Toric degenerations and Newton-Okounkov bodies arising from cluster algebras

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Main references

[FO] F.-Oya,
Newton-Okounkov polytopes of Schubert varieties arising from cluster structures, arXiv:2002.09912v2.

[BCMNC] Bossinger-Cheung-Magee-Nájera Chávez,
Newton-Okounkov bodies and minimal models for cluster varieties, arXiv:2305.04903v1.

1. Intro.

Toric theory

$$(Z(Q), \mathcal{L}(Q)) \longleftrightarrow Q$$

$Z(Q)$: normal proj. toric var.

$\mathcal{L}(Q)$: torus-equiv. ample line bundle

Q: an integral polytope

Want to apply toric theory to non-toric varieties.

$\longrightarrow$ Degenerations to toric varieties are useful
toric degenerations

Q How to construct such degenerations?

A Using cluster algebras!

2. Cluster-theoretic valuations

R : an integral domain w/ a $\mathbb{C}$-alg. str.
$\leq$: a total order on $\mathbb{Z}^m$ s.t. $\mathbb{a} \leq \mathbb{a}'$, $\mathbb{b} \leq \mathbb{b}'$
$(m \in \mathbb{Z}_{>0})$ $\Rightarrow \mathbb{a}+\mathbb{b} \leq \mathbb{a}'+\mathbb{b}'$

Def A valuation on R is a map
$$\nu : R \setminus \{0\} \to \mathbb{Z}^m$$
s.t. for $f, g \in R \setminus \{0\}$ and $c \in \mathbb{C}^\times$,
$$\nu(f \cdot g) = \nu(f) + \nu(g),$$
$$\nu(cf) = \nu(f)$$
$$\nu(f+g) \geq \min\{\nu(f), \nu(g)\} \text{ unless } f+g=0.$$

Ex. $(R = \mathbb{C}(t_1, \ldots, t_m))$

Define $\nu_{\leq}^{\mathrm{low}} : \mathbb{C}(t_1, \ldots, t_m) \setminus \{0\} \to \mathbb{Z}^m$ by

- $\nu_{\leq}^{\mathrm{low}}(f) := (a_1, \ldots, a_m) \Leftrightarrow f = c\, t_1^{a_1} \cdots t_m^{a_m} + \left(\begin{array}{l}\text{terms with higher}\\ \text{exponents w.r.t. } \leq\end{array}\right)$

for $f \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \setminus \{0\}$,
where $c \in \mathbb{C}^\times$.

- $\nu_{\leq}^{\mathrm{low}}(f/g) = \nu_{\leq}^{\mathrm{low}}(f) - \nu_{\leq}^{\mathrm{low}}(g)$

$\rightsquigarrow$ $\nu_{\leq}^{\mathrm{low}}$ is a valuation, called the lowest term valuation

A valuation ν induces a filtration on R by

$$R_{\mathbb{a}} := \{f \in R \setminus \{0\} \mid \nu(f) \geq \mathbb{a}\} \cup \{0\} \subseteq R$$
$(\mathbb{a} \in \mathbb{Z}^m)$ $\cdots \mathbb{C}$-linear subsp.

$\rightsquigarrow \mathrm{gr}_\nu(R) := \bigoplus_{a \in \mathbb{Z}^m} R_a \Big/ \left(\sum_{a < b} R_b\right)$

the associated graded alg.

$\rightsquigarrow$ a degeneration of R

<u>Def</u> An adapted basis for ν is
a $\mathbb{C}$-basis $\mathcal{B}$ of R s.t. $\mathcal{B} \cap R_a$ is
a $\mathbb{C}$-basis of R_a for all a.
$\longrightarrow$ induces a $\mathbb{C}$-basis of $\mathrm{gr}_\nu(R)$.

Let $\mathcal{V}$ be a cluster variety $\mathcal{A}$ or $\mathcal{X}$ w/ $\dim_{\mathbb{C}} \mathcal{V} = m$.

$$\longrightarrow \mathcal{V} = \bigcup_{s:\text{seeds}} \underbrace{\mathcal{V}_s}_{\substack{\shortparallel \\ (\mathbb{C}^\times)^m \\ \shortparallel \\ \mathrm{Spec}(\mathbb{C}[V_{j;s}^{\pm 1} \mid j \in J := \{1,\dots,m\}])}} \overset{\text{Fock-Goncharov dual}}{\longleftrightarrow} \mathcal{V}^\vee = \bigcup_{s:\text{seeds}} \mathcal{V}_s^\vee$$

$(V = A \text{ or } X)$

$J = J_{uf} \sqcup J_{fr}$

unfrozen frozen

Assume the full FG conj. for $\mathcal{V}$, that is,
the upper cluster alg. $\mathrm{up}(\mathcal{V}) := H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$
has the theta func. basis $\{\vartheta_q \mid q \in \mathcal{V}^\vee(\mathbb{Z}^T)\}$,

the semifield $\mathbb{Z}$ with $\oplus = \max$, $\odot = +$

where

$$\mathcal{V}^\vee(\mathbb{Z}^T) = \bigcup_{s:\text{seeds}} \underbrace{\mathcal{V}_s^\vee(\mathbb{Z}^T)}_{\substack{\shortparallel \\ \mathbb{Z}^m}} = \mathcal{V}_s^\vee(\mathbb{Z}^T) = \mathbb{Z}^m$$

glued by the tropicalization
μ_k^T of the mutation μ_k for $\mathcal{V}^\vee$
bijective

Similarly, $\mathcal{V}^\vee(\mathbb{R}^T) = \bigcup_{s:\text{seeds}} \underbrace{\mathcal{V}_s(\mathbb{R}^T)}_{\mathbb{R}^m} = \mathcal{V}_s(\mathbb{R}^T) = \mathbb{R}^m$

$$\tilde{U} \subseteq \mathcal{V}^\vee(\mathbb{R}^T) \ni g \longmapsto g_s \in \mathbb{R}^m \supseteq \tilde{U}_s, \qquad \tilde{U} \longmapsto \tilde{U}_s$$

Case of $\mathcal{V} = \mathcal{A}$

Assume The exchange matrix $\varepsilon_s = (\varepsilon_{i,j}^{(s)})_{i \in J_{uf}, j \in J}$ is of full rank for some seed s ($\Rightarrow$ for all seeds s)

$\Downarrow$ well-defined.

