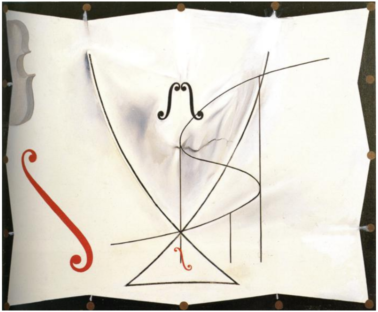
Cluster algebras and symplectic topology III

Summer School on Cluster Algebras 2023



Roger Casals (UC Davis)

August 23rd 2023

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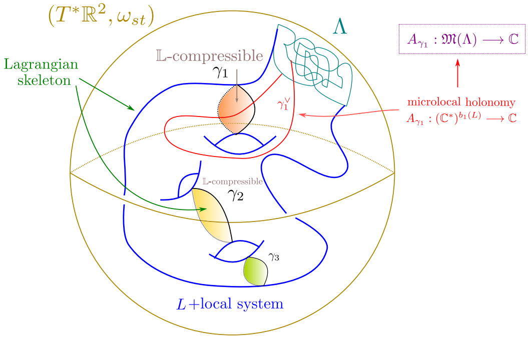
Legendrian Λ $\longrightarrow$ Moduli space $\mathfrak{M}(\Lambda)$

Lagrangian filling L of Λ $\longrightarrow$ Chart $T_L \cong H^1(L, \mathbb{C}^*) \subset \mathfrak{M}(\Lambda)$

$\mathbb{L}$-compressing system $\mathfrak{D}$ for L $\longrightarrow$ Quiver $Q(\mathfrak{D})$ for T_L

Disk $D_i \in \mathfrak{D}$ $\longrightarrow$ Function $A_i : T_L \longrightarrow \mathbb{C}^*$



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3. *Central question*: **surjectivity and injectivity of $\mathfrak{C}^\circ$ and $\mathfrak{C}$?**
   (If $\mathfrak{C}$ surjects, then $\mathfrak{C}^\circ$ surjects onto $\mathrm{Seed}(X(\Lambda_\beta, T))$, subset of $\mathrm{Toric}(X(\Lambda_\beta, T))$.)

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3. In previous work: symplectically realized elements of the cluster modular group and showed that it surjects onto some infinite families in $\text{Seed}(X(\Lambda_\beta, T))$. ($\leftarrow$ braid group actions in $\text{Gr}(k, kn)$, Donaldson-Thomas)

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Today: Focus on surjectivity.

Also, we restrict to Λ_β with $\beta = w_0 \gamma w_0$.
(We write Λ_β to mean $\Lambda_{w_0 \beta w_0}$ onward.)

Main result I: Surjectivity

Theorem (2023)

Let $\Lambda_\beta \subset (\mathbb{R}^3, \xi_{st})$ be the Legendrian link associated to a positive braid word β and $X(\Lambda_\beta)$ its augmentation variety, with one marked point per component. Then the set-theoretic map

$$\mathfrak{C} : Lag^c(\Lambda_\beta) \longrightarrow Seed(X(\Lambda_\beta))$$

is surjective, i.e. each cluster seed is induced by an embedded exact Lagrangian filling endowed with an $\mathbb{L}$-compressing system.

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- Even better: show that **any sequence of cluster mutations can be realized by a sequence of Lagrangian disk surgeries**.
- Non-trivial problem: in algebra mutation automatically removes 2-cycles (by fiat), in geometry this is not at all immediate, e.g. algebraic intersection 0 but geometric intersection 2.

