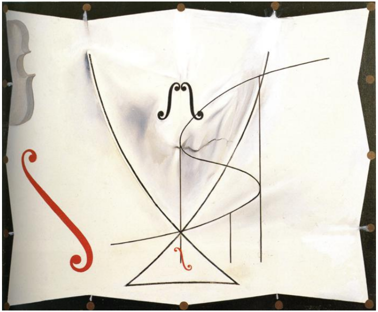
Cluster algebras and symplectic topology II

Summer School on Cluster Algebras 2023



Roger Casals (UC Davis)

August 22nd 2023

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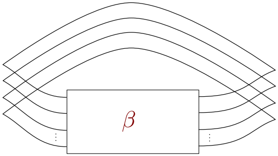

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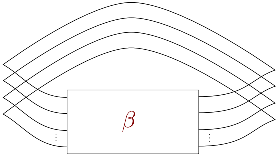

- $\mathfrak{M}(\Lambda_\beta)$ is smooth affine variety:

$$\left\{(\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{l(\beta)} = \mathcal{F}_0) \in (\mathrm{Fl}_m^{aff})^{l(\beta)} : \mathcal{F}_{j-1} \xrightarrow{s_{i_j}} \mathcal{F}_j, \forall j \in [l(\beta)]\right\} / \, \mathrm{GL}_m(\mathbb{C}).$$

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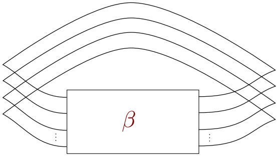

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- *New technique*: **weaves**, a planar diagrammatic calculus to construct and study Lagrangian fillings of Λ_β and their $\mathbb{L}$-compressing systems.

Recap of available ingredients

Symplectic geometry behind $\mathfrak{M}(\Lambda)$

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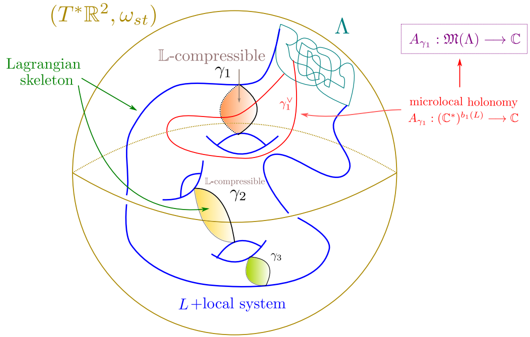
Legendrian Λ	$\longrightarrow$	Moduli space $\mathfrak{M}(\Lambda)$
Lagrangian filling L of Λ	$\longrightarrow$	Chart $T_L \cong H^1(L, \mathbb{C}^*) \subset \mathfrak{M}(\Lambda)$
$\mathbb{L}$-compressing system $\mathfrak{D}$ for L	$\longrightarrow$	Quiver $Q(\mathfrak{D})$ for T_L
Disk $D_i \in \mathfrak{D}$	$\longrightarrow$	Function $A_i : T_L \longrightarrow \mathbb{C}^*$



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   Alternatively, show $\mathbb{C}[\mathfrak{M}(\Lambda_\beta)] \subset A_{\mathfrak{s}_L}$ directly by proving generators z_i of $\mathbb{C}[\mathfrak{M}(\Lambda_\beta)]$ are all cluster variables. ($\leftarrow$ cyclic rotation is quasi-cluster)

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- *Caveat*: the above is done for the case $\beta = w_0 \gamma w_0$: general case follows by an additional localization procedure. (← partial $\mathbb{L}$-compressible systems)

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  Weaves are particular *singular* surfaces in $\mathbb{R}^3$, whose singular set can be completely encoded by planar diagrams (plus permutation labels). These planar diagrams are also referred as *weaves* if context is clear.

