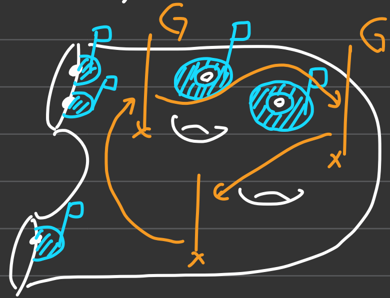
Cluster K_2-structure on the moduli space of decorated twisted G-local systems



$\boxed{\#2}$ Recall:

- $\mathcal{A}_{G,\Sigma}$ = moduli space of decorated twisted G-local systems on Σ

$$\overset{\text{if } \mathbb{M} \subset \partial\Sigma}{\simeq} \left[\mathrm{Hom}^{tw}(\pi_1(T'\Sigma), G) \times (G/U^+)^{\mathbb{M}} \Big/ G \right]$$

- $\mathcal{A}^{\times}_{G,\Sigma} \overset{\text{if } \mathbb{M} \subset \partial\Sigma}{\hookrightarrow} \mathrm{Hom}(\Pi_1(T'\Sigma, \mathbb{B}), G)$

  via (twisted) Wilson lines.

- Reviewed the construction of cluster charts on $\mathcal{A}^{\times}_{G,\Sigma}$.

<u>Today</u>
- ▷ Relation to braid varieties
- ▷ Understand the flip
- ▷ Generalized minors of Wilson lines

§4. Interpolating (decorated) flags

$$\mathcal{A}_G := G/U^+ \xrightarrow{\pi} \mathcal{B}_G := G/B^+ \quad : H\text{-bdl}$$

Recall : for $A_1, A_2 \in \mathcal{A}_G$

$$(A_1, A_2) = g.(\underbrace{h}_{h\text{-distance}}.[U^+], \underbrace{\overline{w}}_{w\text{-distance}}.[U^+])$$

- For $A_1, A_2 \in \mathcal{A}_G$, write $A_1 \xrightarrow{u} A_2$

  if $w(A_1, A_2) = u$

<u>Lem</u> 1) $B_1 \xrightarrow{u} B_2$, $B_2 \xrightarrow{v} B_3$, $l(uv) = l(u) + l(v)$

$\Rightarrow B_1 \xrightarrow{uv} B_3$

2) Conversely, $B_1 \xrightarrow{w} B_3$, $w = uv$, $l(uv) = l(u) + l(v)$

$\Rightarrow B_1 \xrightarrow{u} {}^{\exists !} B_2 \xrightarrow{v} B_3$.

Cor $B \xrightarrow{w} B'$, $w = r_{s_1} \cdots r_{s_m}$: reduced

$\Rightarrow \exists!$ a seq. $B = B_0 \xrightarrow{r_{s_1}} B_1 \xrightarrow{r_{s_2}} \cdots \xrightarrow{r_{s_m}} B_m = B'$.

Non-example $\mathcal{B}_{SL_2} \cong \mathbb{P}^1$.

$$w(B_1, B_2) = \begin{cases} e & \text{if } B_1 = B_2 \\ s_1 & \text{if } B_1 \neq B_2 \end{cases}$$

Let $B = [1:0]$, $B' = [0:1] \in \mathbb{P}^1$

Then $\{ B_1 \mid B \xrightarrow{s_1} B_1 \xrightarrow{s_1} B' \} \cong \mathbb{C}^*$.

... an example of braid variety

(III) Decorated version :

