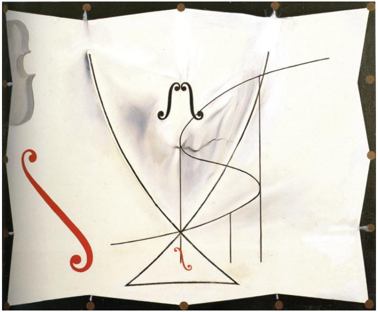
Cluster algebras and symplectic topology I

Summer School on Cluster Algebras 2023



Roger Casals (UC Davis)

August 21st 2023

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- B. An, Y. Bae, I. Le, W. Li, H. Gao, E. Gorsky, M. Gorsky, J. Hughes, E. Lee, A. Roy, J. Simental, L. Shen, M. Sherman-Bennett, D. Treumann, D. Weng, E. Zaslow. . .

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Weaves also used in spectral networks, in higher dimensional contact topology, e.g. Lagrangian concordances, doubling, and more.

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Please ask questions throughout so that we can all learn. Thanks!

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I1. Liouville 1-form $\lambda_{Liouv} := p_1 dq_1 + p_2 dq_2$ and $\omega_{st} := d\lambda_{Liouv}$ is symplectic.
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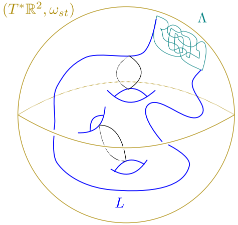
I2. $L \subset T^*\mathbb{R}^2$ exact Lagrangian if $\lambda_{Liouv}|_L$ exact. (← same as Hamiltonian iso.)
Exact Lagrangians must be non-compact, i.e. $\partial L \neq 0$.



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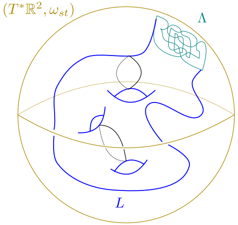

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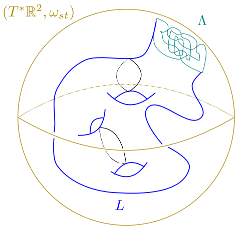

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B2. $\Lambda \subset T^*_\infty\mathbb{R}^2$ link is Legendrian if $T\Lambda \subset \ker\{\lambda_{Liouv}|_{T^*_\infty\mathbb{R}^2}\}$.
(Legendrians Λ are a good boundary condition for exact Lagrangians L: think $\Lambda = \partial L$.)

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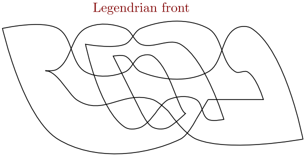

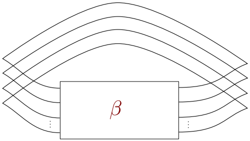

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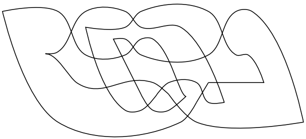

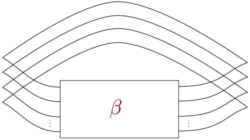

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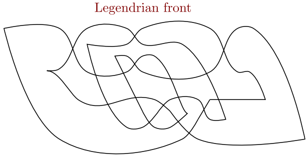

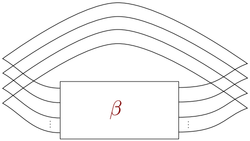

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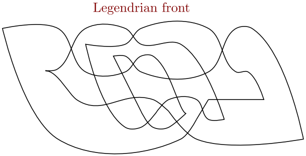

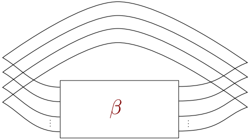

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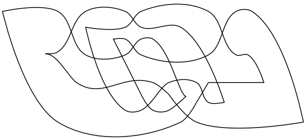

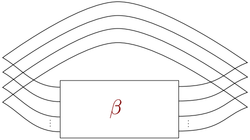

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(iv) Any front **stratifies** plane $\mathbb{R}^2$: partition into $0, 1, 2$-dimensional pieces.

