Azumaya representations of generalized skein algebras

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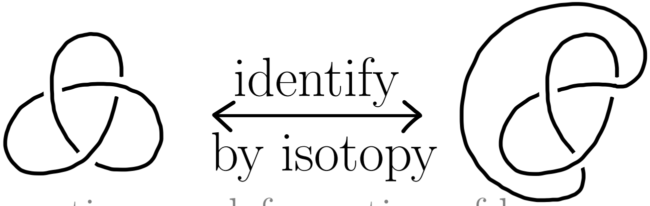
Goal. Some applications from representation theory of skein alg.'s.

Contents.

- From knot theory to skein algebras
- Importance of centers in representation theory
- Stated SL(n)-skein alg.'s in quantum higher Teichmüller sp.'s
- How to compute the highest dimension among irrep.'s

<u>**Knot theory**</u> link $\bigsqcup S^1 \hookrightarrow \Sigma \times (-1,1)$ Σ: oriented surface

knot $=$ 1-component link



continuous deformation of homeomorphisms

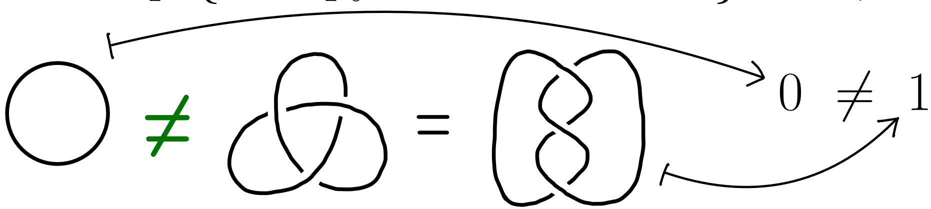
Original question. Given two links L_1 & L_2, show $\begin{cases} L_1 = L_2 \\ \text{or} \\ L_1 \neq L_2 \end{cases}$

To show $L_1 = L_2$, use isotopy concretely.

However, we cannot conclude $L_1 \neq L_2$ only with deformation.

Use an **invariant** to show $L_1 \neq L_2$

a map $\{\text{isotopy classes of links}\} \longrightarrow \mathbb{Z},\ \mathbb{Z}[q^{\pm 1}]$, etc.



<u>Jones polynomial and Kauffman bracket</u>

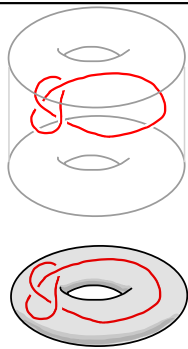
Σ: oriented surf.

$q \in \mathbb{C} \setminus \{0\}$

link in $\Sigma \times (-1, 1)$

$\downarrow$ project

diagram on $\Sigma \times \{0\}$



Kauffman brakcet

$$\langle \times \rangle = q \langle)(\rangle + q^{-1} \langle \asymp \rangle$$

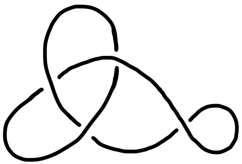
$$\langle D \sqcup \bigcirc \rangle = (-q^2 - q^{-2}) \langle D \rangle$$

$\downarrow$ modify & $\Sigma = \mathbb{R}^2$

Jones polynomial (an invariant of links)

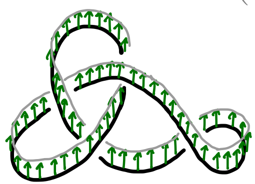
<u>Rmk.</u> Kauffman brakcet is an invariant of framed links