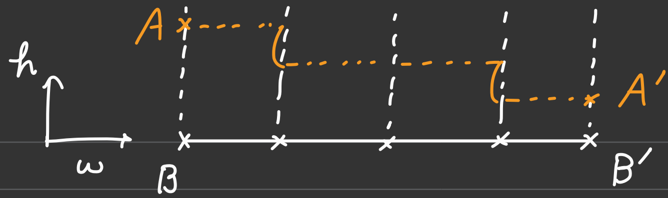
$\forall A, A' \in \mathcal{A}_G$

$\exists!$ a seq. $A = A_m \xrightarrow{r_{s_m}} \cdots \xrightarrow{r_{s_1}} A_0 = A'$

s.t. $h(A_k, A_{k-1}) = \alpha_{s_k}^\vee (c_k)$

Here,

$$\begin{array}{c} \mathcal{A}_G \\ \downarrow \\ \mathcal{B}_G \end{array}$$



$$c_k := \begin{cases} h(A,A')^{\varpi_t} & \text{if } w_{>k}\alpha^\vee_{s_k} = \alpha^\vee_t \text{ is a simple coroot} \\ 1 & \text{otherwise} \end{cases}$$

§5. Relation to the braid variety (cf. Casals' talk)

$$Br^+ := \langle \sigma_s \ (s \in S) \mid \text{braid rel's} \rangle_{\text{monoid}} \ni \beta : \text{"positive braid"}$$

$$\begin{array}{ccc} Br^+ & & \sigma_s \\ \downarrow & & \downarrow \\ W & \ni & r_s \end{array}$$

$W \ni w = r_{s_1} \cdots r_{s_m}$ (red)

$\rightsquigarrow \beta(w) := \sigma_{s_1} \cdots \sigma_{s_m}$ (braid lift)

Def $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell} \in Br^+$

$$X(\beta) := \left[\left\{ (A_0, \cdots, B_\ell) \in \mathcal{A}_G \times \mathcal{B}_G^{\ell-1} \ \middle| \ \begin{tikzcd} B_1 \arrow[r, "r_{s_2}"] & B_2 \arrow[r] & \cdots \arrow[r] & B_{\ell-1} \arrow[d, "r_{s_\ell}"] \\ A_0 \arrow[u, "r_{s_1}"] \arrow[rrr, "\delta(\beta)"] & & & B_\ell \end{tikzcd} \right\} \middle/ G \right]$$

Here, $\delta(\beta) \in W$: Demazure product

s.t. $\delta(\sigma_s) = r_s$, $\delta(\sigma_s \sigma_s) = r_s$

<u>Rem</u> • Isom. class of $X(\beta)$ only depends on β.

• If $\delta(\beta) = w_0$, $X(\beta) \cong \left\{ \begin{tikzcd} B_1 \arrow[r, "r_{s_2}"] & B_2 \arrow[r] & \cdots \arrow[r] & B_{\ell-1} \arrow[d, "r_{s_\ell}"] \\ {[U^+]} \arrow[u, "r_{s_1}"] \arrow[rrr] & & & B^- \end{tikzcd} \right\}$

$= m(\Lambda_\beta)$

<u>Example 1</u> $\beta = \sigma_1 \sigma_1$ $G = SL_2$

$$X(\beta) \cong \left\{ B_1 \;\middle|\; \begin{tikzcd} & B_1 \arrow[dr, "s_1"] & \\ (1,0) \arrow[ur, "s_1"] \arrow[rr, "s_1"'] & & {[0:1]} \end{tikzcd} \right\} = \mathbb{C}^*$$

<u>Example 2</u> Reduced double Bruhat cells [BZ'01]

$$L^{u,v} := U^+ \bar{u} U^+ \cap B^- \bar{v} B^- \xrightarrow{\sim} X(\beta(u)\beta(w_0)\beta(v^{-1}))$$

$$\begin{array}{c} \in \\ g \end{array} \longmapsto \left(\begin{tikzcd} g.B^+ \arrow[r, "w_0"] & g\bar{w}_0 B^+ \\ {[U^+]} \arrow[u, "u"] \arrow[r, "w_0"'] & \bar{w}_0 B^+ \arrow[u, "v"'] \end{tikzcd} \right)$$

$$\begin{tikzcd} \bullet \arrow[r, dash, "w_0"] & \bullet \\ \bullet \arrow[u, "u"] \arrow[r, dash, "w_0"'] & \bullet \arrow[u, "v"'] \end{tikzcd}$$