Main result II: The technical statement

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Let $\Lambda_\beta \subset (\mathbb{R}^3, \xi_{st})$ be the Legendrian link associated to a positive braid word β. Then there exists an embedded exact Lagrangian filling $L \subset (\mathbb{R}^4, \lambda_{st})$ of Λ_β and an $\mathbb{L}$-compressing system Γ for L such that the following holds:

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(i) *If $\mu_{v_\ell} \dots \mu_{v_1}$ is any sequence of mutations, where $v_1, \dots, v_\ell$ are mutable vertices of the quiver $Q(\mathfrak{c}(L,\Gamma))$ associated to the cluster seed $\mathfrak{c}(L,\Gamma)$ of L in $\mathbb{C}[X(\Lambda_\beta)]$, then there exists a sequence of embedded exact Lagrangian fillings L_k of Λ_β, each equipped with an $\mathbb{L}$-compressing system Γ_k, with associated cluster seeds*

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Non-trivial problem (alternative): need to show that an $\mathbb{L}$-compressing system persists under Lagrangian disk surgeries.

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(iii) If one applies Lagrangian disk surgery along a disk whose boundary is a vertex part of a 2-cycle, then it results in an **immersed curve**.

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The issue was the bigon: need to understand when they can be removed!

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- Let Σ be an oriented surface and $\mathcal{C} = \{\gamma_1, \ldots, \gamma_b\}$, $b \in \mathbb{N}$, a collection of embedded oriented closed connected curves $\gamma_i \subset \Sigma$.
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  (or weaker "bigon sides" condition)
- By definition, the quiver $Q(\mathcal{C})$ has vertices the γ_i and arrows their *geometric* intersections.
- We want to build a potential $W(\mathcal{C}) \in HH_0(Q(\mathcal{C}))$ for $Q(\mathcal{C})$ that keeps track of the *polygons* in Σ bounded by curves in $\mathcal{C}$.

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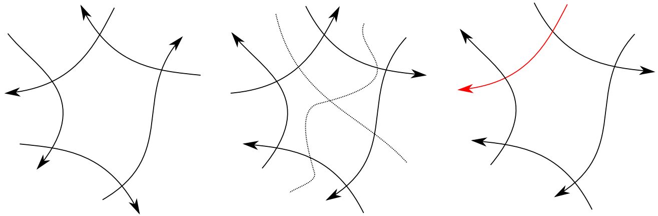
where $\Gamma_\ell^\pm = \{\ell\text{-gons bounded by } \mathcal{C} \text{ which are } \pm\text{-oriented}\}$. (Here σ is sign for $\mathbb{Z}$.)



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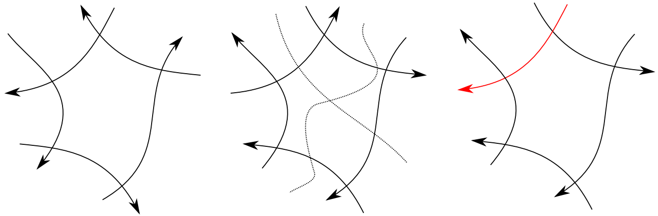

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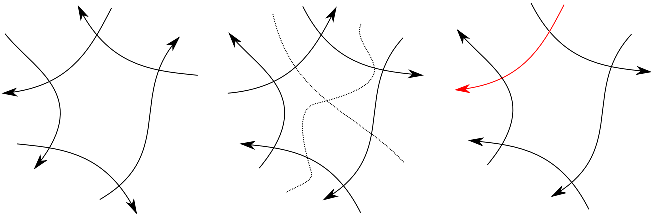

2. By definition, $(Q(\mathcal{C}), W(\mathcal{C}))$ is the curve quiver with potential of $\mathcal{C}$.
3. There is a notion of QP-mutation due to Derksen-Weyman-Zelevinsky (DWZ). Also, we consider QPs up to right-equivalence.

What properties do we need for such curve QPs?

