Mirror symmetry and enumerative geometry of cluster varieties.

"Quivers and curves in higher dimension"

jt with H. Argüz 2308.07270.

Donaldson–Thomas (DT) invariants of quivers with potentials $\leftrightarrow$ Punctured $g=0$ Gromov–Witten invariants of cluster varieties

$\updownarrow$

Stability scattering diagram (Bridgeland)

Enumerative invariants entering the Gross–Siebert mirror symmetry construction.

"Canonical scattering diagram"

$\updownarrow$

Cluster scattering GHKK

Stability scattering diagram $\leftrightarrow$ Cluster scattering

First example: Gross–Pandharipande–Siebert

Gross–Pandharipande

DT $\left[\ \bullet \xrightarrow{n} \bullet\ \right.$ $\longleftrightarrow$ GW invariants of log CY surfaces

s seed
(e_i) basis of N
ω skew-symmetric
$\omega : N \times N \to \mathbb{Z}$

→ Cluster varieties

$\mathcal{X} = \bigcup \operatorname{Spec} \mathbb{C}[M]$

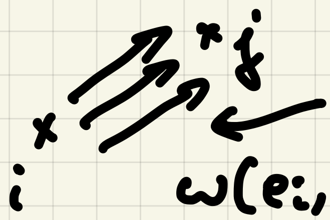
$M = \operatorname{Hom}(N, \mathbb{Z})$

$\mathcal{A} = \bigcup \operatorname{Spec} \mathbb{C}[N]$

$\updownarrow$

Quiver : graph Q
Vertex $\leftrightarrow$ i

$(\omega(e_i, e_j))_{i,j}$



$\omega(e_i, e_j)$ arrows

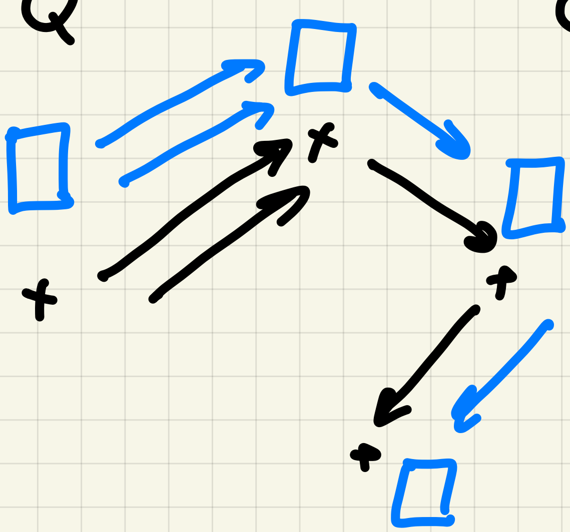
$\omega(e_i, e_j) > 0$

$|\omega(e_i, e_j)|$

$\omega(e_i, e_j) < 0$

↳ Quiver representations

Q



Quiver representation:

. vertex i : V_i finite-dim $\mathbb{C}$-vector space.

. arrow $\alpha : i \to j$

$f_\alpha : V_i \to V_j$ $\mathbb{C}$-linear map.

$V = ((V_i),(f_\alpha)) \to \dim V = (\dim V_i) \in N = \bigoplus_i \mathbb{Z} e_i$

$\left\{ \begin{array}{c} \text{Quiver representations} \\ \text{of dim } \gamma \end{array} \right\} / \text{Isomorphisms}$

Fix $\gamma \in N$

?

King's stability for quiver representations.

Fix $\gamma \in N$ $\quad \gamma^\perp = \{ \theta \in M_{\mathbb{R}} \mid \theta(\gamma) = 0 \}$