Example 3 open Richardson variety

$$\begin{cases} S_w := B^+ \bar{w} B^+ / B^+ = \{ xB^+ \in \mathcal{B}_G \mid B^+ \xrightarrow{w} xB^+ \} \\ S_v^- := B^- \bar{v} B^+ / B^+ = \{ yB^+ \in \mathcal{B}_G \mid B^- \xrightarrow{w_0 v} yB^+ \} \end{cases}$$

$\underline{v \le w}$

$$\begin{array}{ccc} S_v^- \cap S_w & \xrightarrow{\quad \sim \quad} & X(\beta(w)\beta(v^{-1}w_0)) \\ \cup\!\!| & & \cup\!\!| \\ xB^+ & \longmapsto & \left(\begin{tikzcd} & x.B^+ \arrow[dr, "v^{-1}w_0"] & \\ {[U^+]} \arrow[ur, "w"] \arrow[rr, "w_0"'] & & \bar{w}_0.B^+ \end{tikzcd} \right) \end{array}$$

$$\begin{tikzcd} & \bullet \arrow[dr, "v^{-1}w_0"] & \\ \bullet \arrow[ur, "w"] \arrow[rr, dash, "w_0"'] & & \bullet \end{tikzcd}$$

General facts:

- $X(\beta)$: a smooth affine var. of $\dim = |\beta| - \ell(\delta(\beta))$
- $\mathcal{O}(X(\beta))$: a locally acyclic cluster alg.

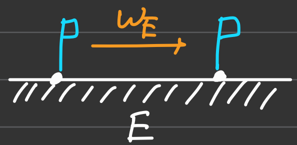
"Demazure weaves $\rightsquigarrow$ clusters

Return to the moduli sp. [IOS'22, §6.2]

<u>Def</u> A <u>boundary W-coloring</u> is a map

$$w : \mathbb{B} \longrightarrow W, \quad E \longmapsto w_E.$$

Let $\mathcal{A}^{w}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ be the subsp.

s.t. 

, $\forall E \in \mathbb{B}$

<u>Prop.</u> $\mathbb{D}_k$: a k-gon $(k \geq 3)$, $w : \mathbb{B} \longrightarrow W$

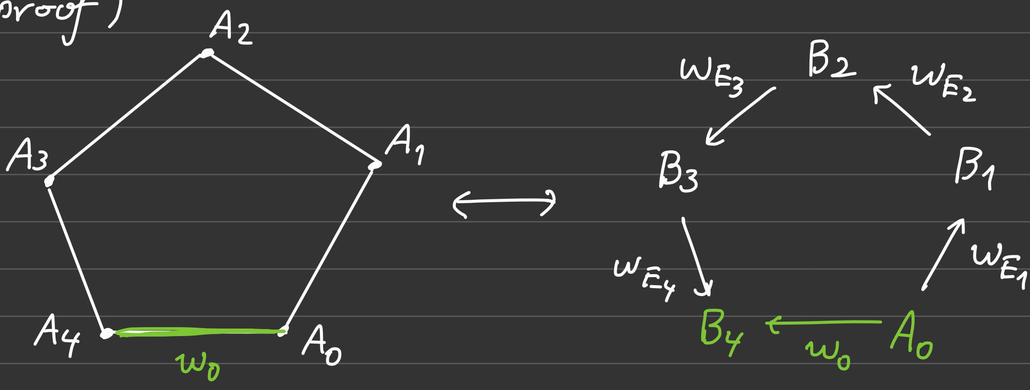
Assume that $w_{E_0} = w_0$ for some $E_0 \in \mathbb{B}$.

Then

$$\mathcal{A}^{w}_{G,\mathbb{D}_k} \cong X(\beta) \times H^{k-1}$$

Here $\beta = \beta(w, E_0) := \beta(w_{E_1}) \cdots \beta(w_{E_k})$.

