@ Summer school on cluster algebras 2023

Cluster K_2-structure on the moduli space of decorated twisted G-local systems

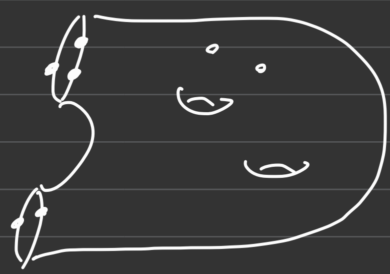
§1. Introduction

- G : simply-conn. semisimple alg. group$/\mathbb{C}$

  e.g. $G = SL_n$ (type A_{n-1})

- $(\Sigma, \mathbb{M})$: marked surface

  $\| $
  Σ



$\rightsquigarrow$ $\mathcal{A}_{G,\Sigma}$: moduli sp. of decorated twisted G-local systems on Σ [FG'06]

$$\mathcal{A}_{G,\Sigma} \xrightarrow{\exists} \left[\mathrm{Hom}(\pi_1(\Sigma), G) / G \right]$$

$\star$ $\mathcal{A}_{G,\Sigma}$ admits a natural <u>cluster K_2-str.</u>:

[FG'06, Le'19, GS'19]

$\exists$ a collection $\{\mathbf{i} = (\{A_i\}_{i\in I}, \varepsilon)\}$ of seeds in $K(\mathcal{A}_{G,\Sigma})$

- giving $\psi_{\mathbf{i}} : \mathcal{A}_{G,\Sigma} \xrightarrow{\sim} (\mathbb{C}^\times)^I$

(cluster coord's)

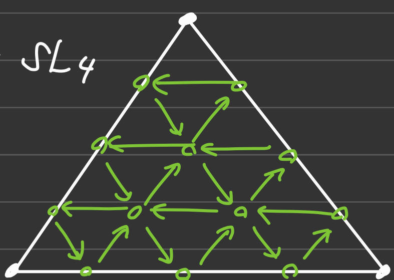
- $\psi_{\mathbf{i}'} \circ \psi_{\mathbf{i}}^{-1}$ being cluster K_2-transf's.

$A_k \cdot A_k' = \prod A_i + \prod A_i$

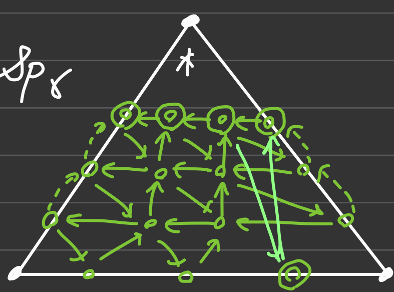
$\exists$ explicit construction $\mathbf{i}_\Delta$

for a decorated triangulation Δ of Σ.

$G = SL_4$



$G = Sp_8$



$\star$ $\mathcal{A}_{SL_2,\Sigma}(\mathbb{R}_{>0}) \cong \widetilde{\mathcal{T}}(\Sigma)$: decorated Teichmüller sp.

[Penner]

* Fock-Goncharov duality : $\mathcal{A}_{G,\Sigma} \longleftrightarrow \mathcal{P}_{G^\vee,\Sigma}$

<u>Theorem</u> (I.-Oya-Shen'22 [IOS'22])

Σ : connected, $\mathbb{M} \subset \partial\Sigma$, $|\mathbb{M}| \geq 2$, $G \neq E_8, F_4, G_2$

$\Rightarrow A_{\mathfrak{g},\Sigma} = U_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$

cluster alg. upper cluster alg. "generic" part

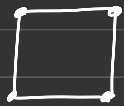
<u>Strategy</u>: $A_{\mathfrak{g},\Sigma} \subseteq U_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^\times_{G,\Sigma}) \subseteq A_{\mathfrak{g},\Sigma}$

- $U_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}_{|i_\Delta|})$,

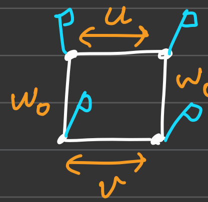
where $\mathcal{A}_{|i_\Delta|} = \bigcup_{i \sim i_\Delta} T_i$: cluster K_2-variety