Hyperplane in $M_{\mathbb{R}}$

Stability parameter for γ: $\boxed{\theta \in \gamma^\perp}$

V of dim γ is θ-stable if $\forall\, 0 \subsetneq V' \subsetneq V$,

$[\theta(\dim V) = \theta(\gamma) = 0]$ $\quad \theta(\dim V') < 0.$

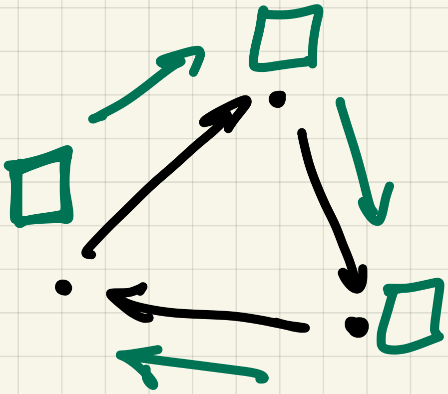
θ-semistable: —— ≤ 0

$\boxed{\mathcal{M}_\gamma^\theta} = \{ \theta\text{-semistable rep of dim } \gamma \} / \sim$

Quasi-projective algebraic variety $/ \mathbb{C}$.

- Projective if Q is acyclic (no oriented cycle)
- In general, introduce a "potential" W

formal linear combination of oriented cycles in Q.



$\mathrm{Tr}\,W : \underbrace{\mathcal{M}_\gamma^\theta} \longrightarrow \mathbb{C}$

(Q, W) $\quad \gamma \in N$, $\theta \in \gamma^\perp$

$\rightsquigarrow$ DT invariant $\Omega_\gamma^\theta \in \mathbb{Z}$

"Virtual count of critical points of $\underline{\mathrm{Tr}\,W}$ on $\underline{\mathcal{M}_\gamma^\theta}$"

$$\Omega_\gamma^\theta = e\left(\mathcal{M}_\gamma^\theta, \phi_{\mathrm{Tr}W}(IC)\right) \in \mathbb{Z}$$

<u>Remark</u>: More general:

DT invariants for any triangulated category Calabi-Yau of dim 3 + Bridgeland stability condition.

$D^b \underline{Coh}(X)$

↰ CY3-fold /

$\mathrm{F}\underline{\mathrm{uk}}(Y)$

↰ CY3-fold /

$\underline{\underline{(Q,W)}} \rightarrow$ Ginzburg dg-algebra / <u>Category CY of dim 3</u>

dg-modules

$$\left.\begin{array}{l}(Q,W)\\ \gamma\in N\\ \theta\in N^{\perp}\end{array}\right\} \to \Omega_\gamma^\theta \in \mathbb{Z}$$

$$\bar{\Omega}_\gamma^\theta = \sum_{\substack{\gamma'\in N\\ \gamma=k\gamma'\\ k\in\mathbb{Z}_{\geq 1}}} \frac{(-1)^{k-1}}{k^2}\,\Omega^\theta_{\gamma'} \in \mathbb{Q}$$