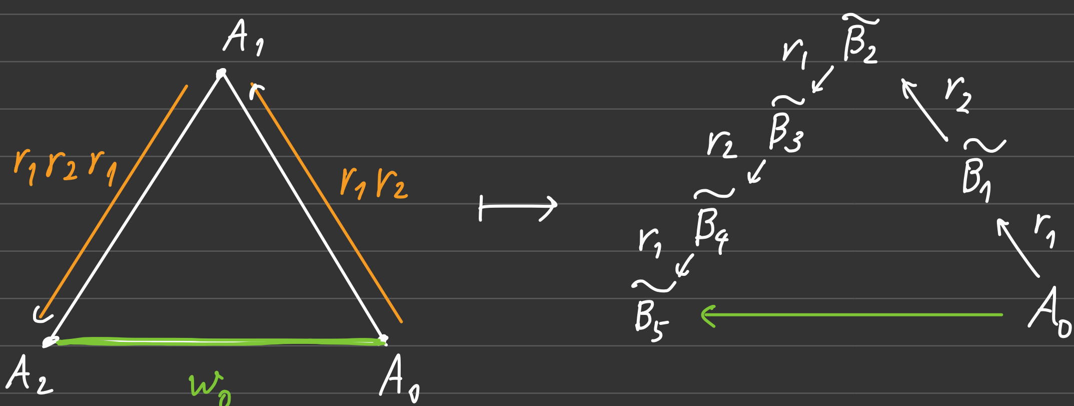
proof)



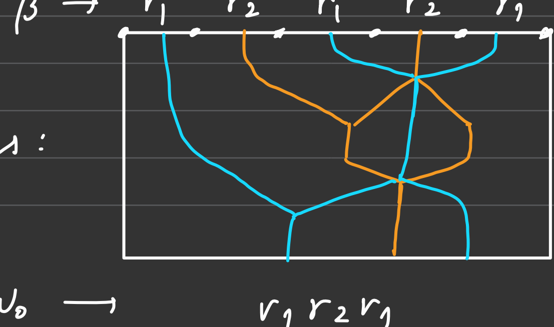
$$+\ h(A_0, A_1), \cdots, h(A_3, A_4)$$

∎

Example (Type A_2)



weave clusters:

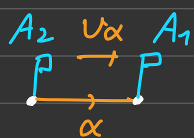


III General case via cutting maps

$\mathbb{B}(\Sigma') = \mathbb{B}(\Sigma) \cup \{\alpha', \alpha''\}$

$$\mathrm{cut}_\alpha : \mathcal{A}^{w}_{G,\Sigma}[\alpha; v_\alpha] \longrightarrow \mathcal{A}^{w \cup \{v_\alpha, v_\alpha^{-1}\}}_{G,\Sigma'}$$

impose:



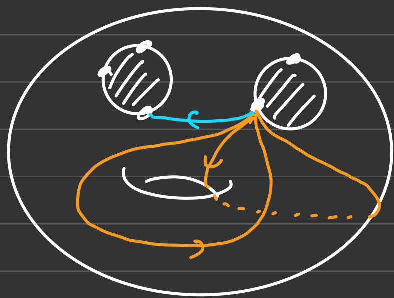
Prop 1) The image of cut_α is $\{h_{\alpha'} = h_{\alpha''}\}$.

2) Each fiber of cut_α is isom. to

$$G_{v_\alpha} := \mathrm{Stab}_G([U^+], \bar{v}_\alpha . B^+) < G.$$

Let Σ be a marked surface w/ $\mathbb{M} \subset \partial\Sigma$

Fix $\mathcal{C}$: a <u>cut system</u> on Σ

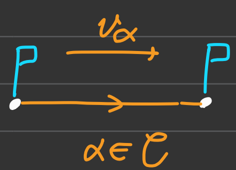


<u>i.e.</u> $\Sigma \setminus \cup\, \mathcal{C} = \mathbb{D}_k$

$$\rightsquigarrow \begin{cases} |\mathcal{C}| = 2g + b - 1 \\ k = 2|\mathcal{C}| + |\mathbb{M}| \end{cases}$$

<u>Def</u> A W-coloring of $\mathcal{C}$ is a map $v : \mathcal{C} \longrightarrow W$.