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Singular surfaces in 3D: wavefront singularities

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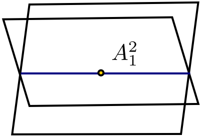
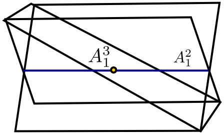
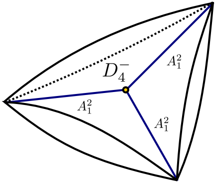
We only draw fronts with these three singularities:







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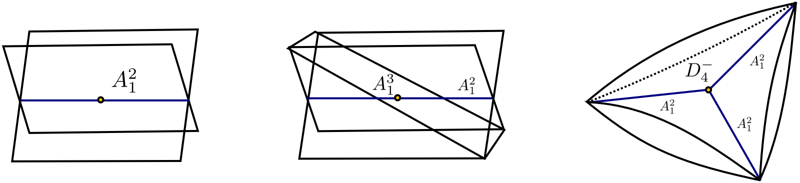

- By definition, a **weave** is any singular surface in $\mathbb{R}^3$ obtained by gluing these three singularities. ($\leftarrow$ n-weave if we use n sheets above.)

  Their (non-crossing) singular set is codimension-2: D_4^- is real part of holomorphic Legendrian singularity. ($\leftarrow$ quadratic differential $z \cdot dz \otimes dz$)

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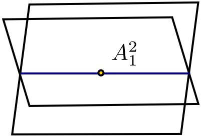

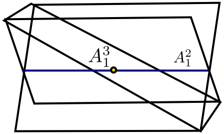

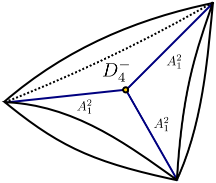

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- *Project from above* to encode **with planar diagrams**.

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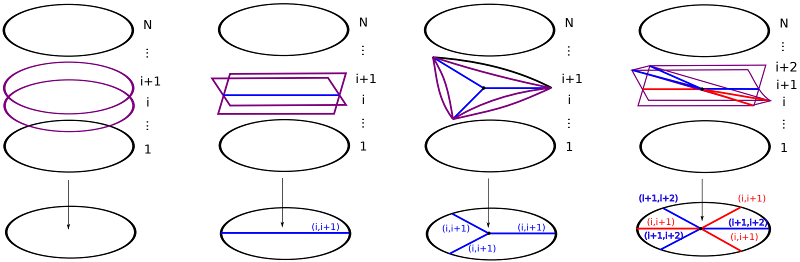

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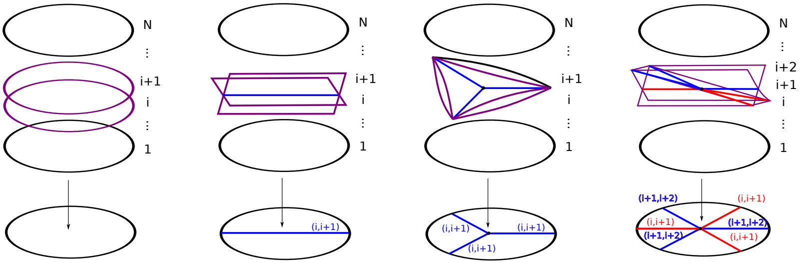

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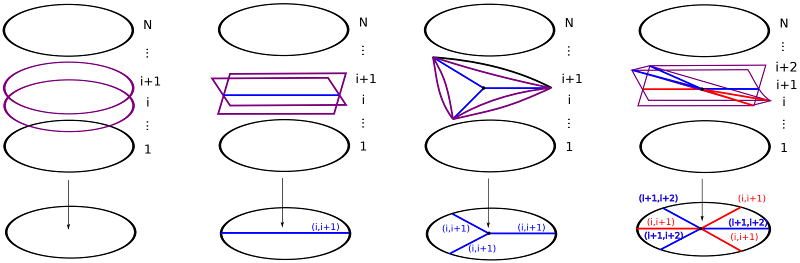