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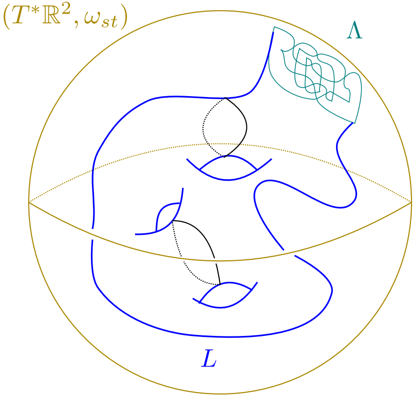

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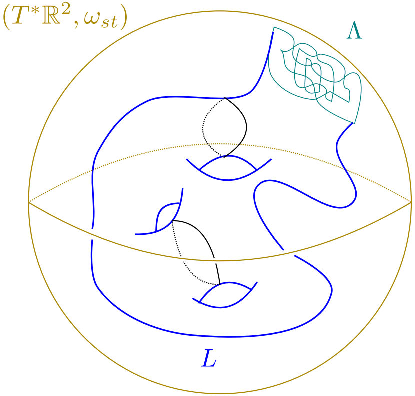

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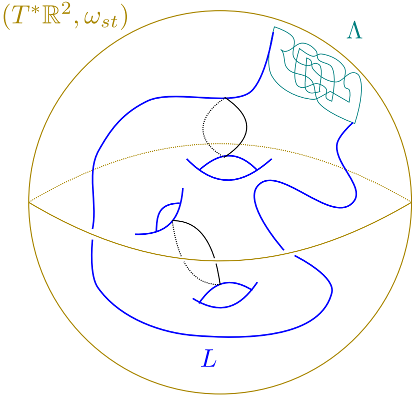

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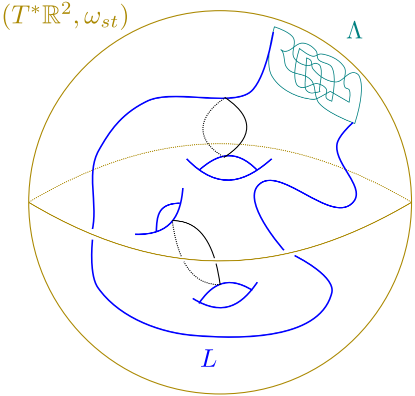
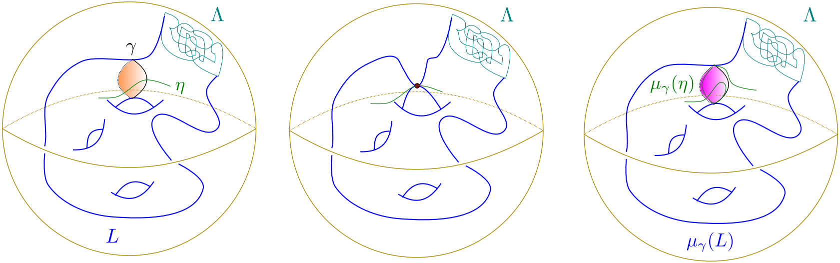
- Describe Lagrangian surfaces in Weinstein 4-folds, objects in Fukaya and sheaf categories, mirror symmetry, etc.



Lagrangian Disk Surgeries

A construction of Lagrangian fillings: **Lagrangian disk surgery**.