We need a comparison between $\mathcal{A}^\times_{G,\Sigma}$ & $\mathcal{A}_{|i_\Delta|}$ up to codim. 2

* "stratifications" of $\mathcal{A}_{G,\Sigma}$ and braid varieties.

e.g. $\Sigma =$ 

$$\begin{array}{c} \mathcal{A}^{w}_{G,\Sigma} \cong H^2 \times G^{u,v} \\ \cap \\ \mathcal{A}_{G,\Sigma} \end{array}$$



$w = (u, w_0, v, w_0)$

Goal

- Geometry of the "generic part" $\mathcal{A}^{\times}_{G,\Sigma}$ §2.
- Cluster structure §3. (§6. §7)
- Stratifications of $\mathcal{A}_{G,\Sigma}$ & braid varieties §4. §5

§2. Geometry of $\mathcal{A}_{G,\Sigma}$

§3. Cluster structure

§4. Interpolation of (decorated) flags

§5. Relation to the braid varieties

§6. Cluster structure on the square

§7. Proof of $\mathcal{A}_{g,\Sigma} = \mathcal{U}_{g,\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$

§2. Geometry of $\mathcal{A}_{G,\Sigma}$

<u>Notation from Lie theory</u>

- G : simply-conn. semisimple alg. group $/\mathbb{C}$

<u>Fix</u> $B^{\pm} < G$: a pair of opposite Borel subgroups

- $H := B^+ \cap B^-$: Cartan subgroup
- $U^{\pm} := [B^{\pm}, B^{\pm}]$: unipotent radical
- $W := N_G(H)/H$: Weyl group

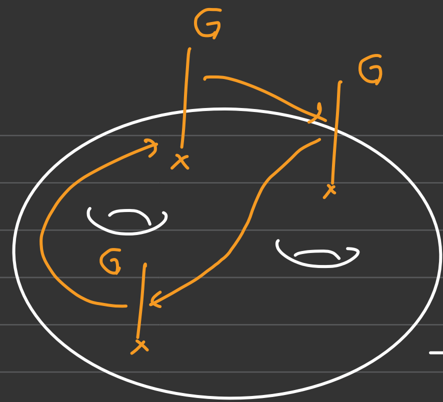
<u>Fix</u> $\{e_s, f_s, \alpha_s^\vee\}_{s\in S}$: Chevalley generators of $\mathfrak{g} = \mathrm{Lie}(G)$

$\rightsquigarrow \varphi_s : SL_2 \longrightarrow G$. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto e_s$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto f_s$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \alpha_s^\vee$

- $\overline{r_s} := \varphi_s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N_G(H)$ $\rightsquigarrow r_s := \overline{r_s} H \in W$

- For $w = r_{s_1} \cdots r_{s_m} \in W$, let $\boxed{\overline{w} := \overline{r_{s_1}} \cdots \overline{r_{s_m}}}$

③ (Twisted) local systems.



<u>Fact</u> (monodromy corresp.)

$$\{\text{flat } G\text{-b'dl on } M\}/\cong \ \xrightarrow{1:1}\ \mathrm{Hom}(\pi_1(M),G)/G$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{L} & \longleftrightarrow & \rho \end{array}$$

$(\Sigma, \mathbb{M})$: a marked surface

$(\mathbb{M} = \underset{\text{int.}}{\mathbb{M}_\circ} \sqcup \underset{\partial.}{\mathbb{M}_\partial})$

$\rightsquigarrow \Sigma^* := \Sigma \setminus \mathbb{M}_\circ$

$\rightsquigarrow T'\Sigma^* := T\Sigma^* \setminus (0\text{-section})$

punctured tangent b'dl

$$0 \longrightarrow \underset{\substack{\| \\ \langle \mathfrak{D} \rangle}}{\pi_1(S^1)} \longrightarrow \pi_1(T'\Sigma^*) \longrightarrow \pi_1(\Sigma^*) \longrightarrow 1 \quad (\text{exact})$$

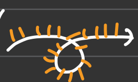
<u>Def</u> A twisted G-local system on Σ

$:\Leftrightarrow$ a G-local system $\mathcal{L}$ on $T'\Sigma^*$ s.t. $\underline{\underline{\rho(\mathfrak{D}) = s_G}}$.