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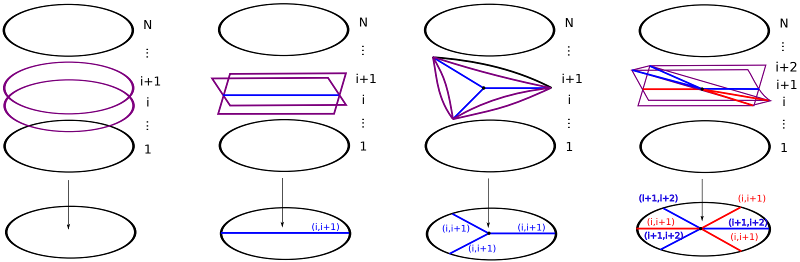

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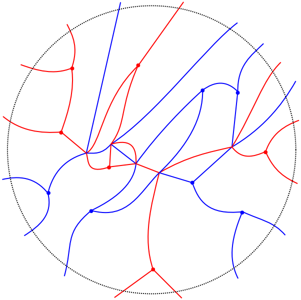
(iv) Fillings of Λ_β: β is braid word around boundary.

More examples



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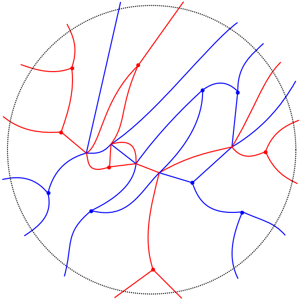

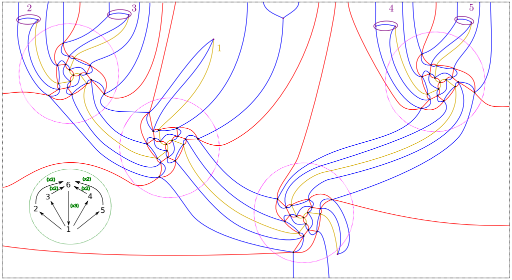

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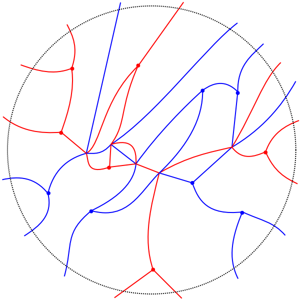

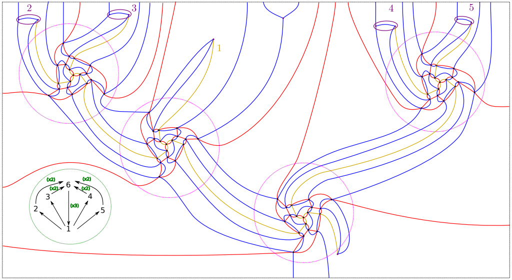

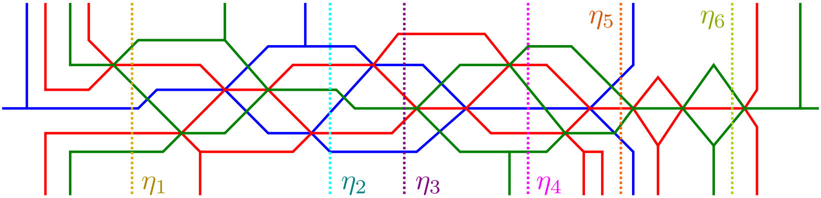

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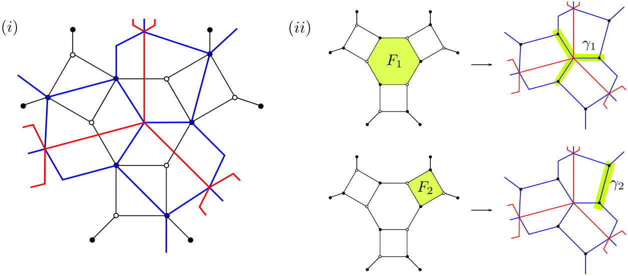