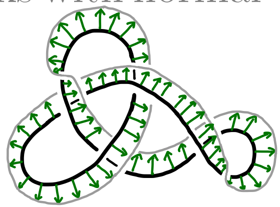
(= links with normal vector fields)



underlying link

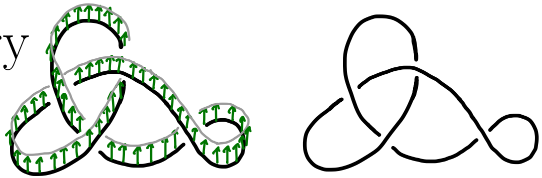
with vertical framing





SL(2)-**skein algebras** Σ: oriented surf., $q \in \mathbb{C} \setminus \{0\}$

Kauffman bracket in knot theory

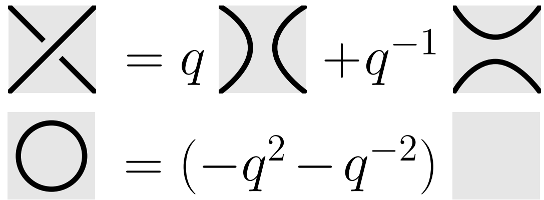


algebraize

$\mathcal{S}_2(\Sigma) = \mathbb{C}\langle \text{isotopy classes of framed links in } \Sigma \times (-1,1) \rangle / (\text{rel's})$

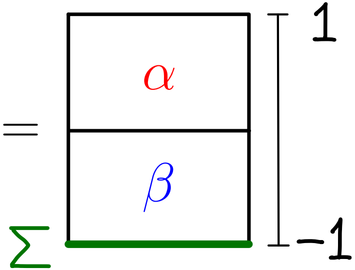
SL(2)-skein algebra of Σ

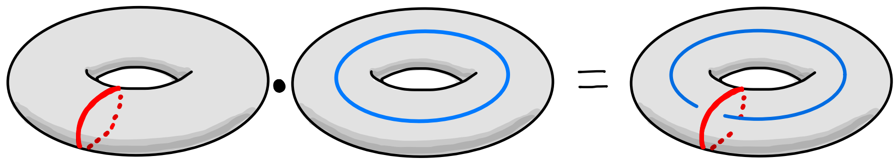
as diagrams



multiplication = stacking

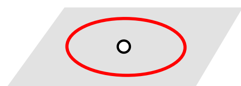
$\alpha \cdot \beta =$





In general, $\mathcal{S}_2(\Sigma)$ is NOT commutative.

Peripheral loops are central.

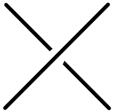
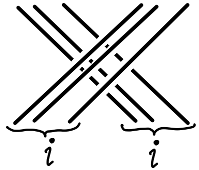


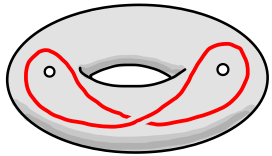
Q. Are there any other central elements?

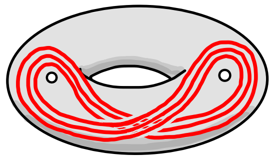
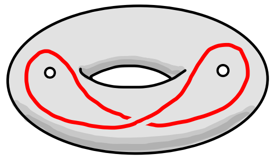
Central elements in skein algebras

$T_N(x) = x \cdot T_{N-1}(x) - T_{N-2}(x), \quad T_1(x) = x, \quad T_0(x) = 2$

1st Chebyshev poly.

z  $\xrightarrow{i \text{ parallel copies}}$ $z^{(i)}$ 

Ex. $T_3(x) = x^3 - 3x$ & $z =$ 

$T_3(z) = z^{(3)} - 3z^{(1)} =$  -3 

Thm. [Frohman–Kania-Bartoszynska–Lê'19]

q : primitive N-th root of 1 (N: odd)

$\mathcal{Z}(S_2(\Sigma))$ is generated by $T_N(z)$ (z: loop) and peripheral loops.

almost Azumaya algebras

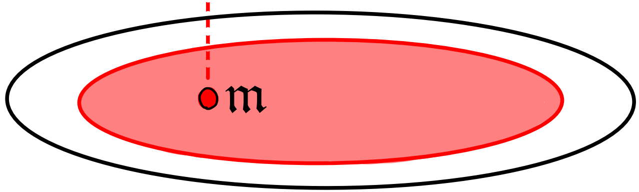
A: $\mathbb{C}$-alg., $\quad \mathcal{Z}(A)$: its center

A: Azumaya algebra $\Rightarrow \forall \mathfrak{m} \in \mathrm{MaxSpec}(\mathcal{Z}(A)),\ A/\mathfrak{m}A \cong \mathrm{M}_D(\mathbb{C})$