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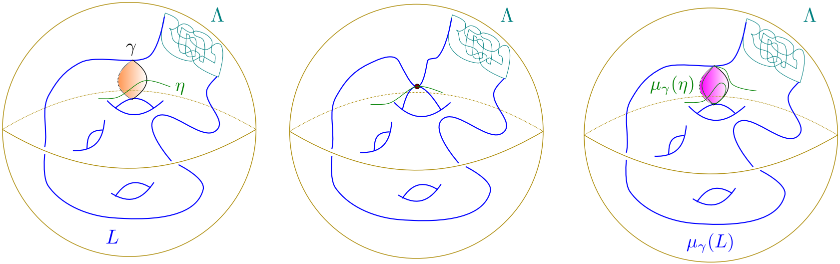

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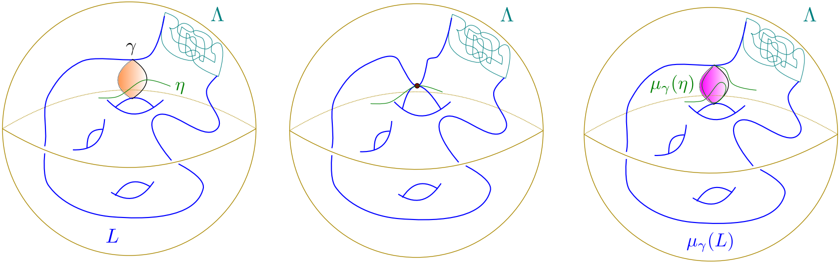

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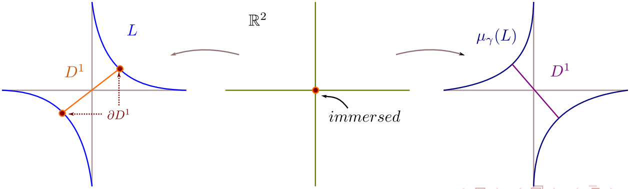
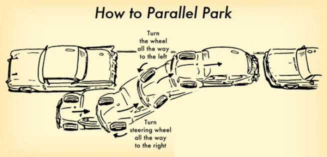
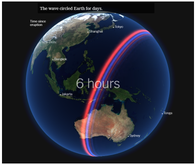


A quick aside for broader context

Bird's-eye tunnel view: Legendrian submanifolds everywhere.





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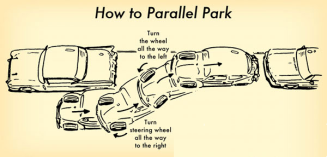

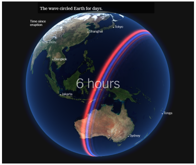

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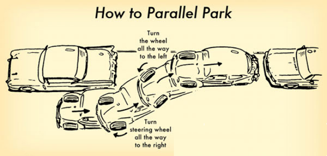

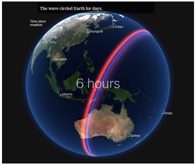

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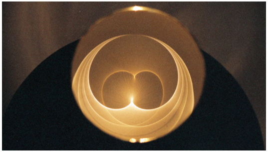
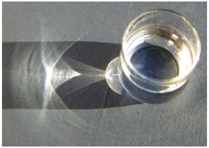
- Detection of Reeb orbits, computation of Floer-theoretic invariants, classification of contact structures, connections to other areas.





How do we study Lagrangians and Legendrians?

Microlocal: (adj) "Local with respect to both space and cotangent space.". Study functions and their *first* derivatives.

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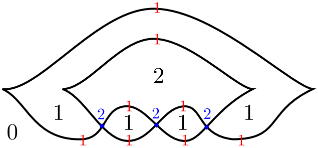

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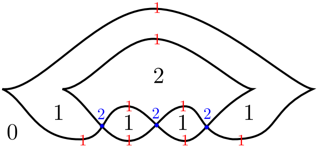

(iii) The right setup: study constructible *sheaves*. The notion of "first derivative" is captured by the *singular support*, pioneered by Mikio Sato.