Let $\mathcal{A}^{W}_{G,\Sigma}[\mathcal{C} ; v] \subset \mathcal{A}^{W}_{G,\Sigma}$ s.t.



$\mathrm{cut}_{\mathcal{C}} := \mathrm{comp}(\mathrm{cut}_\alpha \mid \alpha \in \mathcal{C})$:

$$\mathcal{A}^{W}_{G,\Sigma}[\mathcal{C}, v] \longrightarrow \mathcal{A}^{W_{\mathcal{C}}}_{G,\mathbb{D}_k}$$

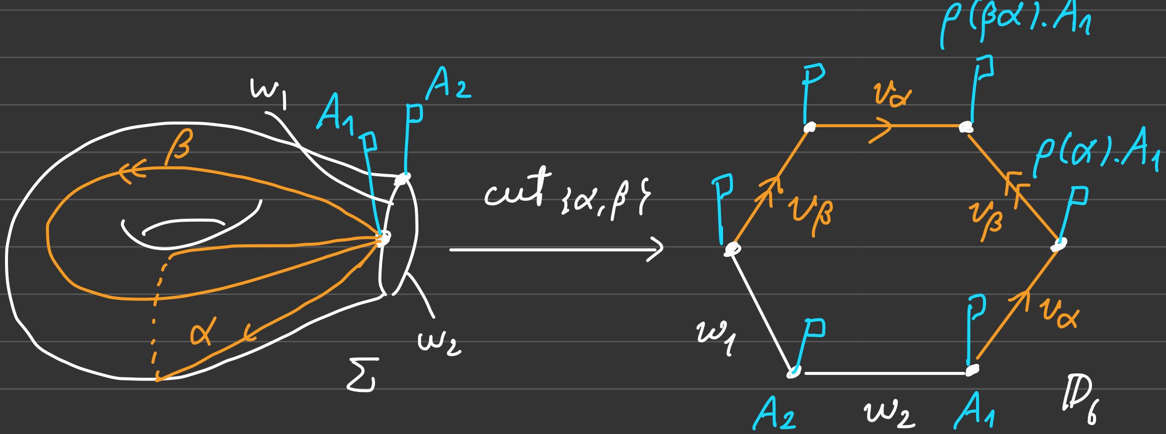
Here $W_{\mathcal{C}} := W \cup \{ v_\alpha, v_\alpha^{-1} \mid \alpha \in \mathcal{C} \}$

(just read off the coloring)

Theorem Assume: $w \cup v$ contains w_0.

$$\mathrm{cut}_{\mathcal{C}} : \mathcal{A}^{w_0}_{G,\Sigma}[\mathcal{C}; v] \twoheadrightarrow \,^{\exists}X(\beta) \times H^{|\mathcal{C}|+|M|-1}$$

Each fiber of $\mathrm{cut}_{\mathcal{C}}$ is isom to $\prod_{\alpha\in\mathcal{C}} G_{v_\alpha}$.



In particular,

$$\dim \mathcal{A}^{w_0}_{G,\Sigma}[\mathcal{C}; v] = \sum_{\alpha\in\mathcal{C}} \underbrace{\dim G_{v_\alpha}}_{\ell(w_0)-\ell(v_\alpha)} + \underbrace{\dim X(\beta)}_{|\beta|-\ell(w_0)} + \text{const.}$$

$$\sim \sum_{\alpha\in\mathcal{C}} \ell(v_\alpha) + \sum_{E\in\mathbb{B}} \ell(w_E) \quad (\text{drop one term.})$$

Anyways :

$\operatorname{codim} \mathcal{A}^{w}_{G,\Sigma}[\mathcal{C}; v]$

$$= \sum_{\alpha \in \mathcal{C}} (\ell(w_0) - \ell(v_\alpha)) + \sum_{E \in \mathcal{B}(\Sigma)} (\ell(w_0) - \ell(w_E))$$

- We also have a modified statement even if $w \cup v$ does not contain w_0.