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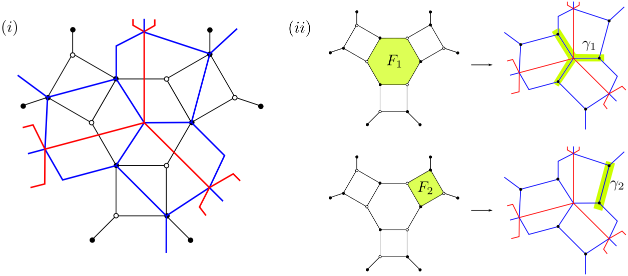

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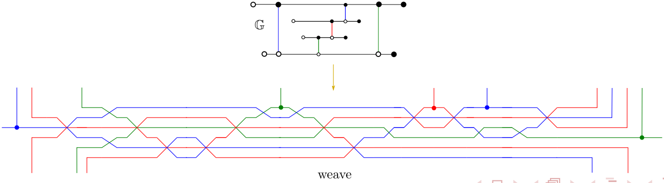

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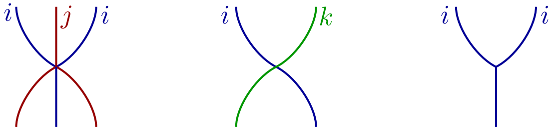

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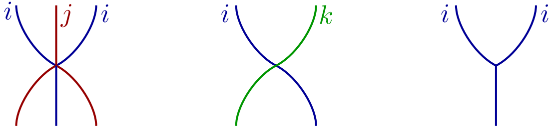

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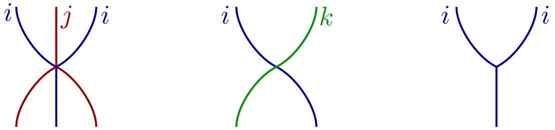

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- Need appropriate basis for $H_1(L,\mathbb{Z})$, to obtain right quiver and cluster variables: *Demazure weaves* provide such basis using *Demazure cycles*.

Demazure weaves

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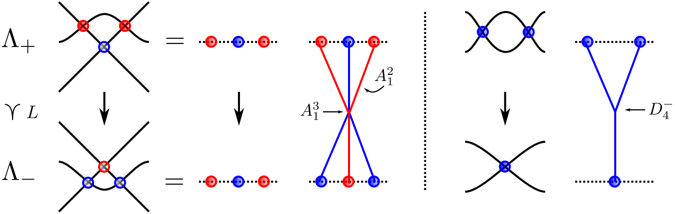
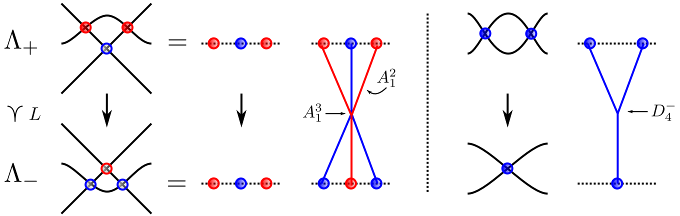


(ii) **Cyclic shift** concordance also useful. In general, given $\beta \in Br_n^+$ with $\delta(\beta) = \Delta$, the **RIII** and D_4^- **moves** above suffice to bring β to Δ.

Demazure weaves

Demazure weaves: encode **spatial fronts** that construct embedded exact Lagrangians. For fillings of Λ_β: β **top** & w_0 **bottom**.

(i) Focus on the **RIII concordance** (A_1^3) and the D_4^- **cobordism**.

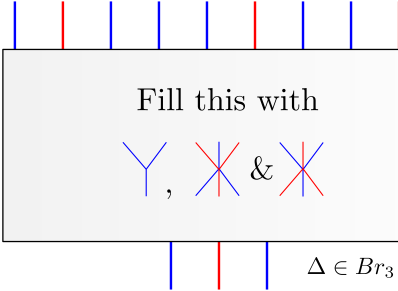


(ii) **Cyclic shift** concordance also useful. In general, given $\beta \in Br_n^+$ with $\delta(\beta) = \Delta$, the **RIII** and D_4^- **moves** above suffice to bring β to Δ.

(iii) Produce embedded exact L fillings of Λ_β via *RIII* and D_4^-.

