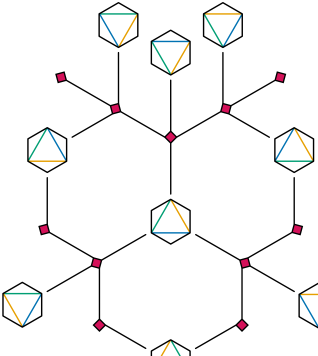
PAINLEVÉ VI, SYMMETRIES, AND CLUSTERS

Spring School on Cluster Algebras 2025

Davide DAL MARTELLO
JSPS Postdoctoral Fellow @ Rikkyo University

DM, **Okamoto's symmetry on the representation space of the sixth Painlevé equation**, *arXiv:2411.17397 (2024)*.



Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

$$\theta_0 := -(\theta_1 + \theta_2 + \theta_3 + \theta_\infty),$$

$$w_k(\theta_i) = \theta_i - \theta_k c_{ki}, \quad (c_{ij}) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

$$\theta_0 := -(\theta_1 + \theta_2 + \theta_3 + \theta_\infty),$$

$$w_k(\theta_i) = \theta_i - \theta_k c_{ki}, \quad (c_{ij}) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$\mathcal{F}(\theta) = \left\{ \mathrm{d} - \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \mathrm{d}\lambda \,\middle|\, A_1 + A_2 + A_3 + A_\infty = \mathbb{0},\ \mathrm{eig}(A_k) = \left\{ \pm \tfrac{\theta_k}{2} \right\} \right\}$$

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

$$\theta_0 := -(\theta_1 + \theta_2 + \theta_3 + \theta_\infty),$$

$$w_k(\theta_i) = \theta_i - \theta_k c_{ki}, \quad (c_{ij}) = \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$\mathcal{F}(\theta) = \left\{ \mathrm{d} - \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \mathrm{d}\lambda \,\middle|\, A_1 + A_2 + A_3 + A_\infty = \mathbb{0},\ \mathrm{eig}(A_k) = \left\{ \pm \tfrac{\theta_k}{2} \right\} \right\}$$

$$P_{VI}(\theta) \quad \longleftrightarrow \quad (A_1, A_2, A_3)(\theta)$$

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

Fact #2 [Inaba-Iwasaki-Saito'04]

All symmetries s_k read as the identity via the $\mathbb{RH}$ map

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

Fact #2 [Inaba-Iwasaki-Saito'04]

All symmetries s_k read as the identity via the $\mathbb{RH}$ map

$$\mathcal{F}(\theta) = \left\{ \mathrm{d} - \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \mathrm{d}\lambda \,\middle|\, A_1 + A_2 + A_3 + A_\infty = \mathbb{0},\ \mathrm{eig}(A_k) = \left\{ \pm \tfrac{\theta_k}{2} \right\} \right\}$$

$$\begin{array}{ccc}
\mathcal{F}(\theta)/\Gamma = \mathcal{A}(\theta) & \xrightarrow{\ s\ } & \mathcal{A}(w(\theta)) \\
\Big\downarrow \mathbb{RH} & & \Big\downarrow \mathbb{RH} \\
\mathcal{R}(\iota)/GL_2(\mathbb{C}) = \mathcal{M}(\iota) & \xrightarrow{\ ???\ } & \mathcal{M}(w(\iota))
\end{array}$$

$$\mathcal{R}(\iota) = \left\{ (M_1, M_2, M_3, M_\infty) \,\middle|\, M_1 M_2 M_3 M_\infty = \mathbb{1},\ \mathrm{eig}(M_k) = \{\iota_k^{\pm 1}\} \right\}$$

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

Fact #2 [Inaba-Iwasaki-Saito'04]