Here, $s_G := \overline{w_0}^{\,2} \in Z(G)$.

e.g. $s_{SL_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -1$, $\quad s_{SL_n} = (-1)^{n-1}$

… "signs"

$\rho\Big($  $\Big) = s_G \cdot \rho\Big($  $\Big)$

<u>Def</u> A decoration of $\mathcal{L}$

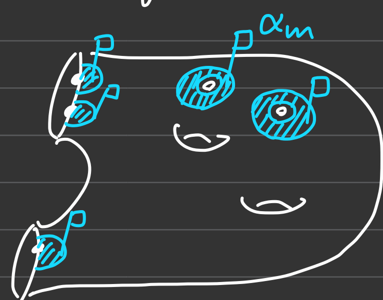
$:\Leftrightarrow$ a flat section α of $\mathcal{L} \times_G (G/U^+)$

$T'\Sigma^*$

(x, v_{out})

defined on a nbd. of $\mathbb{M}$. $\ni x$

<u>What is G/U^+?</u>



① $\mathcal{O}(G/U^+) \cong \bigoplus_\lambda V_\lambda$

(Peter–Weyl)

② $SL_n/U^+ \cong \{(V_\bullet, v_\bullet)\}$, where

$$\begin{cases} V_\bullet : 0 < V_1 < V_2 < \cdots < V_n = \mathbb{C}^n & \dim V_i = i \\ v_\bullet : v_i \in \Lambda^i V_i \setminus \{0\} & (v_n = \mathrm{vol}) \end{cases}$$

③ $\pi : G/U^+ \xrightarrow{H} G/B^+$ — the flag variety

Def $\mathcal{A}_{G,\Sigma}$ parametrizes the pairs $(\mathcal{L}, \alpha)$ up to isomorphisms.

It has a presentation $\mathcal{A}_{G,\Sigma} = [A_{G,\Sigma}/G]$ w/

$$A_{G,\Sigma} \simeq \{ (\rho, (A_m)) \in \mathrm{Hom}^{tw}(\pi_1(T'\Sigma^*), G) \times (G/U^+)^{\mathbb{M}} \text{ s.t. } \rho(\gamma_m).A_m = A_m \text{ for } m \in \mathbb{M}_\circ \}$$

— a quasi-affine variety.

$$\mathcal{O}(\mathcal{A}_{G,\Sigma}) = \mathcal{O}(A_{G,\Sigma})^G.$$

From now on, we assume:

$\mathbb{M} \neq \phi$, $\forall C$: ∂-comp. $\mathbb{M} \cap C \neq \phi$,

$\underbrace{-2\chi(\Sigma^*) + |\mathbb{M}_\partial|}_{}$ > 0.

# of triangles in any ideal triangulation Δ.

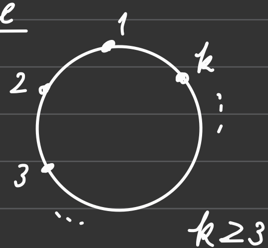
__Rem__ In particular, $\partial\Sigma \neq \emptyset$ or $\mathbb{M}_0 \neq \emptyset$.