Example of a weave filling

Example: Consider $\beta = \sigma_1\sigma_2\sigma_1^3\sigma_2\sigma_1^2\sigma_2^2\sigma_1(\sigma_1\sigma_2\sigma_1)$. Then Λ_β is the max-tb $T(3,4)$. Constructing a Demazure weave for β as follows.



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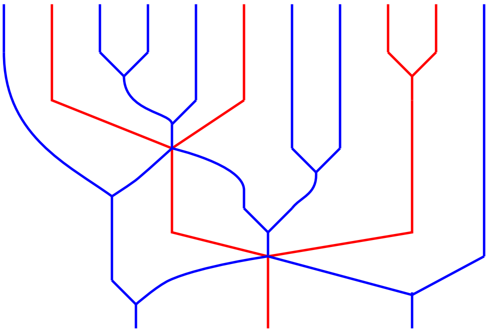

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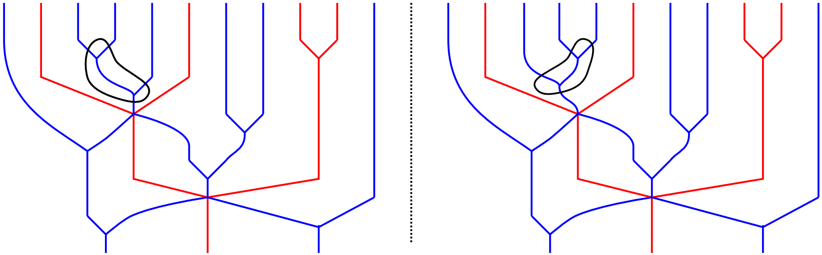
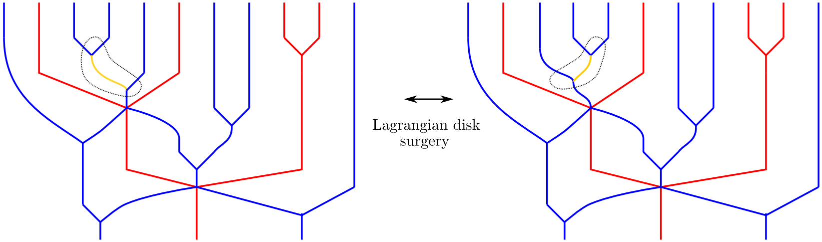
$\exists$ many solutions, typically ∞'ly many if cyclic allowed, e.g. two are



Weave calculus in action

Example continued: Consider $\beta = \sigma_1\sigma_2\sigma_1^3\sigma_2\sigma_1^2\sigma_2^2\sigma_1(\sigma_1\sigma_2\sigma_1)$ as before. These two Lagrangian fillings differ by a disk surgery:



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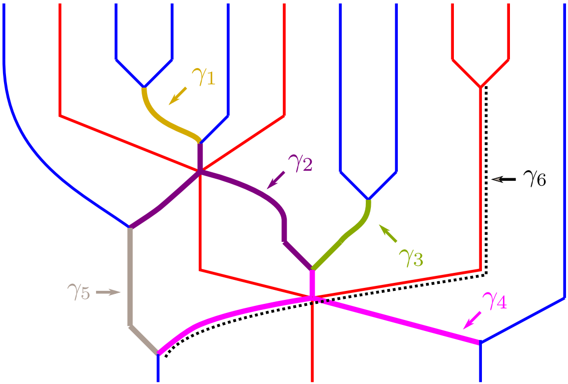
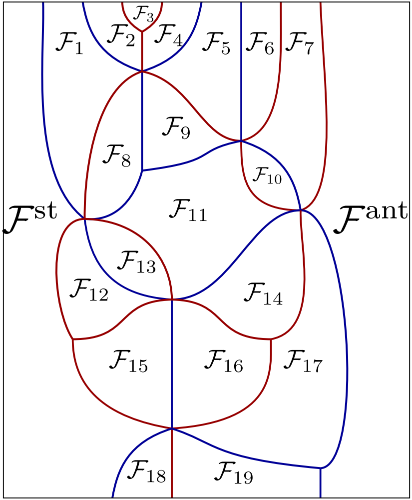