In order to get rid of bigons, we use the following:

Theorem (Hass-Scott Algorithm)

Let $\mathcal{C}_0$ be a configuration with a collection of bigons $\{B_1, \ldots, B_m\}$. Then, for any $i \in [m]$, there exists a sequence of triple point moves and one local bigon move on $\mathcal{C}_0$ that yields a new configuration $\mathcal{C}_1$ such that the collection of bigons of $\mathcal{C}_1$ is $\{B_1, \ldots, B_m\} \setminus \{B_i\}$. □

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- We must **understand behavior of curve QPs under triple point moves and bigon moves**. ($\rightarrow$ change in quiver and potential)

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In order to get rid of bigons, we use the following:

Theorem (Hass-Scott Algorithm)

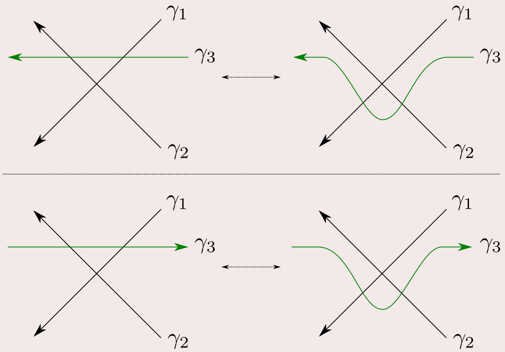
Let $\mathcal{C}_0$ be a configuration with a collection of bigons $\{B_1, \ldots, B_m\}$. Then, for any $i \in [m]$, there exists a sequence of triple point moves and one local bigon move on $\mathcal{C}_0$ that yields a new configuration $\mathcal{C}_1$ such that the collection of bigons of $\mathcal{C}_1$ is $\{B_1, \ldots, B_m\} \setminus \{B_i\}$. □

- We must **understand behavior of curve QPs under triple point moves and bigon moves**. ($\rightarrow$ change in quiver and potential)
- We know that $Q(\mathcal{C})$ changes according to quiver mutation under Lagrangian surgery. We must still show that $W(\mathcal{C})$ changes according to the DWZ's QP-mutation. ($\rightarrow$ see how polygons change)

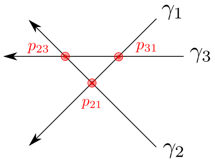
Invariance of Curve QPs under planar moves I

Proposition

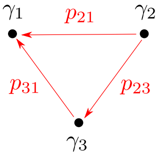
Let $(Q(\mathcal{C}), W(\mathcal{C}))$ be a curve QP associated to $\mathcal{C}$. Then $(Q(\mathcal{C}), W(\mathcal{C}))$ is invariant under triple point moves, up to right-equivalence.



Proof of invariance (triple move): Case I



(i)



$$\begin{pmatrix} 0 & p_{21} + p_{23}p_{31} & p_{31} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{23} & 0 \end{pmatrix}$$

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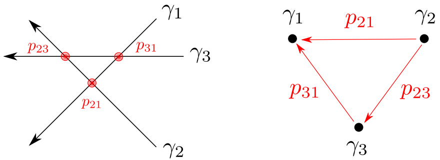
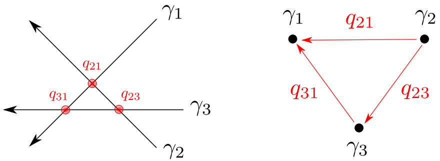


$$\begin{pmatrix} 0 & p_{21} + p_{23}p_{31} & p_{31} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{23} & 0 \end{pmatrix}$$

(ii)



$$\begin{pmatrix} 0 & q_{21} & q_{31} \\ q_{21} - q_{23}q_{31} & 0 & q_{23} \\ q_{31} & q_{23} & 0 \end{pmatrix}$$

Proof of invariance (triple move): Case I (continued)

- Right-equivalence: $p_{21} \mapsto p_{21} - p_{23}p_{31}$ and identity for the rest. Then matrices will match, indeed

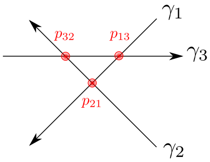
$$\begin{pmatrix} 0 & p_{21} + p_{23}p_{31} & p_{31} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{23} & 0 \end{pmatrix}$$

now becomes

$$\begin{pmatrix} 0 & (p_{21} - p_{23}p_{31}) + p_{23}p_{31} & p_{31} \\ (p_{21} - p_{23}p_{31}) & 0 & p_{23} \\ p_{31} & p_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & p_{21} & p_{31} \\ p_{21} - p_{23}p_{31} & 0 & p_{23} \\ p_{31} & p_{23} & 0 \end{pmatrix},$$

which is the second matrix we had relabeled, thus concludes first case.