$\mathrm{MaxSpec}(\mathcal{Z}(A)) = \{\text{maximal ideals of } \mathcal{Z}(A)\}$ $\qquad \mathrm{M}_D(\mathbb{C})$: matrix algebra

$A/\mathfrak{m}A \cong \mathrm{M}_D(\mathbb{C})$

(almost) continuous family of $\mathrm{M}_D(\mathbb{C})$

$\mathrm{MaxSpec}(\mathcal{Z}(A)) =$ 

almost Azumaya algebra $\Rightarrow \quad \exists$ Azumaya locus $\subset \mathrm{MaxSpec}(\mathcal{Z}(A))$

Zariski open dense

$\mathrm{M}_D(\mathbb{C})$ has the unique irrep. $\mathbb{C}^D$

Q. What is the outside of Azumaya locus?

<u>Unicity theorem</u> $\quad A$: almost Azumaya $\mathbb{C}$-alg.

<u>Thm.</u> [Brown–Gordon] [Frohman–Kania-Bartoszynska–Lê'19]

$$\begin{array}{ccccc} & & & & \mathrm{MaxSpec}(\mathcal{Z}(A)) \\ & & & & \| \\ (1)\ \chi\colon & \{\text{finite dim. irrep's of } A\}/\text{conj.} & \twoheadrightarrow & & \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{Z}(A),\mathbb{C}) \\ & \rotatebox{90}{$\in$} & & & \rotatebox{90}{$\in$} \\ & [\rho] & & & \chi_\rho \ \text{(central character)} \end{array}$$

(2) $\exists\,\mathsf{Azm}(A)$: a Zariski open dense subset of $\mathrm{MaxSpec}(\mathcal{Z}(A))$ s.t.

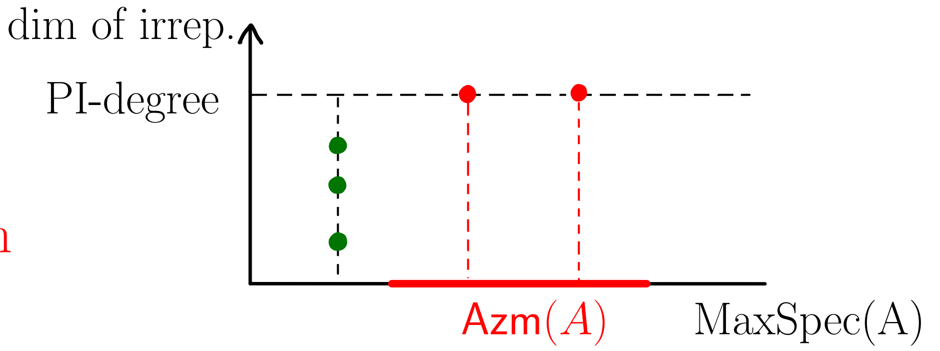
(a) $\forall \tau \in \mathsf{Azm}(A), \quad \chi^{-1}(\tau) = \{\rho_\tau\}$ and $\dim \rho_\tau =$ PI-degree of A

(b) $\forall \tau \notin \mathsf{Azm}(A), \quad$ dim. of irrep. in $\chi^{-1}(\tau) <$ PI-degree of A

For $\tau \in \mathsf{Azm}(A)$

ρ_τ is called

Azumaya representation



Sufficient condition

A is Azumaya $\overset{\text{def}}{\iff}$ A is finitely generated projective $\mathcal{Z}(A)$-module

$$\& \ A \otimes_{\mathcal{Z}(A)} A^{\mathrm{op}} \longrightarrow \mathrm{End}_{\mathcal{Z}(A)} A \text{ is isom.}$$

$$a \otimes b \ \mapsto \ (r \mapsto arb)$$

A is almost Azumaya $\overset{\text{def}}{\iff}$ a localization of A is Azumaya

Sufficient condition

A is Azumaya $\overset{\text{def}}{\Longleftrightarrow}$ A is finitely generated projective $\mathcal{Z}(A)$-module

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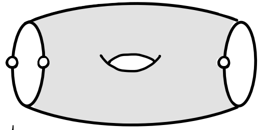
A is almost Azumaya $\overset{\text{def}}{\Longleftrightarrow}$ a localization of A is Azumaya

$$\left.\begin{array}{l}
(1)\ A \text{ is finitely generated as } \mathbb{C}\text{-algebra} \\
(2)\ A \text{ has no zero-divisors} \\
(3)\ A \text{ is finitely generated as } \mathcal{Z}(A)\text{-module}
\end{array}\right\} \Rightarrow A \text{ is almost Azumaya}$$

Fact.

