0, \cdots)$ (here, _ denotes the 0-position), then

$$\mathrm{D}^2(\mathbf{b}) = (\cdots, 0, 1, 2, 3, 4, \underline{5}, 0, 0, 0 \cdots).$$

Categorical crystal for HL category

Let

- $\mathsf{A} = (a_{i,j})_{i,j \in I}$ = affine Cartan matrix
- $U_q(\mathfrak{g})$ = quantum affine alg associated with A (q: indeterminate)
- $U_q'(\mathfrak{g})$ = the subalgebra of $U_q(\mathfrak{g})$ generated by

$$e_i, f_i, K_i := q_i^{\pm h_i} \ (i \in I), \text{ where } q_i := q^{(\alpha_i,\alpha_i)/2}.$$

- $\mathscr{C}_\mathfrak{g}$:= the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules

  Notation $\otimes$: tensor, $\mathrm{hd}(M \otimes N)$= the head of $M \otimes N$

  $\mathscr{D}$:= right dual functor, $\mathscr{D}^{-1}$:= left dual functor
- $\mathscr{C}^0_\mathfrak{g}$:= Hernandez-Leclerc category of $\mathscr{C}_\mathfrak{g}$

Definition

- $\mathcal{D} := \{L_i\}_{i \in I_{\mathrm{fin}}}$ a complete duality datum for $\mathscr{C}^0_{\mathfrak{g}}$
  ($\Leftrightarrow$ L_i are simple modules with certain conditions.)
- $\mathscr{C}_{\mathcal{D}}$:= the smallest full subcategory of $\mathscr{C}^0_{\mathfrak{g}}$ such that
  (a) it contains $\mathcal{F}_{\mathcal{D}}(L)$ for any simple $R_{\mathfrak{g}_{\mathrm{fin}}}$-module L,
  (b) it is stable by taking subquotients, extensions, and tensor products.

$$\begin{array}{ccccc}
 & R_{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} & \xrightarrow{\ \mathcal{F}_{\mathcal{D}}\ } & \mathscr{C}_{\mathcal{D}} \subset \mathscr{C}^0_{\mathfrak{g}} \\
 & \uparrow & & \uparrow \\
B_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \simeq & \{\text{simple } R_{\mathfrak{g}_{\mathrm{fin}}}\text{-modules}\} & \xleftrightarrow{\ 1-1\ } & \{\text{simple modules in } \mathscr{C}_{\mathcal{D}}\} \\
 & b & \longmapsto & \mathcal{L}_{\mathcal{D}}(b)
\end{array}$$

Type of $\mathfrak{g}$	$A_n^{(1)}$ $(n \geq 1)$	$B_n^{(1)}$ $(n \geq 2)$	$C_n^{(1)}$ $(n \geq 3)$	$D_n^{(1)}$ $(n \geq 4)$	$A_{2n}^{(2)}$ $(n \geq 1)$	$A_{2n-1}^{(2)}$ $(n \geq 2)$	$D_{n+1}^{(2)}$ $(n \geq 3)$
Type of $\mathfrak{g}_{\mathrm{fin}}$	A_n	A_{2n-1}	D_{n+1}	D_n	A_{2n}	A_{2n-1}	D_{n+1}

Type of $\mathfrak{g}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$E_6^{(2)}$	$D_4^{(3)}$
Type of $\mathfrak{g}_{\mathrm{fin}}$	E_6	E_7	E_8	E_6	D_4	E_6	D_4

Let

- $\widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty) :=$ the extended crystal of $B_{\mathfrak{g}_{\mathrm{fin}}}(\infty)$.
- For $\mathbf{b} = (b_k)_{k\in\mathbb{Z}} \in \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty)$, define

$$\mathscr{L}_{\mathcal{D}}(\mathbf{b}) := \mathrm{hd}(\cdots \otimes \mathscr{D}^2 M_2 \otimes \mathscr{D} M_1 \otimes M_0 \otimes \mathscr{D}^{-1} M_{-1} \otimes \cdots) \in \mathscr{C}^0_{\mathfrak{g}},$$

where $M_k := \mathcal{L}_{\mathcal{D}}(b_k)$ for $k \in \mathbb{Z}$.