All symmetries s_k read as the identity via the RH map

$$\mathcal{F}(\theta) = \left\{ \mathrm{d} - \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \mathrm{d}\lambda \,\middle|\, A_1 + A_2 + A_3 + A_\infty = \mathbb{0},\ \mathrm{eig}(A_k) = \left\{ \pm \tfrac{\theta_k}{2} \right\} \right\}$$

$$\begin{array}{ccc} \mathcal{F}(\theta)/\Gamma = \mathcal{A}(\theta) & \xrightarrow{\ s\ } & \mathcal{A}(w(\theta)) \\ \Big\downarrow \mathrm{RH} & & \Big\downarrow \mathrm{RH} \\ \mathcal{R}(\iota)/GL_2(\mathbb{C}) = \mathcal{M}(\iota) & \xrightarrow{\ ???\ } & \mathcal{M}(w(\iota)) \end{array}$$

$$\mathcal{R}(\iota) = \left\{ (M_1, M_2, M_3, M_\infty) \,\middle|\, M_1 M_2 M_3 M_\infty = \mathbb{1},\ \mathrm{eig}(M_k) = \{\iota_k^{\pm 1}\} \right\}$$

For $\omega_i = \omega_i(a)$ depending on local $a_k = \mathrm{tr}(M_k) = \iota_k + \iota_k^{-1} = e^{\pi i \theta_k} + e^{-\pi i \theta_k}$, $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_i = \mathrm{tr}(M_j M_k),\ x_1 x_2 x_3 + \sum_i (x_i^2 - \omega_i x_i) - \omega_4 = 0\}$ coordinatizes $\mathcal{M}$ and $w_0(a) \neq a$ but $w_0(\omega) = \omega$, $w_0(x) = x$.

Fact #1 [Okamoto'87]

$P_{VI}(\theta)$ admits a $W(\tilde{D}_4)$-type symmetry group (s_k lifts w_k for all $k = 0, 1, 2, 3, 4$)

Fact #2 [Inaba-Iwasaki-Saito'04]

All symmetries s_k read as the identity via the $\mathbb{RH}$ map

$$\mathcal{F}(\theta) = \left\{ \mathrm{d} - \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - 1} + \frac{A_3}{\lambda - t} \right) \mathrm{d}\lambda \,\middle|\, A_1 + A_2 + A_3 + A_\infty = \mathbb{0},\ \mathrm{eig}(A_k) = \left\{ \pm \tfrac{\theta_k}{2} \right\} \right\}$$

$$\begin{array}{ccc} \mathcal{F}(\theta)/\Gamma = \mathcal{A}(\theta) & \xrightarrow{\ s\ } & \mathcal{A}(w(\theta)) \\ \Big\downarrow \mathbb{RH} & & \Big\downarrow \mathbb{RH} \\ \mathcal{R}(\iota)/GL_2(\mathbb{C}) = \mathcal{M}(\omega) & \xrightarrow{\ \mathrm{id}\ } & \mathcal{M}(\omega) \end{array}$$

$$\mathcal{R}(\iota) = \left\{ (M_1, M_2, M_3, M_\infty) \,\middle|\, M_1 M_2 M_3 M_\infty = \mathbb{1},\ \mathrm{eig}(M_k) = \{\iota_k^{\pm 1}\} \right\}$$

For $\omega_i = \omega_i(a)$ depending on local $a_k = \mathrm{tr}(M_k) = \iota_k + \iota_k^{-1} = e^{\pi i \theta_k} + e^{-\pi i \theta_k}$, $\left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 \,\middle|\, x_i = \mathrm{tr}(M_j M_k),\ x_1 x_2 x_3 + \sum_i (x_i^2 - \omega_i x_i) - \omega_4 = 0 \right\}$ coordinatizes $\mathcal{M}$ and $w_0(a) \neq a$ but $w_0(\omega) = \omega$, $w_0(x) = x$.