$\Rightarrow \pi_1(\Sigma^*)$ is a free group.

$$\Rightarrow \mathcal{A}_{G,\Sigma} \overset{\text{closed}}{\subset} G^{-\chi(\Sigma^*)+1} \times (G/U^+)^{\mathbb{M}}$$

__Example__

$\Sigma =$



$$\Rightarrow \mathcal{A}_{G,\Sigma} \simeq \left[(G/U^+)^k / G \right]$$

$$=: \mathrm{Conf}_k\, G/U^+$$

$$\boxed{\mathcal{O}(\mathrm{Conf}_k\, G/U^+) \simeq \bigoplus_{\lambda_1,\dots,\lambda_k} (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k})^G}$$

- Relative position : $\mathrm{Conf}_2\, G/U^+$
- Cluster coordinates : $\mathrm{Conf}_3\, G/U^+$
- Flips (mutation-equiv.) : $\mathrm{Conf}_4\, G/U^+$

(iii) Relative position (cf. [GS'19, §3.1.6])

__Lemma__ Any pair $(A_1, A_2) \in (G/U^+)^2$ can be translated into a position $(h.[U^+], \overline{w}.[U^+])$ for unique $h \in H$ & $w \in W$.

$\left(\because \text{Bruhat decomp. } G = \bigsqcup_{w \in W} U^- H \overline{w} U^+ \right)$

$$\begin{cases} h(A_1, A_2) := h & (h\text{-distance}) \\ w(A_1, A_2) := w & (w\text{-distance}) \end{cases}$$

depends only on $\pi(A_i) \in G/B^+$

__Def__ (A_1, A_2) : generic $:\Leftrightarrow$ $w(A_1, A_2) = w_0$

__Example__ $SL_2/U^+ \cong \mathbb{C}^2 \setminus \{0\}$ $\quad [U^+] \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\overline{w_0}.[U^+] \longleftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

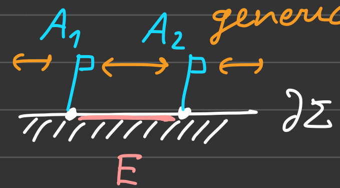
(A_1, A_2) : generic $\Leftrightarrow$ $[A_1] \neq [A_2]$ in $\mathbb{P}^1$.

Lem/Def ([GS19, §3.1.7]) "pinnings"

$$G \xrightarrow{\sim} \{ (A_1, B_2) \in G/U^+ \times G/B^+ \mid w(A_1, B_2) = w_0 \}$$

$$g \longmapsto g.p_{std}, \quad p_{std} := ([U^+], B^-) \qquad =: \mathcal{P}_G$$

Def $\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ subspace s.t.



$$\mathbb{B} := \pi_0(\partial\Sigma \setminus M_\partial) = \{\text{boundary interval}\}$$

For $(\mathcal{L}, \alpha) \in \mathcal{A}^{\times}_{G,\Sigma}$, a section $p_E := (A_1, \pi(A_2))$ of $\mathcal{L} \times_G \mathcal{P}_G$ is associated w/ $E \in \mathbb{B}$.

(III) Wilson lines

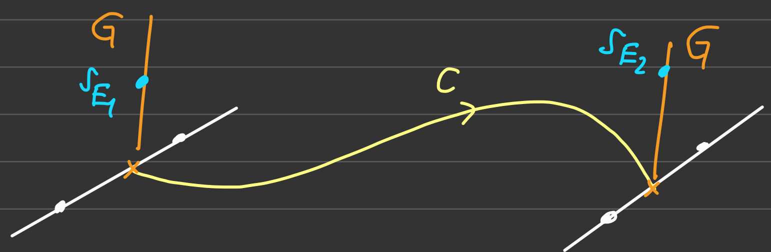
Let $[c]: E_1 \longrightarrow E_2$ be $\overbrace{\text{a path in}}^{\text{homotopy class of}}$ $T'\Sigma^*$.

For $(\mathcal{L}, \alpha) \in \mathcal{A}^{\times}_{G,\Sigma}$, get $p_{E_i} = g_i . p_{std} \in \mathcal{P}_G$ $(i = 1, 2)$

Def $g^{tw}_{[C]}(\mathcal{L},\alpha) := g_1^{-1} g_2$ twisted Wilson line

$g_{[C]}(\mathcal{L},\alpha) := g_1^{-1} g_2 \overline{w_0}$ Wilson line

Topologically:



$\mathcal{L} \cong \mathcal{L} \times_G P_G$

$S_E \longleftrightarrow P_E$

$g^{(tw)}_{[C]} : \mathcal{A}^{\times}_{G,\Sigma} \longrightarrow G$ morphism of stacks.