Categories of sheaves on $\mathbb{R}^2$ singularly supported on a front

The category: $\Lambda \subset T^*_\infty \mathbb{R}^2$ Legendrian, $\mathcal{C}(\Lambda)$ the dg-derived category of decorated constructible sheaves on $\mathbb{R}^2$ with singular support on Λ.*
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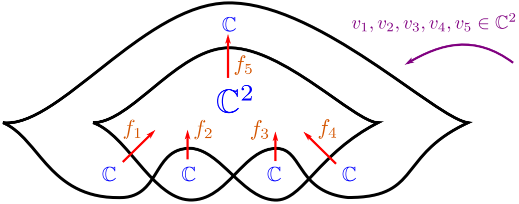
(iii) $\mathfrak{M}(\Lambda) = \{(v_1, v_2, v_3, v_4, v_5) : v_i \in \mathbb{C}^2, \det(v_i, v_{i+1}) = 1, i \in \mathbb{Z}_5\}/PGL_2(\mathbb{C})$



$v_1 = im(f_1)$

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$v_4 = im(f_4)$

$v_5 = ker(f_5)$

Set $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1, z_1)$, $v_4 = (z_4, z_3)$, $v_5 = (z_2, -1)$.
Then $\mathfrak{M}(\Lambda) = \{z_3 + z_1 + z_1 z_3 z_2 = 1\} \subset \mathbb{C}^3_{z_1,z_2,z_3}$.

A potential moduli of Lagrangian Fillings

New look at problem: Study fillings in $T^*\mathbb{R}^2$ using sheaves in $\mathbb{R}^2$

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Thus, $\mathfrak{M}(\Lambda_\beta)$ parametrizes tuples of flags with transversality conditions according to β. The **cluster algebra** will be in the ring $\mathbb{C}[\mathfrak{M}(\Lambda_\beta)]$.

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- **Important**: Lagrangian filling L of Λ gives $(\mathbb{C}^*)^{b_1(L)} \subset \mathfrak{M}(\Lambda)$ chart.
  (Lagr. filling with Abelian local system gives point in $\mathfrak{M}(\Lambda)$. Think $(\mathbb{C}^*)^{b_1(L)} = H^1(L, \mathbb{C}^*)$).

A few examples

- **Trefoil Example**: Then $\mathfrak{M}(\Lambda_{3_1}) = \{z_1 + z_3 + z_1 z_2 z_3 + 1 = 0\} \subset \mathbb{C}^3$, and we have *five* algebraic tori, all given by fillings:

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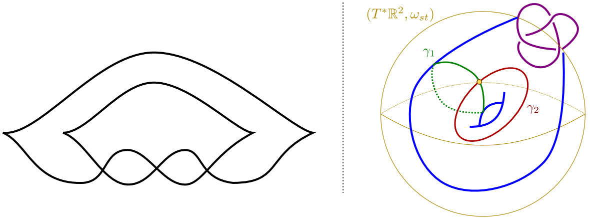

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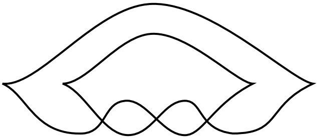

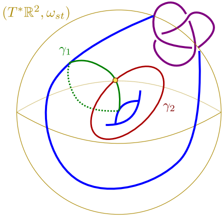

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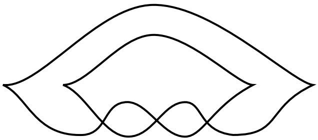

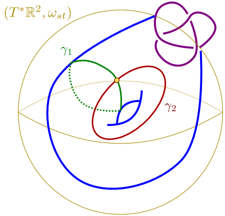

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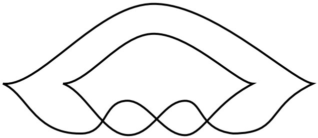

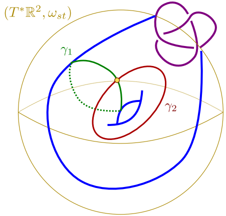

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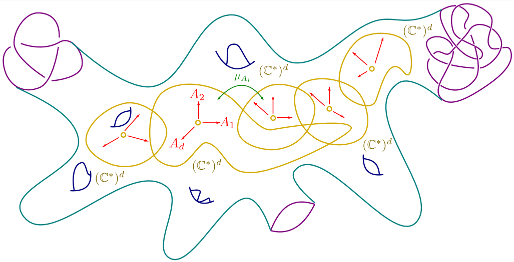
- For β algebraic, also wild character varieties. (now $T^*\Sigma$, cf. P.Boalch's work.)