The cardinality of the minimal generating set of A as $\mathcal{Z}(A)$-module $= (\text{PI-degree of } A)^2$.

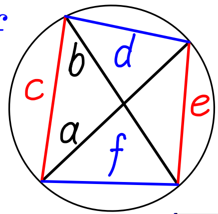
Skein alg. & quantum cluster alg.



Setting. Σ has ∂-punctures & NO interior punctures

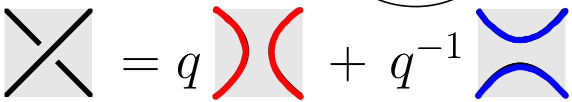
Ptolemy relation $\xrightarrow[\text{algebraize}]{\text{commutative}}$ cluster alg. $\xrightarrow{\text{quantize}}$ quantum cluster alg.

$$a\,b = c\,e + d\,f$$

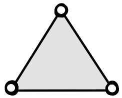


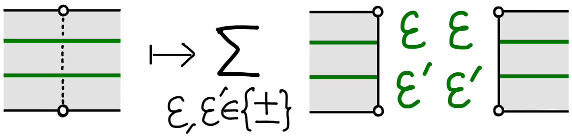
quantize $\downarrow$

[Muller'16] $\updownarrow$



$\xrightarrow{\text{algebraize}}$ generalized skein alg.

$\mathcal{S}_2(\Sigma) \hookrightarrow$ stated skein alg. of $\bigsqcup$  $\xrightarrow{\mathrm{tr}_q^\Delta}$ quantum Teichmüller sp.

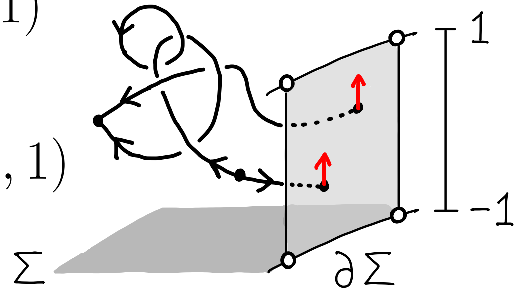


$\exists\,\partial$-punc. $\Rightarrow \ker(\mathrm{tr}_q^\Delta) \neq \emptyset$

- Make the target space larger & modify tr_q^Δ extended ver.
- Take $\mathcal{S}_2^{\mathrm{st}}(\Sigma)/\ker(\mathrm{tr}_q^\Delta)$ reduced ver.

<u>**Jones poly.-like stated SL(2)-skein alg.**</u>

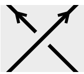
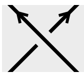
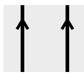
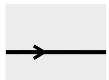
2-web: embedded oriented framed uni-bivalent graphs and loops
with only sink & source in $\Sigma \times (-1,1)$
s.t. vertical framing at univalent
different heights on ∂ edge $\times(-1,1)$

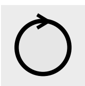
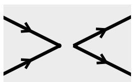
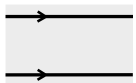
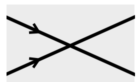


stated 2-web = 2-web with $\{\text{univalent vertices}\} \to \{+,-\}$

$\mathcal{S}_2^{\mathrm{st}}(\Sigma) = \mathbb{C}\langle \text{isotopy classes of stated 2-webs in } \Sigma \times (-1,1)\rangle / (\text{rel's})$

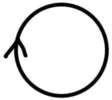
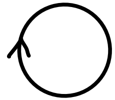
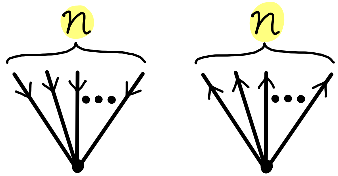
stated SL(2)-skein algebra of Σ equivalent to the previous one