"Lift the diagram to the connection and representation spaces"

$$\begin{array}{ccc} \mathcal{F}(\theta) & \xrightarrow{?} & \mathcal{F}(w_0(\theta)) \\ {\scriptstyle \mathrm{RH}}\downarrow & & \downarrow{\scriptstyle \mathrm{RH}} \\ \mathcal{R}(\iota) & \xrightarrow{?} & \mathcal{R}(w_0(\iota)) \end{array}$$

"Lift the diagram to the connection and representation spaces"

$$\begin{array}{ccc} \mathcal{F}(\theta) & \xrightarrow{\ ?\ } & \mathcal{F}(w_0(\theta)) \\ \Big\downarrow \text{RH} & & \Big\downarrow \text{RH} \\ \mathcal{R}(\iota) & \xrightarrow{\ ?\ } & \mathcal{R}(w_0(\iota)) \end{array}$$

PROBLEM — w_0

"Lift the diagram to the connection and representation spaces"

$$\begin{array}{ccc} \mathcal{F}(\theta) & \xrightarrow{?} & \mathcal{F}(w_0(\theta)) \\ \Big\downarrow{\scriptstyle \mathrm{RH}} & & \Big\downarrow{\scriptstyle \mathrm{RH}} \\ \mathcal{R}(\iota) & \xrightarrow{?} & \mathcal{R}(w_0(\iota)) \end{array}$$

Fact #3 [DM-Mazzocco'24]

$\mathcal{R}(\iota)$ admits the higher Teichmüller coordinatization

$$M_1 = \begin{pmatrix} 0 & \iota_1^{-1} Z_1^{-1} \\ -\iota_1 Z_1 & \iota_1 + \iota_1^{-1} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \iota_2 + \iota_2^{-1} + \iota_2^{-1} Z_2^{-1} & \iota_2 + \iota_2^{-1} + \iota_2^{-1} Z_2^{-1} + \iota_2 Z_2 \\ -\iota_2^{-1} Z_2^{-1} & -\iota_2^{-1} Z_2^{-1} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \iota_3 + \iota_3^{-1} + \iota_3 Z_3 & \iota_3 Z_3 \\ -\iota_3 - \iota_3^{-1} - \iota_3^{-1} Z_3^{-1} - \iota_3 Z_3 & -\iota_3 Z_3 \end{pmatrix},$$

over the A_3-type cluster Poisson algebra with $\mathcal{X}$-coordinate set $\{Z_1, Z_2, Z_3\}$.

Middle convolution MC_ν

For $\nu \in \mathbb{C}\backslash\{0,1\}$,

$$GL(V)^p \ni \mathbf{M} = (M_1, \ldots, M_p) \xrightarrow{\oplus} (N_1, \ldots, N_p) \xrightarrow{/} (\tilde{N}_1, \ldots, \tilde{N}_p) = \mathbf{N} \in GL(V^p/U)^p$$

- $$N_i = \begin{pmatrix} \mathbb{1} & \mathbb{0} & & \ldots & & & \\ \mathbb{0} & \ddots & & & & & \\ \vdots & & \mathbb{1} & & & & \\ \nu(M_1 - \mathbb{1}) & \ldots & \nu(M_{i-1} - \mathbb{1}) & \nu M_i & M_{i+1} - \mathbb{1} & \ldots & M_p - \mathbb{1} \\ & & & & \mathbb{1} & & \vdots \\ & & & & & \ddots & \mathbb{0} \\ & & & \ldots & & \mathbb{0} & \mathbb{1} \end{pmatrix}$$
- $\dim(U) = p + 1$ maximal at $\nu \in \mathrm{eig}(M_\infty)$

Middle convolution MC_ν

For $\nu \in \mathbb{C}\backslash\{0,1\}$,

$$GL(V)^p \ni \mathbf{M} = (M_1,\dots,M_p) \xrightarrow{\oplus} (N_1,\dots,N_p) \xrightarrow{/} (\tilde{N}_1,\dots,\tilde{N}_p) = \mathbf{N} \in GL(V^p/U)^p$$

