Toric degenerations and Newton-Okounkov bodies arising from cluster algebras

Recall $\mathcal{V} = \mathcal{A}$ or $\mathcal{X}$ w/ $\dim_{\mathbb{C}} \mathcal{V} = m$ 2023/08/23

Assumption

- the full Fock-Goncharov conjecture for $\mathcal{V}$
- ($\mathcal{V} = \mathcal{A}$) the exchange matrix ε_s is of full rank for all seeds s.
- ($\mathcal{V} = \mathcal{X}$) consider the skew-symmetric case for simplicity.

$\mathcal{V} = \mathcal{A}$

$\leq_s^{\mathrm{ref}}$ ... a total order on $\mathbb{Z}^m$ which refines the opposite order $\leq_s^{\mathrm{op}}$ of the dominance order $\leq_s$

$\rightsquigarrow$ a g-vector valuation g_s on $\mathrm{up}(\mathcal{A})$

$\rightsquigarrow$ a g-vector valuation g_s on $\mathbb{C}(\mathcal{A})$

$= v^{\mathrm{low}}_{\leq_s^{\mathrm{ref}}}$ on $= \mathbb{C}(A_{1,s}, \ldots, A_{m,s})$

$\mathcal{V} = \mathcal{X}$

$\trianglelefteq^{\mathrm{ref}}$ ... a total order on $\mathbb{Z}^m$ which refines $\trianglelefteq$

$\rightsquigarrow$ a c-vector valuation c_s on $\mathrm{up}(\mathcal{X})$, which is the restriction of a c-vector valuation c_s on $\mathbb{C}(\mathcal{X})$ defined by $c_s := v^{\mathrm{low}}_{\trianglelefteq^{\mathrm{ref}}}$ on $\mathbb{C}(X_{1,s}, \ldots, X_{m,s})$.

$\ni X_{k,s'}$

Remark (see Bossinger–Frías-Medina–Magee–Nájera Chávez 2020)
The unfrozen part of $c_s(X_{k,s'})$ coincides with the usual c-vector in [Nakanishi-Zelevinsky 2012].

3. Newton-Okounkov bodies arising from cluster structures

$$v_s := \begin{cases} g_s & (\text{if } \mathcal{V} = \mathcal{A}) \\ c_s & (\text{if } \mathcal{V} = \mathcal{X}) \end{cases}$$

Z ⋯ an irred. normal proj. var. $/\mathbb{C}$
whose open subsch. is isom. to V
up to codim. ≥ 2 part.

$\mathcal{L}$ ⋯ a very ample line bundle on Z.

$\tau \in H^0(Z, \mathcal{L})$ ⋯ a nonzero section

$$\rightsquigarrow H^0(Z, \mathcal{L}^{\otimes k}) \hookrightarrow \mathbb{C}(Z) = \mathbb{C}(V)$$
$$\sigma \longmapsto \sigma/\tau^k$$
for each $k \in \mathbb{Z}_{\geq 0}$.

<u>Def</u> (Lazarsfeld–Mustata 2009, Kaveh–Khovanskii 2012)

$$\underbrace{S}_{\text{semigroup}} = S(Z, \mathcal{L}, v_S, \tau) := \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, v_S(\sigma/\tau^k)) \mid \sigma \in H^0(Z, \mathcal{L}^{\otimes k}) \setminus \{0\}\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}^m$$

$\underbrace{C}_{\text{convex}} = C(Z, \mathcal{L}, v_S, \tau)$ ⋯ the smallest real closed cone containing S.

$\Delta = \Delta(Z, \mathcal{L}, v_S, \tau) := \{a \in \mathbb{R}^m \mid (1, a) \in C\}$

Newton–Okounkov body

a compact convex set (a convex body)

<u>Thm</u> (Anderson 2013) (Cond. 1) $\Rightarrow$ Δ is a rational conv. polytope.

If $S(Z, \mathcal{L}, v_S, \tau)$ is finitely generated, then there exists a flat morph. $\pi : \mathfrak{X} \to \mathbb{C}$ s.t.

· $\pi^{-1}(t) \simeq Z$ (for all $t \in \mathbb{C}^\times$),

$\cdot\ \pi^{-1}(0) \simeq \mathrm{Proj}(\underbrace{\mathbb{C}[S(Z, \mathcal{L}, V_s, \tau)]}_{\text{a } \mathbb{Z}_{\geq 0}\text{-graded } \mathbb{C}\text{-alg.}})$

a not necessarily normal toric var.

(Cond. 2) $k a \in S$ for some $k \in \mathbb{Z}_{>0}$, then $a \in S$.

If $S(Z, \mathcal{L}, V_s, \tau)$ is saturated in addition,

$$\mathrm{Proj}(\mathbb{C}[S(Z, \mathcal{L}, V_s, \tau)]) = \underbrace{X(\Delta(Z, \mathcal{L}, V_s, \tau))}_{\text{normal proj. toric var. corr. to } \Delta.}$$

In this case, π is called a toric degeneration

As $t \to 0$, $Z = \pi^{-1}(t)$ degenerates into $\pi^{-1}(0)$

$$\pi^{-1}(0) = X(\Delta)$$

Case of $V = \emptyset$

Thm (Berenstein-Fomin-Zelevinsky (2005), Williams (2013))
Flag varieties and their Schubert subvarieties give examples of Z for $V = \emptyset$

Thm [FO]

- For such Z, $\Delta(Z, \mathcal{L}, \mathfrak{g}_s, \tau)$ satisfies (Cond 1) and (Cond 2).
- $\Delta(Z, \mathcal{L}, \mathfrak{g}_s, \tau)$ realizes Berenstein-Littelmann-Zelevinsky's string polytopes and Nakashima-Zelevinsky's polyhedral realization of crystal bases

Case of $\mathcal{V} = \mathcal{X}$

Thm (Rietsch-Williams 2019)

- Grassmann varieties $\mathrm{Gr}_k(\mathbb{C}^n)$ give examples of Z for $\mathcal{V} = \mathcal{X}$.
- $\Delta(\mathrm{Gr}_k(\mathbb{C}^n), \mathcal{L}, C_s, \tau)$ satisfies (Cond 1) and (Cond 2).
- $\Delta(\mathrm{Gr}_k(\mathbb{C}^n), \mathcal{L}, C_s, \tau)$ realizes Rietsch's superpotential polytopes.