$q^{\frac{1}{2}}$  $- q^{-\frac{1}{2}}$  $= (q - q^{-1})$  ,  $= q^{\frac{3}{2}}$  ,

 $= (q + q^{-1})$,  $= -q \Big($  $- q^{-\frac{1}{2}}$  $\Big)$,

and rel.'s around $\partial\Sigma$

stated SL(2)**-skein to stated** SL(n)**-skein**

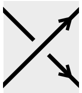
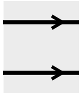
	SL(2)	SL(n)
loop		
graph	 uni-bivalent	 uni-n-valent
states	$\{+, -\}$	$\{1, 2, \ldots, n\}$
q-parameter	$q^{1/2}$	$q^{1/n}$
relations	in the last slide	in the **next** slide

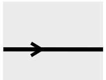
n-web

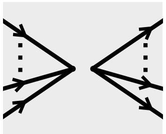
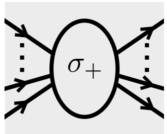
$\mathcal{S}_n^{\mathrm{st}}(\Sigma) = \mathbb{C}\langle \text{isotopy classes of stated } n\text{-webs in } \Sigma \times (-1, 1)\rangle/(\text{rel's})$

stated SL(n)-skein algebra of Σ

Defining relations of stated $\mathrm{SL}(n)$-skein alg.

(1) $q^{\frac{1}{n}}$  $- q^{-\frac{1}{n}}$  $= (q - q^{-1})$  ,

(2)  $= q^{\frac{n^2-1}{n}}$  , (3)  $= (-1)^{n-1}\dfrac{q^n - q^{-n}}{q - q^{-1}}$,

(4)  $= (-q)^{\frac{n(n-1)}{2}} \cdot \sum_{\sigma\in\mathfrak{S}_n}(-q^{\frac{1-n}{n}})^{\text{length of } \sigma}$ 

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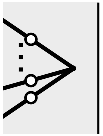
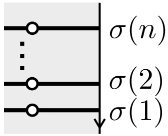
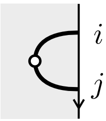
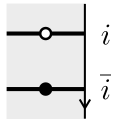
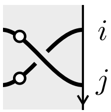
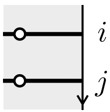
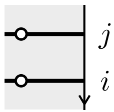
(5)  $= q^{\frac{n+1-2n^2}{4}} \sum_{\sigma\in\mathfrak{S}_n}(-q)^{\text{length of } \sigma}$ 

(6) 

$= \delta_{\bar{j},i}(-q)^{n-i}q^{\frac{n-1}{2n}}$, (7)  $= \sum_{i=1}^{n}(-q)^{-n+\bar{i}}q^{\frac{-n+1}{2n}}$ 

(8) 

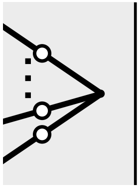
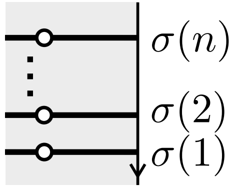
$= q^{-\frac{1}{n}}\Bigg(\delta_{j<i}(q - q^{-1})$ 

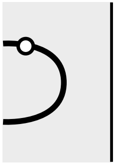
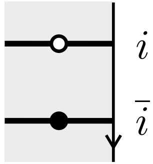
$+ q^{\delta_{i,j}}$ 

$\Bigg)$.

$\circ$ and $\bullet$ denote opposite orientations

Defining relations of stated $\mathrm{SL}(n)$-skein alg.