The toric chart from a weave

In practice, sheaf quantization of L through its weave goes as follows:



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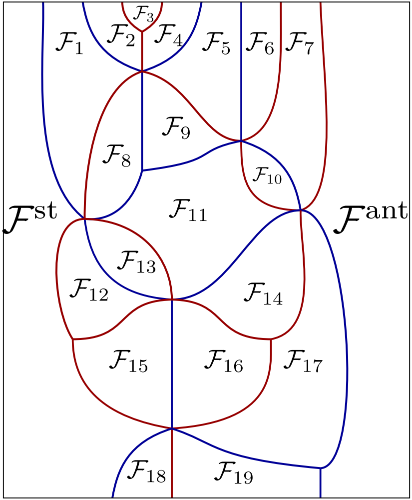

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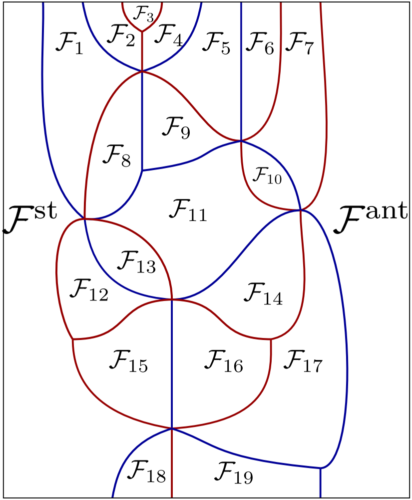

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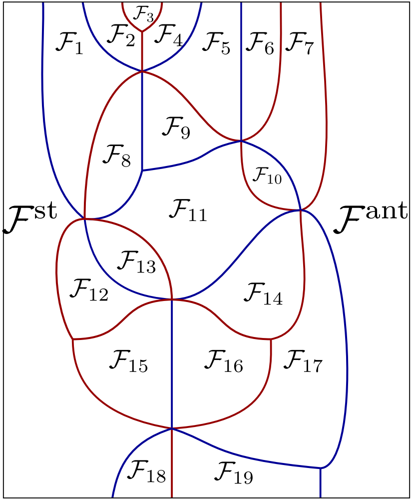

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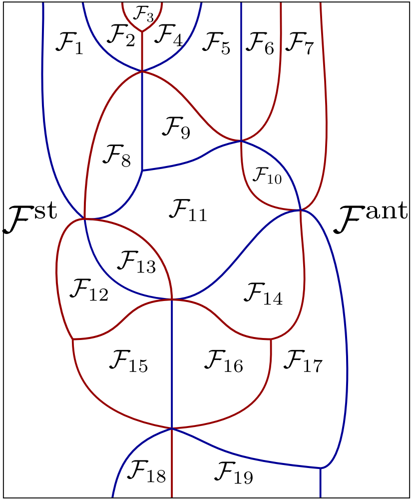
- The cluster variables A_i measure transversality of flags along the relative cycle dual to the Lusztig cycle.



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First upshot: For Legendrian links $\Lambda_\beta \subset (T^*_\infty\mathbb{R}^2, \ker \lambda_{\text{st}})$, **weaves** construct many Lagrangian fillings with $\mathbb{L}$-compressible systems.