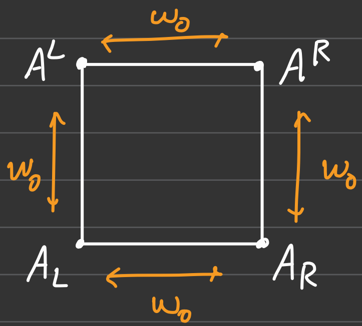
<u>Remark</u> $\operatorname{cut}_{\mathcal{C}} : \mathcal{A}^{w}_{G,\Sigma}[\mathcal{C}, w_0] \xrightarrow{\sim} X(\beta) \times H^{|\mathcal{C}| + |M| + 1}$

<u>Q.</u> Does each stratum have cluster K_2-str. s.t. $\operatorname{cut}_{\mathcal{C}}$ is a cluster fibration? [Le - Fraser]

<u>cf.</u> $G = \bigsqcup_{u,v \in W} G^{u,v}$ "cluster stratification"

§6. Cluster str. on the square

$\mathcal{A}^{\times}_{G,\mathbb{D}_4}$:



Let $\mathbf{s}$ be a double reduced word of (w_0, w_0).

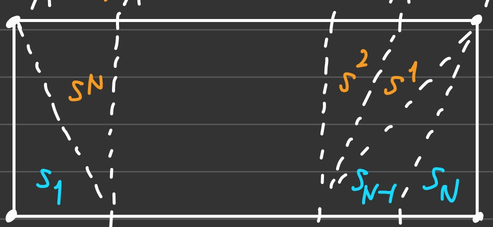
$\mathbf{s}^{\bullet} = (s^1, \dots, s^N)$

e.g. $(1, \bar{1}, \bar{2}, 2, 1, \bar{1})$

$\mathbf{s}_{\bullet} = (s_1, \dots, s_N)$

$$A^L = A^N \xrightarrow{s^N} A^{N-1} \xrightarrow{s^{N-1}} \cdots \xrightarrow{s^2} A^1 \xrightarrow{s^1} A^0 = A^R$$

$\rightsquigarrow$



$\leftarrow$ "triangulation" w.r.t. $\mathbf{s}$.

$$A_L = A_0 \xleftarrow{s_1^*} A_1 \xleftarrow{s_2^*} \cdots \xleftarrow{s_{N-1}^*} A_{N-1} \xleftarrow{s_N^*} A_N = A_R$$

<u>Recall</u> $\mathcal{O}(\mathrm{Conf}_2\, G/U^+) \cong \bigoplus_{\lambda} (V_\lambda \otimes V_\lambda^*)^G$

Let $\Delta_s \in (V_{\varpi_s} \otimes V_{\varpi_s}^*)^G$ s.t. $\Delta_s([U^+], \overline{w_0}[U^+]) = 1$.

Get a collection $\left\{ \Delta_s(A^k, A_\ell) \;\middle|\; \exists \begin{array}{c} A^k \\ \vdots \\ A^\ell \end{array}, \; s \in S \right\}$

<u>Note</u>: $\Rightarrow \Delta_s(A^k, A_\ell) = \Delta_s(A^k, A_{\ell+1})$

for $s \neq s_\ell^*$

<u>Frozen variables</u>

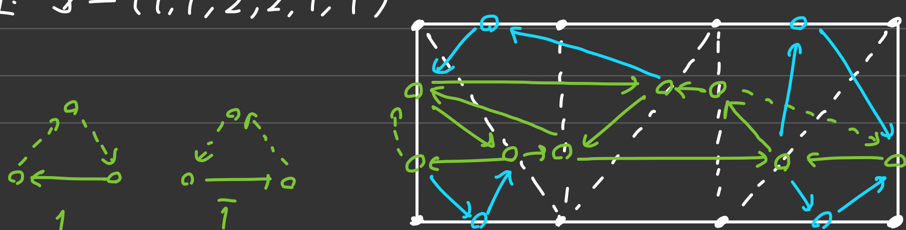
on the "top": $h(A^{k-1}, A^k)^{\varpi_t} = h(A^L, A^R)^{\varpi_t}$