(5)  $= q^{\frac{n+1-2n^2}{4}} \sum_{\sigma\in\mathfrak{S}_n} (-q)^{\text{length of } \sigma}$ 

(7)  $= \sum_{i=1}^{n} (-q)^{1-i} q^{\frac{-n+1}{2n}}$ 

Important (5) & (7) $\Rightarrow$ an n-web = prod. and sum of **arc**s

$q^{1/n}$: a primitive m'-th root of 1

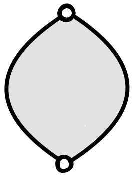
Thm. [Bonahon–Higgins'23]

Threading of a loop by 'Chebyshev polynomials' $\in \mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$

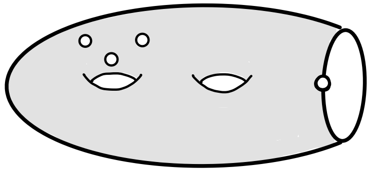
Prop. [Wang'23] The 'm'-th power' of an arc $\in \mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$

Why stated SL(n)-skein alg.?

quantum coordinate ring

- $\mathcal{S}_n^{\mathrm{st}}($  $) \cong \mathcal{O}_q(\mathrm{SL}(n))$

[Costantino–Lê'22]($n = 2$)
[Lê–Sikora'21]($n \geq 3$)

- $\Sigma =$  $\Rightarrow \mathcal{S}_n^{\mathrm{st}}(\Sigma) \cong$ the quantum moduli alg.

[Alekseev–Grosse–Schomerus'95]
[Baseilhac–Faitg–Roche'23]

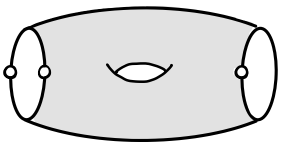
- $\exists$ extended quantum trace map

$\mathrm{tr}_q^{\Delta} \colon \mathcal{S}_n^{\mathrm{st}}(\Sigma) \longrightarrow$ extended Fock–Goncharov alg. [Fock–Goncharov'06,'09]

in (quantum) higher Teichmüller theory

- ◦ Σ has ∂-punctures and NO interior punctures $\Rightarrow \mathrm{tr}_q^{\Delta}$ is injective

Today's setting



[Bonahon–Wong'11, Lê'19]($n=2$),
[Kim'20,'21, Douglus'24]($n=3$), [Lê–Yu'23]($n \geq 4$)

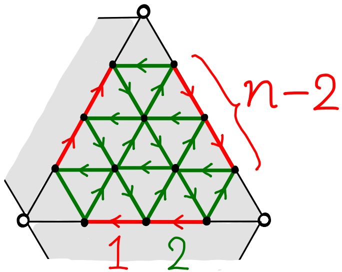
Quantum tori in higher Teichmüller theory

P: $m \times m$ anti-symmetric matrix

$\mathbb{T}(\mathsf{P}) := \mathbb{C}\langle x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_m^{\pm 1}\rangle / (x_i x_j = q^{\mathsf{P}_{ij}} x_j x_i)$
quantum torus

triangulation $\rightsquigarrow$ weighted quiver $\rightsquigarrow$ anti-symm. matrix

Δ



$\mathsf{Q}^{\Delta}_{ij} := (\# i \bullet\!\!\rightarrow\!\!\bullet j - \# i \bullet\!\!\leftarrow\!\!\bullet j)$
$\times$ weight of $i \bullet\!\!-\!\!\bullet j$

extended Fock–Goncharov alg.

Thm. [Lê–Yu'23] $\mathcal{A}_q(\Delta) \xrightarrow{\cong} \mathcal{X}_q^{\mathrm{bl}}(\Delta) \subset \mathcal{X}_q(\Delta) := \mathbb{T}(\mathsf{Q}^{\Delta})$

Rmk. $\exists$ anti-symm. matrix P^{Δ} s.t. $\mathcal{A}_q(\Delta) := \mathbb{T}(\mathsf{P}^{\Delta})$

extended $\mathcal{A}$-quantum torus

sandwiched property. $\mathcal{A}_q^{+}(\Delta) \subset \mathcal{S}_n^{\mathrm{st}}(\Sigma) \subset \mathcal{A}_q(\Delta)$

Unicity theorem for stated $\mathrm{SL}(n)$-skein alg.