First appearance of cluster algebras

Theorem (Simplified main result)

Let $\Lambda_\beta \subset (\mathbb{R}^3, \xi_{st})$ be the Legendrian link associated to a positive braid word β. Then $\mathbb{C}[\mathfrak{M}(\Lambda_\beta)]$ is a cluster algebra and, in many cases, every cluster seed is known to come from a Lagrangian filling.



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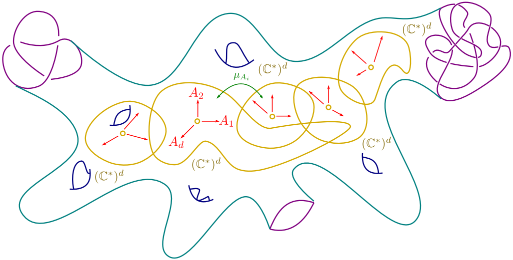

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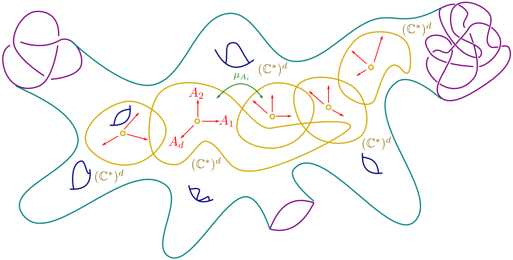

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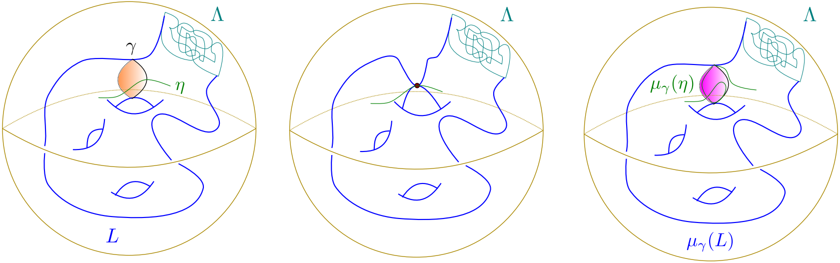
All these ingredients will be described **symplectically**: *back to geometry*!

Inspiration from Lagrangian Disk Surgeries

Recall **Lagrangian surgery**: inputs Lagrangian filling and disk.
Outputs *another* Lagrangian filling and with disk. It is *involutive*.



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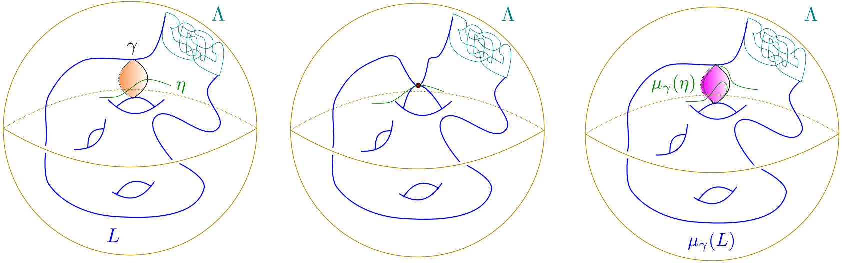