if $w_{>k}\, \alpha_{s_k}^\vee = \alpha_t^\vee$: simple

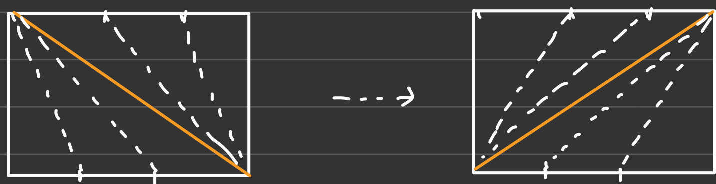
on the "bottom": $h(A_\ell, A_{\ell-1})^{\varpi_u^*} = h(A_R, A_L)^{\varpi_u^*}$

if $w_{>\ell}^*\, \alpha_{s_\ell^*}^\vee = \alpha_u^\vee$: simple

<u>e.g.</u> $\mathbf{s} = (1, \bar{1}, \bar{2}, 2, \bar{1}, 1)$

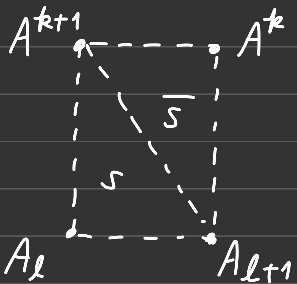


Flip :



$(\delta_\circ, \delta^\circ)$ $\qquad\qquad$ $(\delta^\circ, \delta_\circ)$

Elementary operation : $s\bar{s} \longleftrightarrow \bar{s}s$

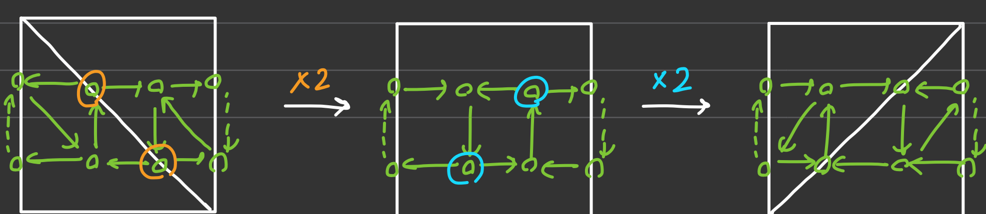


$$\Delta_s(A^{k+1}, A_{\ell+1})\,\Delta_s(A^k, A_\ell)$$
$$= \Delta_s(A^{k+1}, A_\ell)\,\Delta_s(A^k, A_{\ell+1})$$
$$+ (\text{frozen}) \cdot \prod_{t \neq s} \Delta_s(A^k, A_\ell)^{-C_{st}}$$

$(s\bar{t} = \bar{t}s$ for $s \neq t)$

e.g. $1\,2\,1\,\bar{2}\,\bar{1}\,\bar{2} = 1\,\underline{2\,\bar{2}}\,\underline{1\,\bar{1}}\,\bar{2} \xrightarrow{\times 2} 1\,\bar{2}\,2\,\bar{1}\,1\,\bar{2}$

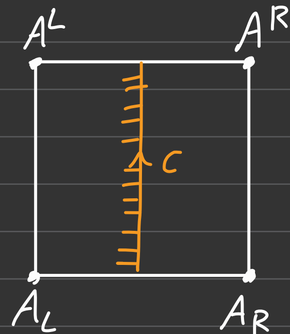
$= \bar{2}\,\underline{1\,\bar{1}}\,\underline{2\,\bar{2}}\,1 \xrightarrow{\times 2} \bar{2}\,\bar{1}\,1\,\bar{2}\,2\,1 = \bar{2}\,\bar{1}\,\bar{2}\,1\,2\,1$



(11) Generalized minors of simple Wilson lines

<u>Prop</u> $\Delta_{w_{>l}\varpi_s,\, w^{>k}\varpi_s}(g_{[c]})$

$$= \frac{\Delta_s(A^k, A_l)}{h(A_R, A_L)^{[w_{>l}\varpi_s]_+^*}\, h(A^L, A^R)^{[w^{>k}\varpi_s]_+}}$$