4. GHKK's positive sets

Consider $\mathcal{V} = \mathcal{A}$.

Write
$$\vartheta_{q_1} \cdot \vartheta_{q_2} = \sum_{q \in \mathcal{A}^\vee(\mathbb{Z}^T)} \underbrace{\alpha(q_1, q_2, q)}_{\in \mathbb{C}} \vartheta_q$$

Def (GHKK 2018)

A closed subset $\Xi \subseteq \mathcal{A}^\vee(\mathbb{R}^T)$ is positive if for all $d_1, d_2 \in \mathbb{Z}_{\geq 0}$, $q_1 \in d_1\Xi(\mathbb{Z})$, $q_2 \in d_2\Xi(\mathbb{Z})$ and $q \in \mathcal{A}^\vee(\mathbb{Z}^T)$ s.t. $\alpha(q_1, q_2, q) \neq 0$,

it follows that $q \in (d_1 + d_2)\Xi(\mathbb{Z})$,

where

$$d\Xi(\mathbb{Z}) := \begin{cases} (d\Xi) \cap \mathcal{A}^\vee(\mathbb{Z}^T) & \text{if } d > 0 \\ \{q \in \mathcal{A}^\vee(\mathbb{Z}^T) \mid (0, q) \in \overline{\{(r, p) \mid r \in \mathbb{R}_{\geq 0}, p \in r\Xi\}}\} & \text{if } d = 0. \end{cases}$$

If Ξ is bounded, then this is $\{0\}$.

$\rightsquigarrow \tilde{S}_\Xi := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \bigoplus_{q \in d\Xi(\mathbb{Z})} \mathbb{C}\,\vartheta_q x^d$

indeterminate

is a $\mathbb{C}$-subalg. of $\mathrm{up}(\mathcal{A})[x]$.

<u>Thm</u> (Cheung–Magee–Nájera Chávez 2022)

Ξ is positive $\iff$ Ξ is broken line convex, that is, for every $q_1, q_2 \in \Xi \cap \mathcal{A}^\vee(\mathbb{Q}^T)$, each segment of a broken line with endpoints q_1, q_2 is contained in Ξ

<u>GHKK's toric degenerations</u> (under some assump.)

Let $\mathcal{A}_{\mathrm{prin}}$ be the $\mathcal{A}$-cluster variety with principal coefficients.

<u>Properties</u>

- There exists a natural mor.

$$\pi : \mathcal{A}_{\mathrm{prin}} \to (\mathbb{C}^\times)^m = \mathrm{Spec}(\mathbb{C}[N])$$

s.t. $\pi^{-1}(e) \simeq \mathcal{A}$

unit element

$\rightsquigarrow \mathrm{up}(\mathcal{A}_{\mathrm{prin}})$ is a $\mathbb{C}[N]$-alg.

- Let $\rho : \mathcal{A} \simeq \pi^{-1}(e) \hookrightarrow \mathcal{A}_{\mathrm{prin}}$

$\rightsquigarrow (\rho^\vee)^T : \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T) \to \mathcal{A}^\vee(\mathbb{R}^T)$

Let $\Xi \subseteq \mathcal{A}^\vee_{\mathrm{prin}}(\mathbb{R}^T)$ be a full-dim. bounded, rationally defined positive polytope.

$\rightsquigarrow (\rho^\vee)^T(\Xi) \subseteq \mathcal{A}^\vee(\mathbb{R})$ is positive

We set positive

$$\widetilde{\Xi} := \Xi + (N \otimes_{\mathbb{Z}} \mathbb{R})$$

$$\rightsquigarrow \widetilde{S}_{\widetilde{\Xi}} \subseteq \mathrm{up}(\mathcal{A}_{\mathrm{prin}})[x]$$

Thm (GHKK 2018)

- $\mathcal{X}' := \mathrm{Proj}(\widetilde{S}_{\widetilde{\Xi}})$ gives a flat family $\mathcal{X}' \to (\mathbb{C}^\times)^m \ni z$
- the fiber $\mathcal{X}'_z$ compactifies $\mathcal{A}$.
- For each seed s, $\mathcal{X}' \to (\mathbb{C}^\times)^m$ extends to a flat family $\mathcal{X} \to \mathbb{C}^m$
  s.t. the central fiber $\mathcal{X}_0 = X((\rho^\vee)^T(\Xi)_s)$

Thm [FO]

$(\rho^\vee)^T(\Xi)_s$ is a Newton-Okounkov body $\Delta(\mathcal{X}_e, \mathcal{O}(1), g_s, x)$.

Q. How about the converse?

When is $\Delta(Z, \mathcal{L}, g_s, \tau)$ a positive set?

[BCMNC]

gave a sufficient cond. for $\Delta(Z, \mathcal{L}, g_s, \tau)$ to be positive.