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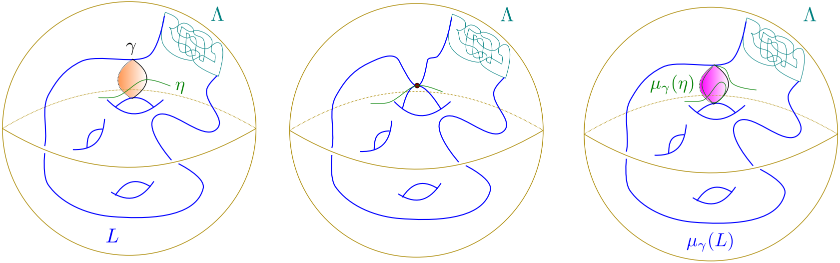

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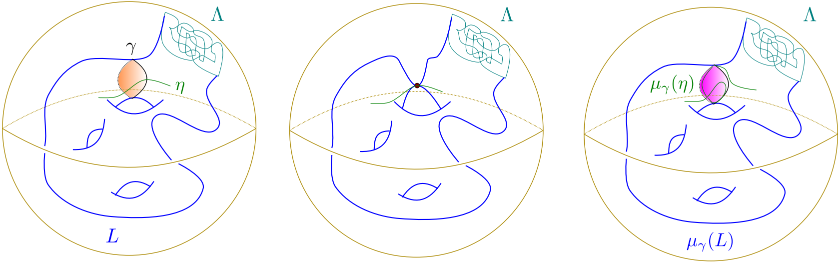

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   If we encode this change via intersection quiver we get *quiver mutation*!

$\mathbb{L}$-compressible systems for Lagrangian fillings

Definition

A maximal $\mathbb{L}$-compressible system for L is a collection of Lagrangian disks $\{D_1, \ldots, D_{b_1(L)}\}$ properly embedded in $T^*\mathbb{R}^2 \setminus L$ such that the embedded curves $\partial\overline{D}_1, \ldots \partial\overline{D}_{b_1(L)} \subset L$ are a basis of $H_1(L, \mathbb{Z})$.

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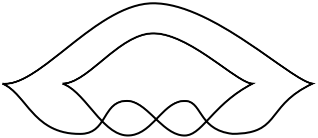
Example. Λ max-tb Legendrian link of isolated singularity $f : \mathbb{C}^2 \to \mathbb{C}$. Then a real morsification $f_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ gives a Lagrangian filling $L(f_{\mathbb{R}})$ and a maximal $\mathbb{L}$-compressible system.
(L is "Lagrangian Milnor fiber" and D_i are "vanishing thimbles".)

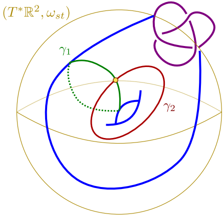
$\mathbb{L}$-compressible systems for Lagrangian fillings

Definition

A maximal $\mathbb{L}$-compressible system for L is a collection of Lagrangian disks $\{D_1, \ldots, D_{b_1(L)}\}$ properly embedded in $T^*\mathbb{R}^2 \setminus L$ such that the embedded curves $\partial \overline{D}_1, \ldots \partial \overline{D}_{b_1(L)} \subset L$ are a basis of $H_1(L, \mathbb{Z})$.

Example. Λ max-tb Legendrian link of isolated singularity $f : \mathbb{C}^2 \to \mathbb{C}$. Then a real morsification $f_\mathbb{R} : \mathbb{R}^2 \to \mathbb{R}$ gives a Lagrangian filling $L(f_\mathbb{R})$ and a maximal $\mathbb{L}$-compressible system.
(L is "Lagrangian Milnor fiber" and D_i are "vanishing thimbles".)