Theorem ([Kashiwara-P.])

(i) Let $\mathcal{B}(\mathfrak{g}) :=$ the set of simple modules in $\mathscr{C}^0_{\mathfrak{g}}$.

$$\Phi_{\mathcal{D}} \colon \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \xrightarrow{1-1} \mathcal{B}(\mathfrak{g}), \qquad \mathbf{b} \mapsto \mathscr{L}_{\mathcal{D}}(\mathbf{b}).$$

($\Rightarrow$ new parametrization of $\mathcal{B}(\mathfrak{g})$!)

(ii) For any $t \in \mathbb{Z}$ and $\mathbf{b} \in \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty)$, we have

$$\Phi_{\mathcal{D}}(\mathrm{D}^t(\mathbf{b})) = \mathscr{D}^t(\Phi_{\mathcal{D}}(\mathbf{b})),$$

Theorem ([Kashiwara-P.]) (continued)

(iii) For $M \in \mathcal{B}(\mathfrak{g})$ and $(i,k) \in \widehat{I}_{\mathrm{fin}}$, define

$$\widetilde{\mathcal{F}}_{i,k}(M) := \mathrm{hd}((\mathscr{D}^k \mathsf{L}_i) \otimes M) \quad \text{and} \quad \widetilde{\mathcal{E}}_{i,k}(M) := \mathrm{hd}(M \otimes (\mathscr{D}^{k+1} \mathsf{L}_i)).$$

Then, for $(i,k) \in \widehat{I}_{\mathrm{fin}}$ and $\mathbf{b} \in \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty)$, we have

$$\Phi_{\mathcal{D}}(\widetilde{F}_{i,k}(\mathbf{b})) = \widetilde{\mathcal{F}}_{i,k}(\Phi_{\mathcal{D}}(\mathbf{b})), \qquad \Phi_{\mathcal{D}}(\widetilde{E}_{i,k}(\mathbf{b})) = \widetilde{\mathcal{E}}_{i,k}(\Phi_{\mathcal{D}}(\mathbf{b})).$$

(v) $\mathcal{B}(\mathfrak{g})$ has an extended crystal structure and

$$\Phi_{\mathcal{D}} : \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \simeq \mathcal{B}(\mathfrak{g}) \qquad \text{as an extended crystal.}$$

This is denoted by $\mathcal{B}_{\mathcal{D}}(\mathfrak{g})$.

(vi) $\mathcal{B}_{\mathcal{D}}(\mathfrak{g})$ has an $\widehat{I}_{\mathrm{fin}}$-colored graph structure ($\simeq \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty)$).

Braid group action on $\widehat{B}(\infty)$

Question

Can we extend the Saito crystal reflections on $B(\infty)$ to the extended crystal $\widehat{B}(\infty)$?

Let A be of finite type. For any $i \in I$, we set

$$
\begin{aligned}
{}_iB(\infty) &:= \{b \in B(\infty) \mid \varepsilon_i(b) = 0\}, \\
B_i(\infty) &:= \{b \in B(\infty) \mid \varepsilon_i^*(b) = 0\}.
\end{aligned}
$$

The Saito crystal reflections on the crystal $B(\infty)$ are defined as follows

$$
\begin{aligned}
\mathrm{T}_i : {}_iB(\infty) &\to B_i(\infty), & \mathrm{T}_i(b) &:= \tilde{f}_i^{\varphi_i^*(b)} \tilde{e}_i^{*\,\varepsilon_i^*(b)}(b), \\
\mathrm{T}_i^* : B_i(\infty) &\to {}_iB(\infty), & \mathrm{T}_i^*(b) &:= \tilde{f}_i^{*\,\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)}(b).
\end{aligned}
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$$ \mathrm{T}_i^* : B_i(\infty) \to {}_iB(\infty), \qquad \mathrm{T}_i^*(b) := \tilde{f}_i^{*\,\varphi_i(b)} \tilde{e}_i^{\varepsilon_i(b)}(b). $$