Def (Qin 2017)
For each seed s, define a partial order $\leq_s$ on $\mathbb{Z}^J = \mathbb{Z}^m$ by dominance order

$$g' \leq_s g \iff g' - g \in \sum_{j \in J_{uf}} \mathbb{Z}_{\geq 0} (\varepsilon_{j,e}^{(s)})_{e \in J}.$$

Thm (see GHKK 2018)

For each seed s and $g \in \mathcal{A}^\vee(\mathbb{Z}^T)$,

$$\vartheta_g \in A_{1,s}^{g_1} \cdots A_{m,s}^{g_m} + \sum_{\substack{g' = (g_1', \dots, g_m') \in \mathbb{Z}^m \\ g' <_s g_s}} \mathbb{Z} A_{1,s}^{g_1'} \cdots A_{m,s}^{g_m'}$$

where $g_s = (g_1, \dots, g_m) \in \mathbb{Z}^m$

$\longrightarrow$ ϑ_g is pointed for s.

We write $g_s(\vartheta_g) := g_s = (g_1, \dots, g_m)$
the g-vector of ϑ_g

Def ([FO])

Fix a total order $\leq_s^{\mathrm{ref}}$ on $\mathbb{Z}^m$ refining the opposite order $\leq_s^{\mathrm{op}}$ of $\leq_s$.

Define a valuation g_s on $\mathrm{up}(\mathcal{A})$ by

g-vector valuation

$$g_s(c_1\theta_{q_1}+\cdots+c_k\theta_{q_k}) = \underbrace{\min_{1\leq \ell\leq k}}_{\text{w.r.t. } \leq_s^{\mathrm{ref}}} g_s(\theta_{q_\ell})$$
$(c_1,\dots,c_k\in\mathbb{C}^\times)$

Prop

- $g_s(\mathrm{up}(\mathcal{A})\setminus\{0\}) = \{g_s(\theta_q) \mid q\in\mathcal{A}^\vee(\mathbb{Z}^T)\}$
- $\{\theta_q \mid q\in\mathcal{A}^\vee(\mathbb{Z}^T)\}$ is an adapted basis for g_s.

Since $\mathbb{C}(\mathcal{A})$ is the fraction field of $\mathrm{up}(\mathcal{A})$, the valuation g_s can be uniquely extended to a valuation g_s on $\mathbb{C}(\mathcal{A})$ by

$$g_s\left(\frac{\sigma}{\tau}\right) := g_s(\sigma) - g_s(\tau) \text{ for } \sigma,\tau\in \mathrm{up}(\mathcal{A}).$$

Prop Under $\mathbb{C}(\mathcal{A}) = \mathbb{C}(A_{1,s},\dots,A_{m,s})$, we have $g_s = \nu^{\mathrm{low}}_{\leq_s^{\mathrm{ref}}}$

Remark
- g_s can be defined without using theta functions.
- There exist other adapted basis for g_s coming from cluster theory (a common triangular basis, a generic basis, ...)

Case of $\mathcal{V}=\mathcal{X}$

For simplicity, we consider only the skew-symmetric case

Def Define a partial order $\trianglelefteq$ on $\mathbb{Z}^J$ by $c \trianglelefteq c' \Leftrightarrow c' - c \in \mathbb{Z}^m_{\geq 0}$

Thm (see GHKK 2018)

$$\theta_g \in X_{1,s}^{c_1} \cdots X_{m,s}^{c_m} + \sum_{\substack{c'=(c'_1,\dots,c'_m)\in\mathbb{Z}^m \\ g_s \ntrianglelefteq c'}} \mathbb{Z} X_{1,s}^{c'_1} \cdots X_{m,s}^{c'_m}$$

$(g_s = (c_1, \dots, c_m))$

We write $c_s(\theta_g) := g_s$

c-vector of θ_g.

Def (see [BCMNC])

Fix a total order $\trianglelefteq^{\mathrm{ref}}$ on $\mathbb{Z}^J$ which refines $\trianglelefteq$.

$\rightarrow$ Define a valuation c_s on $\mathrm{up}(\mathcal{X})$ by

c-vector valuation

$$c_s(a_1\theta_{g_1} + \cdots + a_k\theta_{g_k}) = \min_{1\leq l\leq k} c_s(\theta_{g_l})$$

$(a_1, \dots, a_k \in \mathbb{C}^\times)$

c_s satisfies properties similar to g_s

Let $p : \mathcal{A} \to \mathcal{X}$ be the cluster ensemble map.

$\rightsquigarrow p^* : \mathrm{up}(\mathcal{X}) \to \mathrm{up}(\mathcal{A})$

Under p^*, g_s and c_s are related by the tropicalization of $p^\vee : \mathcal{X}^\vee \to \mathcal{A}^\vee$.