- $$N_i = \begin{pmatrix} \mathbb{1} & \mathbb{0} & & \dots & & & \\ \mathbb{0} & \ddots & & & & & \\ \vdots & & \mathbb{1} & & & & \\ \nu(M_1-\mathbb{1}) & \dots & \nu(M_{i-1}-\mathbb{1}) & \nu M_i & M_{i+1}-\mathbb{1} & \dots & M_p-\mathbb{1} \\ & & & & \mathbb{1} & & \vdots \\ & & & & & \ddots & \mathbb{0} \\ & & & \dots & & \mathbb{0} & \mathbb{1} \end{pmatrix}$$
- $\dim(U) = p+1$ maximal at $\nu \in \operatorname{eig}(M_\infty)$

$P_{VI} \implies \dim(V) = 2,\ p = 3$ so that $\bar{\nu} \in \operatorname{eig}(M_\infty) \implies \dim(V^3/U) = 2.$

$$\tilde{\mathbf{M}} := (\tilde{M}_1, \tilde{M}_2, \tilde{M}_3) = MC_{\bar{\nu}}(\mathbf{M}(Z)) \implies \begin{cases} \operatorname{eig}(\tilde{M}_k) = w_0(\iota_k^{\pm 1}) \\ \operatorname{tr}(\tilde{M}_j \tilde{M}_k) = x_i. \end{cases}$$

Middle convolution MC_ν

For $\nu \in \mathbb{C}\backslash\{0,1\}$,

$$GL(V)^p \ni \mathbf{M} = (M_1,\dots,M_p) \xrightarrow{\oplus} (N_1,\dots,N_p) \xrightarrow{/} (\tilde{N}_1,\dots,\tilde{N}_p) = \mathbf{N} \in GL(V^p/U)^p$$

- $$N_i = \begin{pmatrix} \mathbb{1} & \mathbb{0} & & \dots & & & \\ \mathbb{0} & \ddots & & & & & \\ \vdots & & \mathbb{1} & & & & \\ \nu(M_1-\mathbb{1}) & \dots & \nu(M_{i-1}-\mathbb{1}) & \nu M_i & M_{i+1}-\mathbb{1} & \dots & M_p-\mathbb{1} \\ & & & & \mathbb{1} & & \vdots \\ & & & & & \ddots & \mathbb{0} \\ & & & \dots & & \mathbb{0} & \mathbb{1} \end{pmatrix}$$
- $\dim(U) = p+1$ maximal at $\nu \in \mathrm{eig}(M_\infty)$

$P_{VI} \implies \dim(V)=2,\ p=3$ so that $\bar{\nu} \in \mathrm{eig}(M_\infty) \implies \dim(V^3/U)=2.$

$$\tilde{\mathbf{M}} := (\tilde{M}_1,\tilde{M}_2,\tilde{M}_3) = MC_{\bar{\nu}}(\mathbf{M}(Z)) \implies \begin{cases} \mathrm{eig}(\tilde{M}_k) = w_0(\iota_k^{\pm 1}) \\ \mathrm{tr}(\tilde{M}_j\tilde{M}_k) = x_i. \end{cases}$$

$$\mathcal{R}(\iota) \xrightarrow{MC_{\bar{\nu}}} \mathcal{R}(w_0(\iota))$$

- $\tilde{\mathrm{M}}$ is highly non-canonical. Nevertheless, $\exists!$ basis completion such that

$$\begin{cases} \iota_k \longmapsto \tilde{\iota}_k, \\ Z_i \longmapsto \tilde{Z}_i := \dfrac{1 + Z_i + Z_i Z_{i+1}}{Z_{i+1}(1 + Z_{i+1} + Z_i Z_{i+2})}, \quad i \in \mathbb{Z}_3 \end{cases}$$

matches, as entry-wise action, $MC_{\bar{\nu}}$.