From $\mathbb{L}$-compressible systems to cluster seeds

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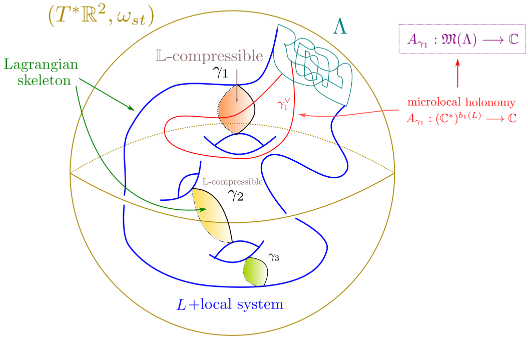

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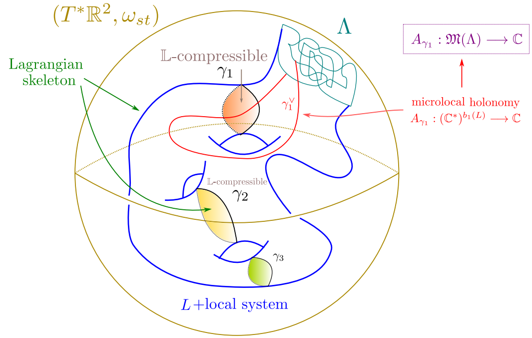

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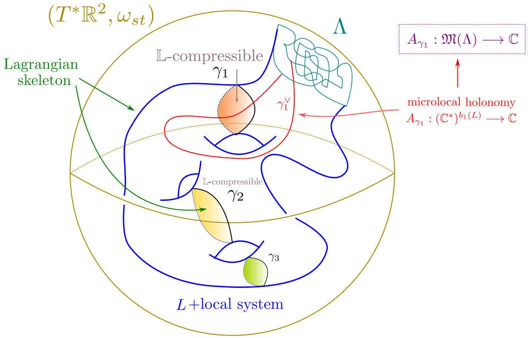

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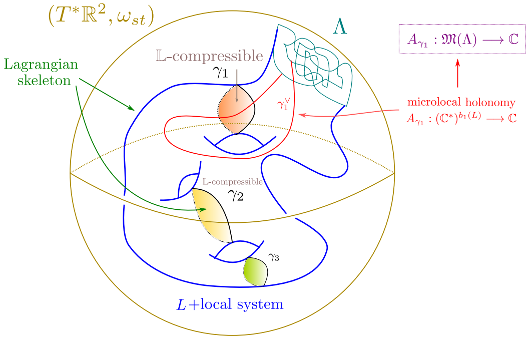

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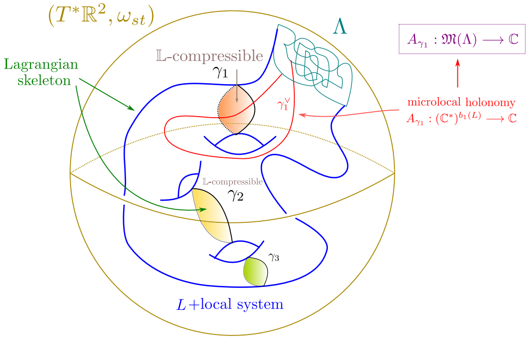

Another use: $\mathbb{L}$-compressible systems $\mathfrak{D}$ give skeleta $L \cup \mathfrak{D}$ for $(T^*\mathbb{R}^2, \Lambda)$ and $\mathrm{End}(\mathfrak{D}^* \oplus L^*)$ generate $\mathfrak{W}(T^*\mathbb{R}^2, \Lambda)$. ($\rightarrow$ bounded t-structures)

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Summarizing table of ingredients

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- Regularity of mutated $\mu_{D_j}(A_i)$ + generators of $\mathbb{C}[\mathfrak{M}(\Lambda)]$ are cluster. (Starfish lemma gives $\mathbb{C}[\mathfrak{M}(\Lambda)] = \mathcal{U}$, then need $\mathcal{A} = \mathcal{U}$. Also codim-2 argument.)

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  Require a minimal amount of *immersed* disks. ($\rightarrow$ correct quiver)
  Immersed disks lead to non-vanishing $A_i \in \mathbb{C}[\mathfrak{M}(\Lambda)]$. ($\rightarrow$ frozen variables)

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- All this, weaves and more, in the next lecture!

The end of the beginning.

Thanks a lot!