Thm. [Lê–Yu'23] $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$ satisfies (1) & (2)

The rest is (3) finite generation as $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$-module

Thm 1. [KW'24] $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$ is finitely generated as $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$-module

$\Rightarrow$ Unicity thm. can be applied to $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$

Strategy. Show finite generation of $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$ over the subalg. generated by m'-th powers of arcs

Prop. [Wang'23] $q^{1/n}$: a primitive m'-th root of 1

The m'-th power of an arc $\in \mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$

Next. We compute the PI-degree

$\forall \tau \in \mathsf{Azm}(A)$, $\exists!$ irrep. ρ_τ and $\dim \rho_\tau =$ PI-degree of A

$|$minimal generating set of A as $\mathcal{Z}(A)$-module$| =$ (PI-degree of $A)^2$

Comparing centers

P: $m \times m$ anti-symmetric matrix

$\mathbb{T}(\mathsf{P}) := \mathbb{C}\langle x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_m^{\pm 1}\rangle / (x_i x_j = q^{\mathsf{P}_{ij}} x_j x_i)$

$\mathbb{T}^+(\mathsf{P}) := \mathbb{C}\langle x_1, x_2, \cdots, x_m\rangle /$ same relation

its positive part

$\mathbb{T}^+(\mathsf{P}^\Delta)$ $\qquad\qquad$ $\mathbb{T}(\mathsf{P}^\Delta)$

Thm. [Lê–Yu'23] $\mathcal{A}_q^+(\Delta) \subset \mathcal{S}_n^{\mathrm{st}}(\Sigma) \subset \mathcal{A}_q(\Delta)$

$x_i z = z x_i \Rightarrow z x_i^{-1} = x_i^{-1} z$ $\quad$ implies $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma)) \subset \mathcal{Z}(\mathcal{A}_q(\Delta))$,

i.e. $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma)) = \mathcal{S}_n^{\mathrm{st}}(\Sigma) \cap \mathcal{Z}(\mathcal{A}_q(\Delta))$

Thm 2. [KW'24] $\quad q^{1/n}$: a primitive m'-th root of 1

We described $\mathcal{Z}(\mathcal{A}_q(\Delta))$ explicitly. We also described $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$.

Prop. [KW'24] Thm. implies PI-deg. of $\mathcal{S}_n^{\mathrm{st}}(\Sigma) =$ PI-deg. of $\mathcal{A}_q(\Delta)$

PI-degree = the highest dim. among finite dim. irrep.'s

$|$minimal generating set of A as $\mathcal{Z}(A)$-module$| = (\text{PI-degree of } A)^2$

Strategy.

$\exists \mathsf{P}^\Delta,\ \mathcal{A}_q(\Delta) = \mathbb{T}(\mathsf{P}^\Delta)$ quantum torus

Step 1. Decompose P^Δ and describe some block matrices explicitly

Step 2. Describe the conditions of the center as vectors

Step 3. Take the quotient and compute the cardinality

$r(\Sigma) := \#(\partial\text{-punctures}) + \#(\text{components of } \partial\overline{\Sigma}) + 2g - 2$

$t:$ # components of $\partial\overline{\Sigma}$ with even number of ∂-punctures

Thm 3. [KW'24]

PI-deg. of $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$ = PI-deg. of $\mathcal{A}_q(\Delta) = \sqrt{d^{r(\Sigma)-t} m^{(n^2-1)r(\Sigma)-t(n-1)}}$

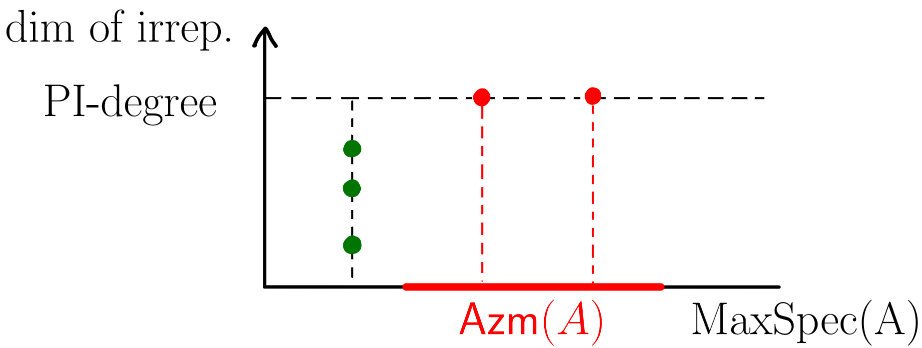
$d = \gcd(m', n),\ m = m'/d$

Rmk. For $n = 2$, it recovers [Yu'23].