Prop ([IOS'22]) If $M_0 = \phi$,

$$g^{tw}_{\bullet} : \mathcal{A}^{\times}_{G,\Sigma} \underset{\text{closed}}{\hookrightarrow} \mathrm{Hom}(\underline{\Pi_1(T'\Sigma^*, \mathbb{B})}, G)$$

fund. groupoid w/ obj. $= \mathbb{B}$.

— on affine variety

<u>Cor</u> $\mathcal{M}_\circ = \phi \Rightarrow \mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$ is generated by matrix coefficients of Wilson lines.

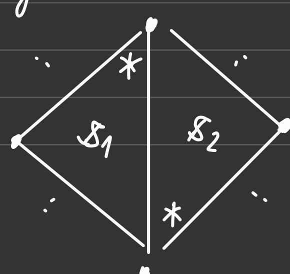
$$\bigoplus_\lambda V_\lambda^* \otimes V_\lambda \xrightarrow{\sim} \mathcal{O}(G)$$

$$f \otimes v \longmapsto (c^\lambda_{f,v} : g \longmapsto \langle f, g.v \rangle_{V_\lambda})$$

§3. Construction of cluster charts (on the generic part) [GS'19]

A <u>decorated triangulation</u> $\boldsymbol{\Delta} = (\Delta, m_\Delta, s_\Delta)$ of Σ consists of:

1) Δ : an ideal triangulation of Σ
2) $m_\Delta = (m_T)_{T \in t(\Delta)}$: choice of corners
3) $s_\Delta = (s_T)_{T \in t(\Delta)}$: reduced words of w_0



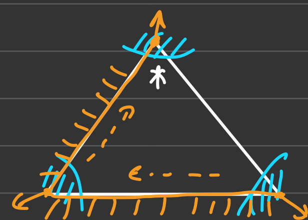
Then we get:

1) Restriction morphism $\psi_\Delta : \mathcal{A}_{G,\Sigma} \longrightarrow \prod_T \mathcal{A}_{G,T}$.

$\exists$ a reconstruction from $\mathcal{A}^\times_{G,T}$.

2) Isomorphism $f_{m_T} : \mathcal{A}_{G,T} \xrightarrow{\sim} \mathrm{Conf}_3\, G/U^+$

w.r.t. the corner m_T.



3) Cluster chart $A_{s_T} : \mathrm{Conf}_3\, G/U^+ \longrightarrow \mathbb{C}^{l(w_0)+2r}$

+ (weighted) quiver

Then the collection

$$A_\Delta := \bigcup_T \psi_\Delta^* f_{m_T}{}^* A_{s_T} =: \{A_i\}_{i\in I}$$

defines a cluster chart $A_\Delta : \mathcal{A}_{G,\Sigma} \xrightarrow{\sim} (\mathbb{C}^*)^I$

⑪ Cluster chart on $\mathrm{Conf}_3\, G/U^+$ [GS'19, §9.1]

Recall : $\mathcal{O}(\mathrm{Conf}_3\, G/U^+) = \bigoplus_{\lambda,\mu,\nu} (V_\lambda \otimes V_\mu \otimes V_\nu)^G$

<u>Idea</u> : Pick up $A_{\lambda,\mu,\nu} \in \underbrace{(V_\lambda \otimes V_\mu \otimes V_\nu)}$

for triples (λ,μ,ν) s.t. dim. $(\text{////}) = 1$

<u>Lemma</u> $\dim (V_\lambda \otimes V_\mu \otimes V_\nu)^G = 1$

if $w.\lambda = \nu^* - \mu$ for some $w \in W$

Given a reduced word $\boldsymbol{s} = (s_1, \cdots, s_N)$ of w_0,

Set $w_{>k} := r_{s_N} \cdots r_{s_{k+1}} \in W$

$$I(\boldsymbol{s}) := \left\{ (\varpi_s, [w_{>k}.\varpi_s]_-, [w_{>k}.\varpi_s]_+^*) \;\middle|\; \begin{matrix} s \in S \\ k = 0, \cdots, N \end{matrix} \right\}$$