Let

$$ {}_i\pi(b) := \tilde{e}_i^{\max}(b) \quad \text{and} \quad \pi_i(b) := \tilde{e}_i^{*\max}(b) \qquad \text{for } b \in B(\infty), $$

and we set

$$ \widetilde{\mathrm{T}}_i := \mathrm{T}_i \circ {}_i\pi : B(\infty) \longrightarrow B_i(\infty) \subset B(\infty), $$
$$ \widetilde{\mathrm{T}}_i^* := \mathrm{T}_i^* \circ \pi_i : B(\infty) \longrightarrow {}_iB(\infty) \subset B(\infty). $$

Definition (R_i and R_i^*)

For $i \in I$ and $\mathbf{b} = (b_k)_{k\in\mathbb{Z}} \in \widehat{B}(\infty)$, we define

$$\mathrm{R}_i(\mathbf{b}) = (b'_k)_{k\in\mathbb{Z}} \qquad \text{and} \qquad \mathrm{R}_i^*(\mathbf{b}) = (b''_k)_{k\in\mathbb{Z}}$$

by

$$b'_k := \tilde{f}_i^{*\varepsilon_i(b_{k-1})}\left(\widetilde{\mathrm{T}}_i(b_k)\right) \qquad \text{and} \qquad b''_k := \tilde{f}_i^{\varepsilon_i^*(b_{k+1})}\left(\widetilde{\mathrm{T}}_i^*(b_k)\right)$$

for any $k \in \mathbb{Z}$ respectively. Thus we have the maps

$$\mathrm{R}_i : \widehat{B}(\infty) \longrightarrow \widehat{B}(\infty) \qquad \text{and} \qquad \mathrm{R}_i^* : \widehat{B}(\infty) \longrightarrow \widehat{B}(\infty).$$

Theorem (P.) Let $i \in I$.

- R_i and R_i^* are bijective.
- R_i and R_i^* are inverse to each other.
- $\mathrm{R}_i \circ \mathrm{D} = \mathrm{D} \circ \mathrm{R}_i$ and $\mathrm{R}_i^* \circ \mathrm{D} = \mathrm{D} \circ \mathrm{R}_i^*$.
- For any $\mathbf{b} \in \widehat{B}(\infty)$, we have

$$\widehat{\mathrm{wt}}(\mathrm{R}_i(\mathbf{b})) = s_i(\widehat{\mathrm{wt}}(\mathbf{b})) \quad \text{and} \quad \widehat{\mathrm{wt}}(\mathrm{R}_i^*(\mathbf{b})) = s_i(\widehat{\mathrm{wt}}(\mathbf{b})).$$

Recall that $\mathsf{A} = (a_{i,j})_{i,j \in I}$ is a Cartan matrix of finite type. For $i, j \in I$ with $i \neq j$, we set

$$m(i,j) := \begin{cases} 2 & \text{if } a_{i,j}a_{j,i} = 0, \\ 3 & \text{if } a_{i,j}a_{j,i} = 1, \\ 4 & \text{if } a_{i,j}a_{j,i} = 2, \\ 6 & \text{if } a_{i,j}a_{j,i} = 3. \end{cases}$$

We denote by $\mathfrak{B}$ the generalized braid group (or Artin-Tits group) defined by the generators r_i ($i \in I$) and the following defining relations:

$$\underbrace{r_i r_j r_i r_j \cdots}_{m(i,j) \text{ factors}} = \underbrace{r_j r_i r_j r_i \cdots}_{m(j,i) \text{ factors}} \qquad \text{for } i, j \in I \text{ with } i \neq j.$$