Summary

Thm 1. $\mathcal{S}_n^{\mathrm{st}}(\Sigma)$ is finitely generated as $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$-module

Unicity thm. $\{$finite dim. irrep's of $A\}/\text{conj.} \twoheadrightarrow \mathrm{MaxSpec}(\mathcal{Z}(A))$



$$\mathcal{A}_q^+(\Delta) \subset \mathcal{S}_n^{\mathrm{st}}(\Sigma) \subset \mathcal{A}_q(\Delta) \Longrightarrow \mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma)) = \mathcal{S}_n^{\mathrm{st}}(\Sigma) \cap \mathcal{Z}(\mathcal{A}_q(\Sigma, \lambda))$$

Thm 2. $q^{1/n}$: a primitive m'-th root of 1

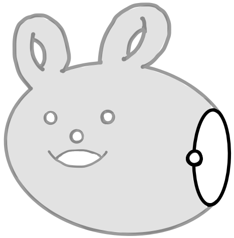
We described $\mathcal{Z}(\mathcal{A}_q(\Delta))$ explicitly. We also described $\mathcal{Z}(\mathcal{S}_n^{\mathrm{st}}(\Sigma))$.

Thm 3. PI-deg. of $\mathcal{S}_n^{\mathrm{st}}(\Sigma) = \sqrt{d^{r(\Sigma)-t} m^{(n^2-1)r(\Sigma)-t(n-1)}}$

$$d = \gcd(m', n),\ m = m'/d$$

Application & similar results

We can access to representation theory of quantum moduli alg.

$\Sigma =$  $\Rightarrow \mathcal{S}_n^{\mathrm{st}}(\Sigma) \cong$ the quantum moduli alg.

$$\overline{\mathcal{S}}_n^{\mathrm{st}}(\Sigma) = \mathcal{S}_n^{\mathrm{st}}(\Sigma)/(\text{kernel of (original) quantum trace map})$$

reduced stated SL(n)-skein alg.

injective quantum trace map

$$\overline{\mathrm{tr}}_q^{\Delta} : \overline{\mathcal{S}}_n^{\mathrm{st}}(\Sigma) \hookrightarrow \overline{\mathcal{A}}_q(\Delta) \xrightarrow{\cong} \overline{\mathcal{X}}_q^{\mathrm{bl}}(\Delta) \subset \overline{\mathcal{X}}_q(\Delta)$$

original Fock–Goncharov alg.

in (quantum) higher Teichmüller sp.

Thm 4. Similar results for reduced stated SL(n)-skein alg.'s

almost Azumaya, $\mathcal{Z}(\overline{\mathcal{A}}_q(\Delta))$, $\mathcal{Z}(\overline{\mathcal{S}}_n^{\mathrm{st}}(\Sigma))$, PI-deg.

What's next?

$\overline{\mathcal{S}}^{\mathrm{st}}_n(\Sigma)$: reduced stated SL($n$)-skein algebra of Σ.

Each conn. comp. of Σ has $\partial\Sigma \neq \emptyset$ and NO interior punctures

today's setting

$\underline{n=2}$ $\overline{\mathcal{S}}^{\mathrm{st}}_2(\Sigma) =$ quantum cluster alg. [Muller'16]

$\underline{n=3}$ $\overline{\mathcal{S}}^{\mathrm{st}}_3(\Sigma) \subset$ quantum cluster alg. [Ishibashi–Yuasa'22]

Conj. $\overline{\mathcal{S}}^{\mathrm{st}}_3(\Sigma) =$ quantum cluster alg.

$\overline{\mathcal{S}}^{\mathrm{st}}_n(\Sigma) =$ quantum cluster alg. for any n ?

if it is true $\Rightarrow$ representation theory of quantum higher cluster alg.

Prob. • Describe the Azumaya locus explicitly.

• Give a geometric meaning of Auzmaya representations.