THE "CANONICAL" FORM

- $\tilde{\mathsf{M}}$ is highly non-canonical. Nevertheless, $\exists!$ basis completion such that

$$\begin{cases} \iota_k \longmapsto \tilde{\iota}_k, \\ Z_i \longmapsto \tilde{Z}_i := \dfrac{1 + Z_i + Z_i Z_{i+1}}{Z_{i+1}(1 + Z_{i+1} + Z_i Z_{i+2})}, \quad i \in \mathbb{Z}_3 \end{cases}$$

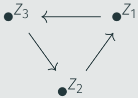
  matches, as entry-wise action, $MC_{\bar{\nu}}$.

- Moreover, $\tilde{Z}_i = \mu_{w_0}(Z_i)$ for the sequence of mutations

$$\mu_{w_0} := (23)\mu_1\mu_3\mu_2\mu_1 = (\mu_2\mu_3\mu_2\mu_3\mu_2)\mu_1\mu_3\mu_2\mu_1$$

$\mathcal{X}$-mutations

$$\mu_k^* Z_i' = \begin{cases} Z_k^{-1} & i = k, \\ Z_i \left(1 + Z_k^{-\operatorname{sgn}(\epsilon_{ik})}\right)^{-\epsilon_{ik}} & i \neq k, \end{cases}$$



CHARACTERIZATION #1

The sequence $\mu_{w_0} = (23)\mu_1\mu_3\mu_2\mu_1$ is singled out by preserving the triangular quiver ∇.

Each q-Painlevé equation arises from a mutation-periodic quiver, with its symmetry group realized in terms of mutations and permutations. Such μ_{w_0} gives the first instance of this phenomenon in the continuous differential world.

The sequence $\mu_{w_0} = (23)\mu_1\mu_3\mu_2\mu_1$ is singled out by preserving the triangular quiver $\triangledown$.

Each q-Painlevé equation arises from a mutation-periodic quiver, with its symmetry group realized in terms of mutations and permutations. Such μ_{w_0} gives the first instance of this phenomenon in the continuous differential world.

Many sequences preserve $\triangledown$—e.g., $(13)\mu_2\mu_1\mu_3\mu_2$.
$\triangledown \in [A_3] \implies$ finite type $\implies$ geometry:

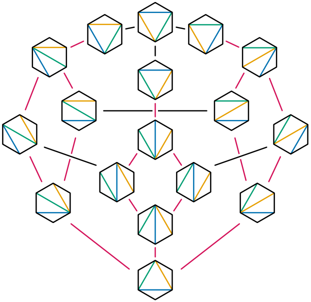
The sequence $\mu_{w_0} = (23)\mu_1\mu_3\mu_2\mu_1$ is singled out by preserving the triangular quiver $\triangledown$.

Each q-Painlevé equation arises from a mutation-periodic quiver, with its symmetry group realized in terms of mutations and permutations. Such μ_{w_0} gives the first instance of this phenomenon in the continuous differential world.

Many sequences preserve $\triangledown$—e.g., $(13)\mu_2\mu_1\mu_3\mu_2$.

$\triangledown \in [A_3] \implies$ finite type $\implies$ geometry:

$$\begin{cases} \{Z_1, Z_2, Z_3\} & \longleftrightarrow \quad \text{colored triangulation of } \{6\} \\ \mu_k & \longleftrightarrow \quad \text{flip of diagonal for } Z_k \end{cases}$$



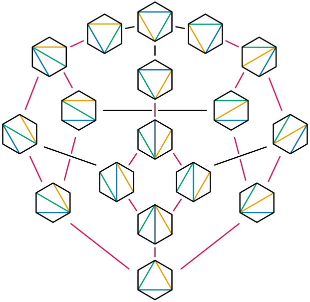
The sequence $\mu_{w_0} = (23)\mu_1\mu_3\mu_2\mu_1$ is singled out by preserving the triangular quiver ∇.