Proof of invariance (triple move): Case II



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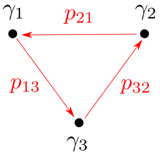

$$\begin{pmatrix} 0 & p_{21} & p_{13} \\ p_{21} & 0 & p_{32} \\ p_{13} & p_{32} & 0 \end{pmatrix}, \quad \text{plus monomial } p_{13}p_{32}p_{21}$$

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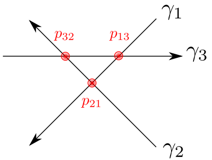

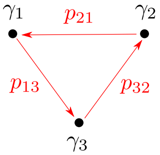

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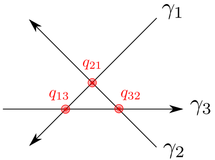

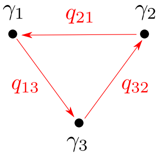

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Bigon moves

Eliminating bigons: Extracting the reduced part

By [DWZ], every (Q, W) breaks into a *trivial* and *reduced* parts: $(Q_{triv}, W_{triv}) \oplus (Q_{red}, W_{red})$. (Intuitively, *trivial* contains 2-cycles seen by W.)

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Proposition

Let $(Q(\mathcal{C}), W(\mathcal{C}))$ be a curve QP associated to $\mathcal{C}$ and $\mathcal{C}_{red}$ the result of applying the Hass-Scott algorithm removing all bigons. Then

$$(Q(\mathcal{C}_{red}), W(\mathcal{C}_{red})) = (Q(\mathcal{C})_{red}, W(\mathcal{C})_{red})$$

is the reduced part of $(Q(\mathcal{C}), W(\mathcal{C}))$.

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Therefore, in the context of curve QP, we know that extracting the reduced part of curve QP is achieved by removing bigons.

Curve QPs under Lagrangian disk surgery

Disk surgery: Inducing QP-mutations

By [DWZ], QP-mutation consists of a quiver mutation *without eliminating* 2-cycles, a change in W, and then taking the reduced part.

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$$(Q(\mu_\gamma(\mathcal{C})), W(\mu_\gamma(\mathcal{C}))) = (\mu_\gamma(Q(\mathcal{C})), \mu_\gamma(W(\mathcal{C}))).$$

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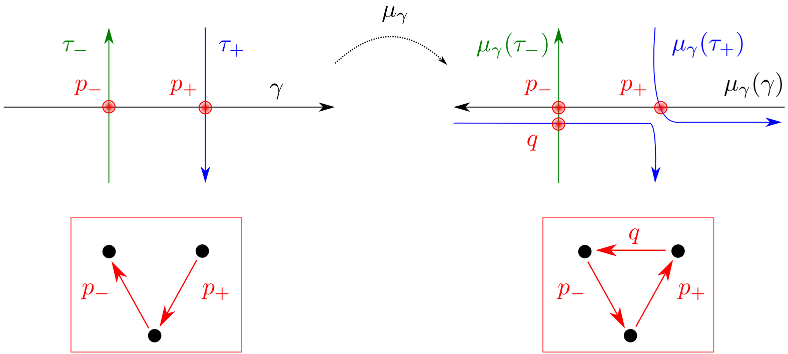
Therefore, in the context of curve QP, performing a γ-exchange (e.g. from Lagrangian disk surgery) is a QP-mutation.