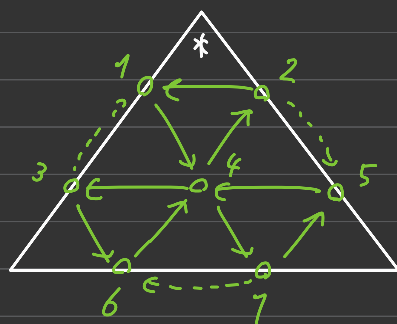
$$\cup \; \{ (0, \varpi_s, \varpi_s^*) \mid s \in S \}$$

Here for $\lambda = \sum_{s \in S} m_s \varpi_s$, $[\lambda]_\pm := \sum_{s \in S} [\pm m_s]_+ \varpi_s$.

$= [\lambda]_+ - [\lambda]_-$

For $(\lambda,\mu,\nu)\in I(\delta)$, $\dim \underline{(V_\lambda\otimes V_\mu\otimes V_\nu)}^G = 1$.

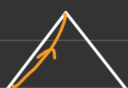
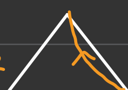
Choose $A_{\lambda,\mu,\nu} \in$ w/ a normalization.



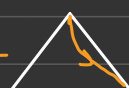
Example

1) Type A_2, $\delta = (1,2,1)$

$I(\delta) = \{$ $(\varpi_2,\varpi_1,0)$, $(\varpi_2,0,\varpi_1)$,

A_1  A_2 

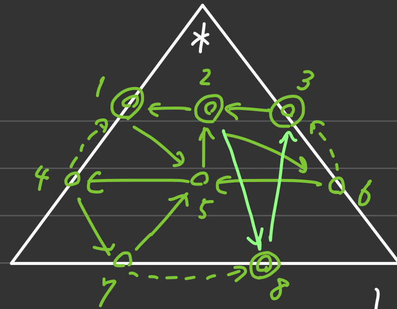
$(\varpi_1,\varpi_2,0)$, $(\varpi_1,\varpi_1,\varpi_1)$, $(\varpi_1,0,\varpi_2)$,

A_3  A_4  A_5 

$(0,\varpi_2,\varpi_1)$, $(0,\varpi_1,\varpi_2)$

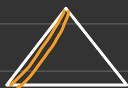
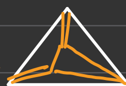
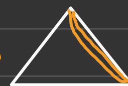
A_6  A_7  $\}$

2) Type C_2 , $\mathbf{s}=(1,2,1,2)$

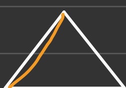
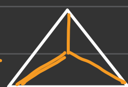
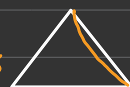


$I(\mathbf{s}) = \{$

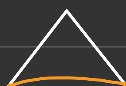
$(\varpi_2,\varpi_2,0)$, $(\varpi_2,\varpi_2,2\varpi_1)$, $(\varpi_2,0,\varpi_2)$

A_1  A_2  A_3 

$(\varpi_1,\varpi_1,0)$, $(\varpi_1,\varpi_2,\varpi_1)$, $(\varpi_1,0,\varpi_1)$

A_4  A_5  A_6 

$(0,\varpi_2,\varpi_2)$, $(0,\varpi_1,\varpi_1)$

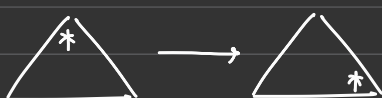
A_7  A_8 

$\}$

... cluster str. is not symmetric under rotations!

Need to show the mutation-equivalence for:

1) flips 

2) rotations 

3) changes of words $\mathbf{s}_T \longrightarrow \mathbf{s}'_T$