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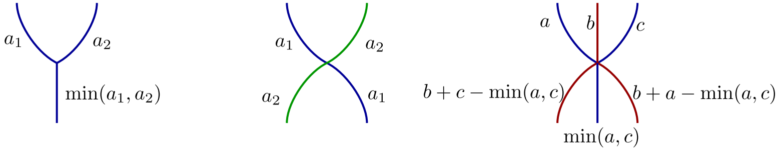

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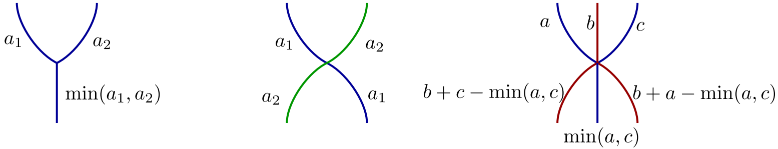

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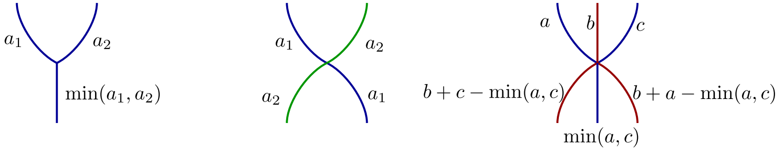

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(iii) Thus, Demazure weave for Λ_β gives an initial seed. Even better: the **flag moduli** of the weave gives the toric chart T_L in $\mathfrak{M}(\Lambda_\beta)$.

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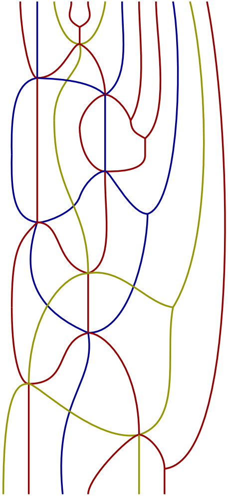
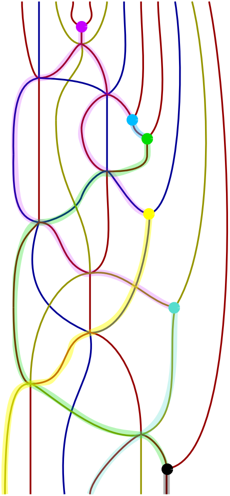

Structural result of weaves

Theorem (Framework for Weave Calculus)

Let $\beta \in Br_n^+$ be a braid and $\mathfrak{w}$ a Demazure weave from β to Δ. Then $\mathfrak{w}$ defines an exact Lagrangian filling $L(\mathfrak{w})$ of Λ_β. Furthermore:

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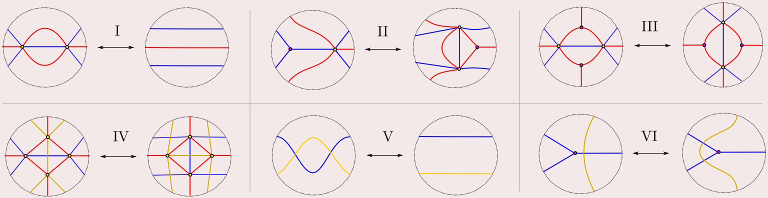

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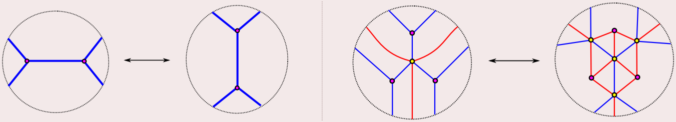

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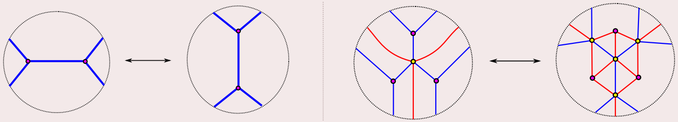

(iv) *Any two such weaves with same boundary conditions* **connected by** *a sequence of* **equivalences and mutations**.