Each q-Painlevé equation arises from a mutation-periodic quiver, with its symmetry group realized in terms of mutations and permutations. Such μ_{w_0} gives the first instance of this phenomenon in the continuous differential world.

Many sequences preserve ∇—e.g., $(13)\mu_2\mu_1\mu_3\mu_2$.
$\nabla \in [A_3] \implies$ finite type $\implies$ geometry:

$$\begin{cases} \{Z_1, Z_2, Z_3\} & \longleftrightarrow \quad \text{colored triangulation of } \{6\} \\ \mu_k & \longleftrightarrow \quad \text{flip of diagonal for } Z_k \end{cases}$$

- The resulting graph matches the skeleton of $\mathbb{A}_3^c$
- Mutation formulae match this geometry



Theorem [DM'24]

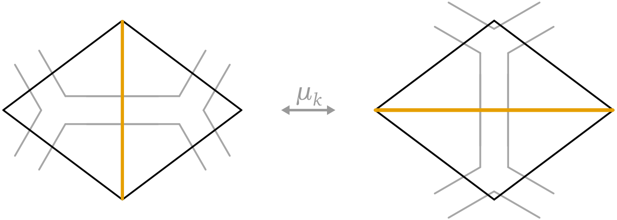
The mutation formula attached to a sequence of flips on $\mathbb{A}_3^c$ is path-independent. In particular, $\{\tilde{Z}\}$ arises as the rotation of π on equilateral triangulations.

In finite type, this result invites to consider colorful generalized associahedra as the natural geometric locus for labeled seeds

CHARACTERIZATION #2

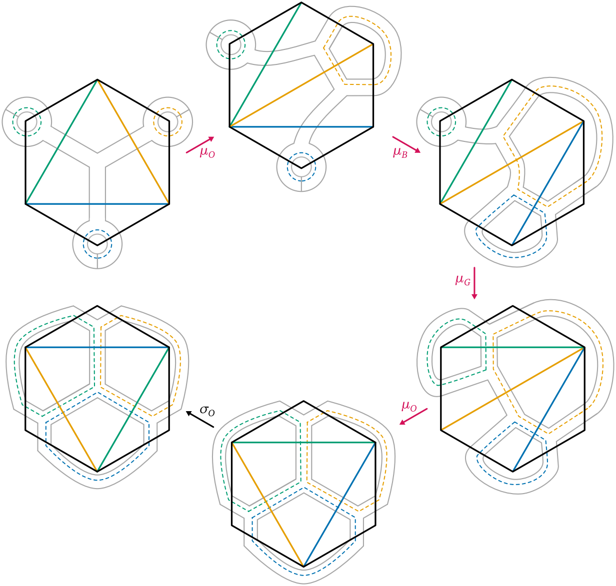
A triangulation of $\{6\}$ dualizes to a *spine* $\Gamma_{0,4}$, i.e., a 3-valent fat graph living on $\Sigma_{0,4}$.

$$\begin{cases} \{Z_1, Z_2, Z_3\} & \longleftrightarrow \quad \text{colored triangulation of } \{6\} \quad \overset{\vee}{\longleftrightarrow} \quad \text{spine } \Gamma_{0,4} \\ \mu_k & \longleftrightarrow \quad \text{flip of diagonal for } Z_k \quad \overset{\vee}{\longleftrightarrow} \quad \text{flip of fat edge for } Z_k \end{cases}$$



Theorem [DM'24]

The rotation of π on equilateral triangulations of $\{6\}$ dualizes to the *inside-out* operation on star-shaped spines of $\Sigma_{0,4}$.



Current toolkit [Chekhov-Mazzocco-Rubtsov'16], that realizes analytic continuation via (gen.) $\mathcal{A}$-mutations, forbids to mutate at edges with self-glued ends. Allowing it, we codify symmetry via $\mathcal{X}$-mutations—advancing the cluster state of the art for P_{VI}.