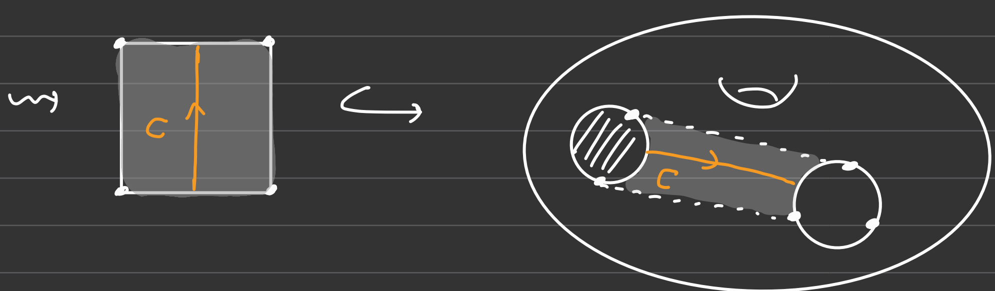


In particular, it is a cluster monomial.

§7. Proof of $\mathcal{A}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$

$[c]: E_1 \longrightarrow E_2$ simple $:\Longleftrightarrow$ $E_1 \neq E_2$ &

c has no self-intersection.



* Computation of $g_{[c]}$ is localized to the square.

<u>Prop</u> ([IOS'22])

- Σ is conn., $\mathbb{M} \subset \partial\Sigma$, $|\mathbb{M}| \geq 2$
- $G \neq E_8, F_4, G_2$

$\Rightarrow \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$ is gen'ed by gen. minors of simple Wilson lines.

Then we know:

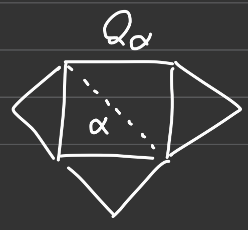
$$\mathcal{A}_{\mathfrak{g},\Sigma} \subseteq \mathcal{U}_{\mathfrak{g},\Sigma} \overset{?}{=} \underset{\langle \text{gen. minors of simple Wilson lines} \rangle}{\underset{\parallel}{\mathcal{O}(\mathcal{A}^\times_{G,\Sigma})}} \subseteq \mathcal{A}_{\mathfrak{g},\Sigma}$$

<u>claim</u> $\mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$

Let Δ be a triangulation, pick $\alpha \in e(\Delta)$

$$\mathcal{A}^{\Delta;\alpha}_{G,\Sigma} := \mathcal{A}^\times_{G,\Sigma}[e(\Delta)\setminus\{\alpha\} ; w_0]$$

$$\overset{\text{cl.}}{\subset} \prod_T \mathcal{A}^\times_{G,T} \times \mathcal{A}^\times_{G,Q_\alpha}$$



$\mathcal{O}(\mathcal{A}^\times_{G,T})$ & $\mathcal{O}(\mathcal{A}^\times_{G,Q_\alpha})$ are upper cluster alg's

[BFZ'05]

By gluing them, we see $\mathcal{O}(\mathcal{A}^{\Delta:\alpha}_{G,\Sigma})$ is an upper cluster alg.

<u>claim</u> : complement of $\bigcup_{\alpha\in e(\Delta)} \mathcal{A}^{\Delta:\alpha}_{G,\Sigma}$ has codim ≥ 2.

$\because$

$$\text{complement} = \bigcup_{\alpha,\beta\in e(\Delta)} \left(\bigsqcup_{u,v\neq w_0} \mathcal{A}^\times_{G,\Sigma}[\alpha,\beta : u,v] \right)$$

$\boxed{///}$

$$\therefore \mathcal{O}(\mathcal{A}^\times_{G,\Sigma}) = \bigcap_{\alpha\in e(\Delta)} \mathcal{O}(\mathcal{A}^{\Delta:\alpha}_{G,\Sigma})$$

upper bound thm.

$$= \bigcap_{\alpha\in e(\Delta)} \mathcal{U}^{\Delta:\alpha}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma}$$

some CV's are freezed

//