Theorem (P.) The bijections R_i and R_i^* satisfy the braid group relations for $\mathfrak{B}$, i.e., for $i, j \in I$ with $i \neq j$,

$$\underbrace{\mathrm{R}_i\mathrm{R}_j\mathrm{R}_i\mathrm{R}_j\cdots}_{m(i,j)\text{ factors}} = \underbrace{\mathrm{R}_j\mathrm{R}_i\mathrm{R}_j\mathrm{R}_i\cdots}_{m(j,i)\text{ factors}} \quad \text{and} \quad \underbrace{\mathrm{R}_i^*\mathrm{R}_j^*\mathrm{R}_i^*\mathrm{R}_j^*\cdots}_{m(i,j)\text{ factors}} = \underbrace{\mathrm{R}_j^*\mathrm{R}_i^*\mathrm{R}_j^*\mathrm{R}_i^*\cdots}_{m(j,i)\text{ factors}}.$$

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(Idea of Proof) PBW crystals.

The braid group $\mathfrak{B}$ acts on the extended crystal $\widehat{B}(\infty)$ as follows:

$$r_i \cdot \mathbf{b} := \mathrm{R}_i(b) \quad \text{and} \quad r_i^{-1} \cdot \mathbf{b} := \mathrm{R}_i^*(b) \qquad \text{for } i \in I \text{ and } \mathbf{b} \in \widehat{B}(\infty).$$

$\rightsquigarrow$ **$\widehat{B}(\infty)$ is a $\mathfrak{B}$-set!**

Note As a $\mathfrak{B}$-set, $\widehat{B}(\infty)$ is not transitive.
Question As a $\mathfrak{B}$-set, is $\widehat{B}(\infty)$ faithful?

Properties

- For any $i \in I$, we have $\mathrm{R}_i \circ \zeta = \zeta \circ \mathrm{R}_{\zeta(i)}$ and $\mathrm{R}^*_i \circ \zeta = \zeta \circ \mathrm{R}^*_{\zeta(i)}$, where $\zeta : i \mapsto i^*$. (Here $\alpha_{i^*} = -w_0 \alpha_i$)

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- For any reduced expression $w_0 = s_{i_1} \dots s_{i_\ell}$, we have

$$\mathrm{R}_{i_1} \cdots \mathrm{R}_{i_\ell} = \mathrm{D} \circ \zeta \qquad \text{and} \qquad \mathrm{R}_{i_1}^* \cdots \mathrm{R}_{i_\ell}^* \mathbf{i} = \mathrm{D}^{-1} \circ \zeta.$$

  In particular, unless the Cartan matrix A is of type A_n ($n \in \mathbb{Z}_{>1}$), D_n (n is odd) or E_6, we have

$$\mathrm{R}_{i_1} \cdots \mathrm{R}_{i_\ell} = \mathrm{D} \qquad \text{and} \qquad \mathrm{R}_{i_1}^* \cdots \mathrm{R}_{i_\ell}^* = \mathrm{D}^{-1}.$$

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- Similarity for the extended crystal $\widehat{B}(\infty)$ exists.

Connection to HL category

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Connection to HL category

R_i is a crystal-theoretic shadow of the conjectural autofunctor [KKOP] on the Hernandez-Leclerc category $\mathscr{C}_{\mathfrak{g}}^0$, i.e.,

we assume that the conjectural autofunctors $\mathscr{R}_i$ ($i \in I_{\mathrm{fin}}$) exist in $\mathscr{C}_{\mathfrak{g}}^0$. Then we have

$$\Phi_{\mathcal{D}}(\mathrm{R}_i(\mathbf{b})) = \mathscr{R}_i(\Phi_{\mathcal{D}}(\mathbf{b})) \qquad \text{for any } \mathbf{b} \in \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \text{ and } i \in I_{\mathrm{fin}},$$

where $\Phi_{\mathcal{D}} \colon \widehat{B}_{\mathfrak{g}_{\mathrm{fin}}}(\infty) \xrightarrow{\sim} \mathcal{B}_{\mathcal{D}}(\mathfrak{g})$ is the extended crystal isomorphism.

THANK YOU