Sketch of the argument I

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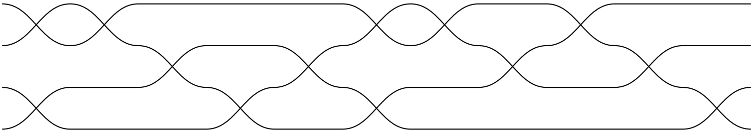

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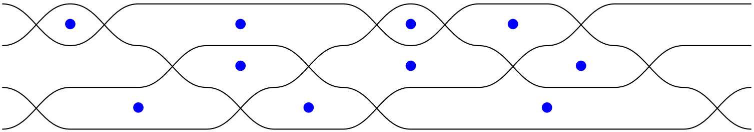

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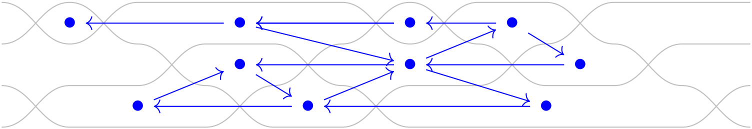

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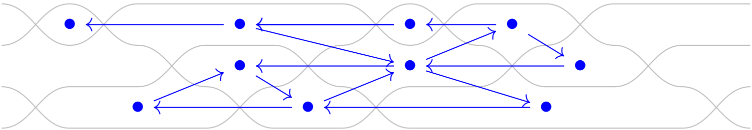

2. The cluster variables are the minor giving the transversality of the leftmost flag with each of the other flags. They are microlocal merodromies along relative cycles, dual to the Lusztig cycles. Therefore, they define **global regular functions**.

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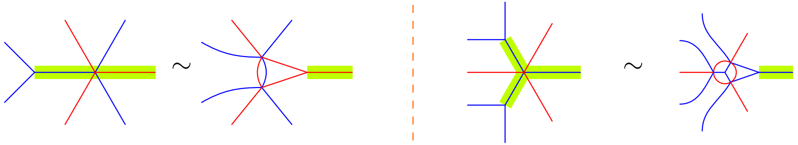
3. To apply starfish, **first simplify geometrically**. Make all Y-cycles into short cycles via sequence of weaves equivalences:



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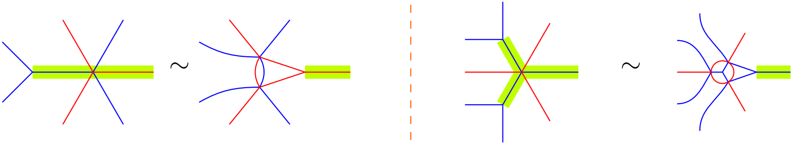
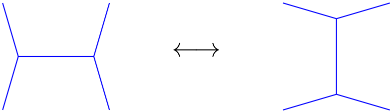


4. Then apply Lagrangian disk surgeries using weave mutations:



Direct computation then shows that mutated variable is regular: the configuration of flags is such that when the denominator vanishes in the new transversality condition, so does the numerator.

Sketch of the argument III

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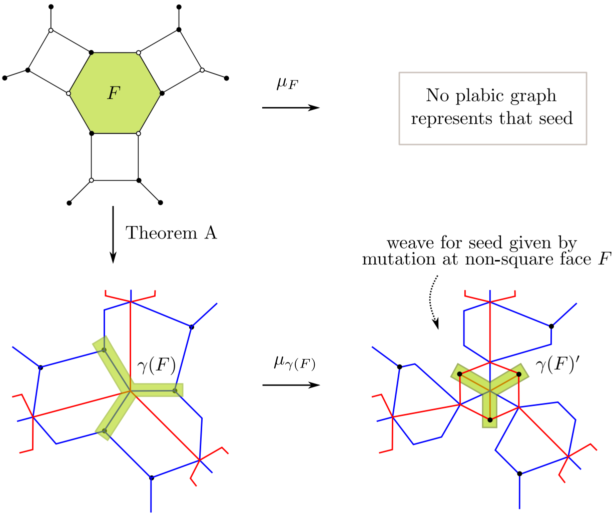
7. Several details behind the scenes: factoriality of $\mathbb{C}[\mathfrak{M}(\Lambda_\beta)]$, irreducibility of A_i and codimension-2 argument with non-free weaves. $\square$

A last hurrah: a non-Plücker seed with weaves

Here is a simple example for a positroid in Gr(3, 6):



Only one lecture to go, and mostly self-contained!

Thanks a lot, see you tomorrow!