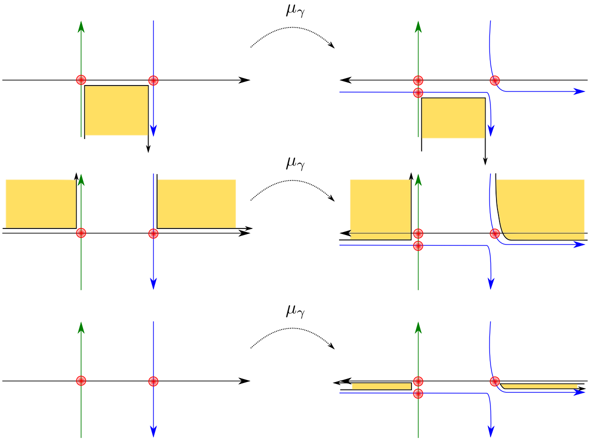
Example of QP-mutation from γ-exchange

Let us work out the change in the quiver in a simple scenario:



Example of QP-mutation from γ-exchange (continued)

The change in polygons in this scenario:



Steps for surjectivity

1. Construct a filling L and an $\mathbb{L}$-compressing system $\mathfrak{D}$ such that the associated curve QP $(Q(\mathfrak{D}), W(\mathfrak{D}))$ is **non-degenerate**.

   Non-degeneracy guarantees that no 2-cycles ever appear when mutating $(Q(\mathfrak{D}), W(\mathfrak{D}))$, *so you can mutate forever*. How is this achieved?

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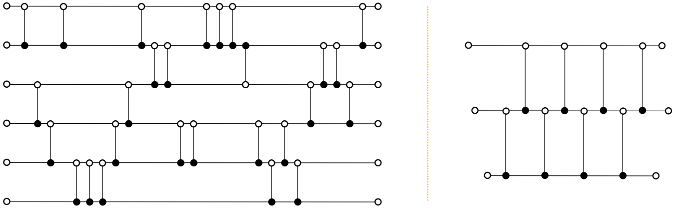
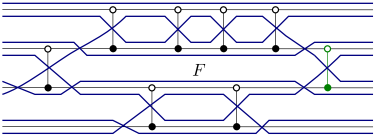
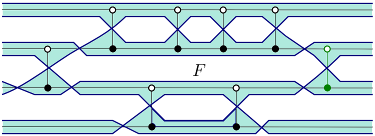
2. The construction uses *conjugate surfaces* associated to plabic fences:



Steps for surjectivity II

3. Use that *conjugate surface can be made an embedded exact Lagrangian filling* and plabic faces give $\mathbb{L}$-compressing disks. ($\leftarrow$ *weaves work too*)





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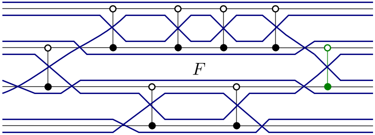

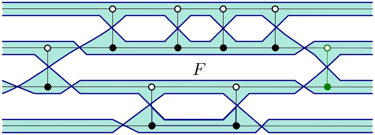

4. Prove that the resulting curve QP is **rigid**, which implies non-degenerate. (Rigid is intuitively that there are no non-trivial infinitesimal deformations: trace space of Jacobian algebra is just the ground ring.)

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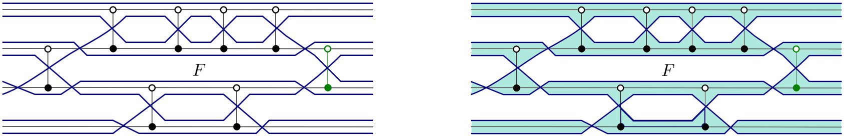

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   This is achieved via induction, using an interesting combinatorial property of these quivers: the rightmost vertex can always be turned into a source/sink via mutations. ($\leftarrow$ triangular extensions)

A few questions

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5. Generalize this program for a general Λ. This includes building the right $\mathbb{L}$-compressing systems *and* understanding what it means for a dg-category (or at least a D^--stack) to be a cluster algebra.

We reached the end.

Thanks a lot for attending these lectures!