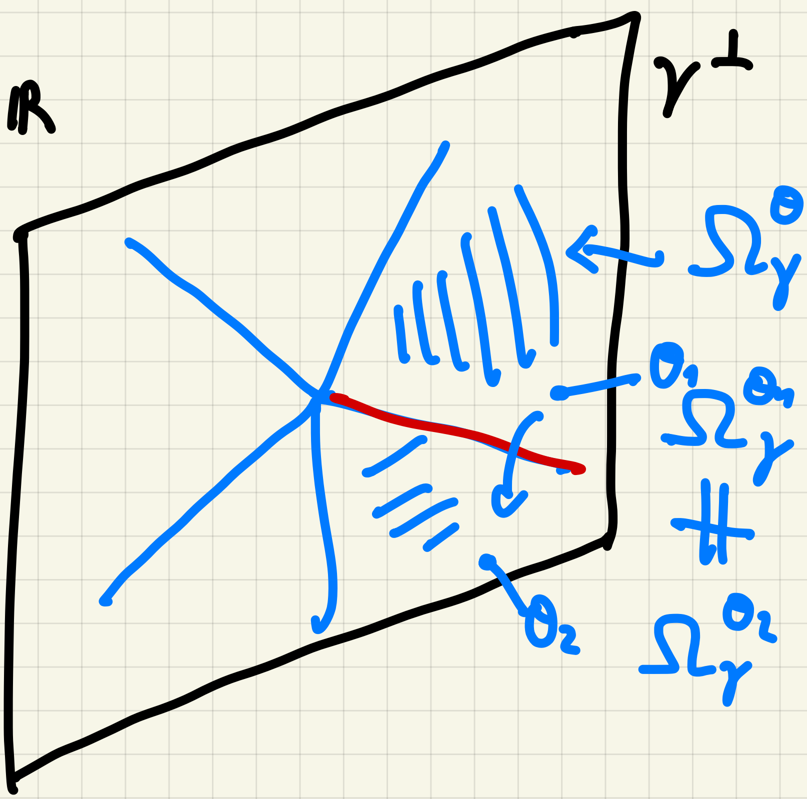
Finite sum

$$\{\Omega_\gamma^\theta\} \leftrightarrow \{\bar{\Omega}_\gamma^\theta\}$$

$\Omega_\gamma^\theta \leftarrow$ Stability $\theta\in\gamma^\perp$

Wall-crossing with respect to θ.

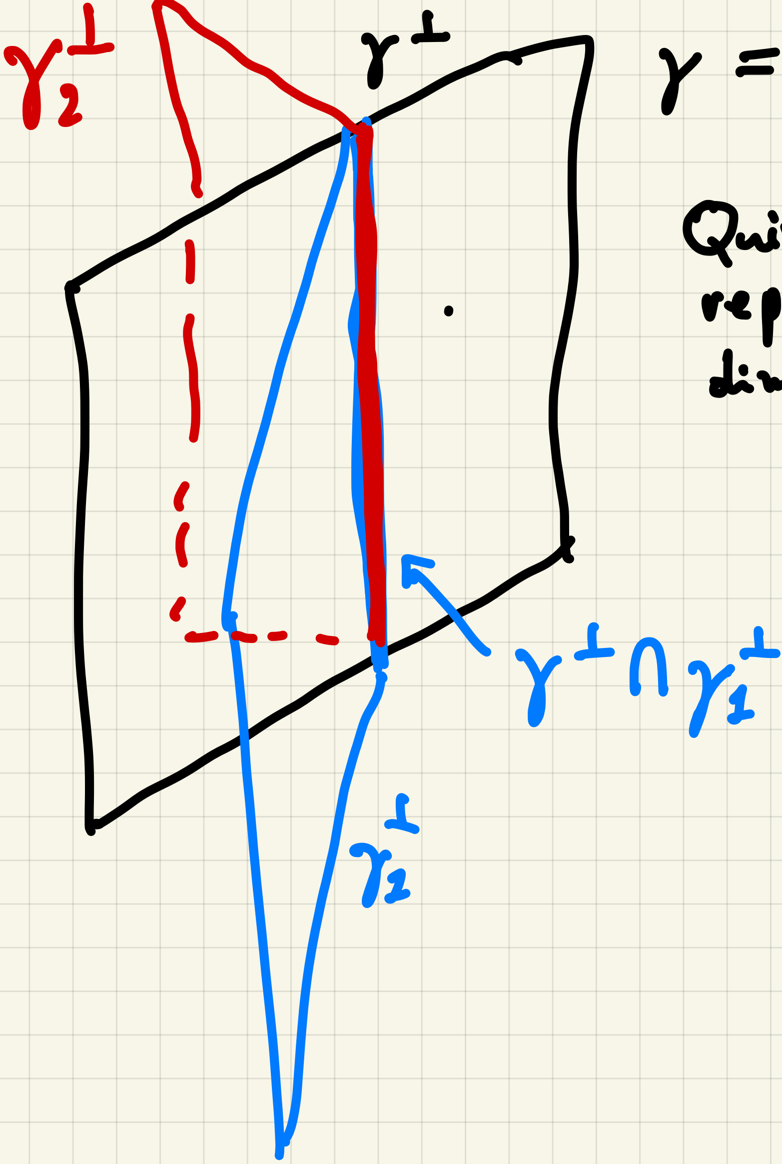
$M_{\mathbb{R}}$



Universal wall-crossing formula

Joyce-Song

Kontsevich-Soibelman.



$\gamma = \gamma_1 + \gamma_2$ $\quad$ $\gamma = \gamma_1 + \cdots + \gamma_n$

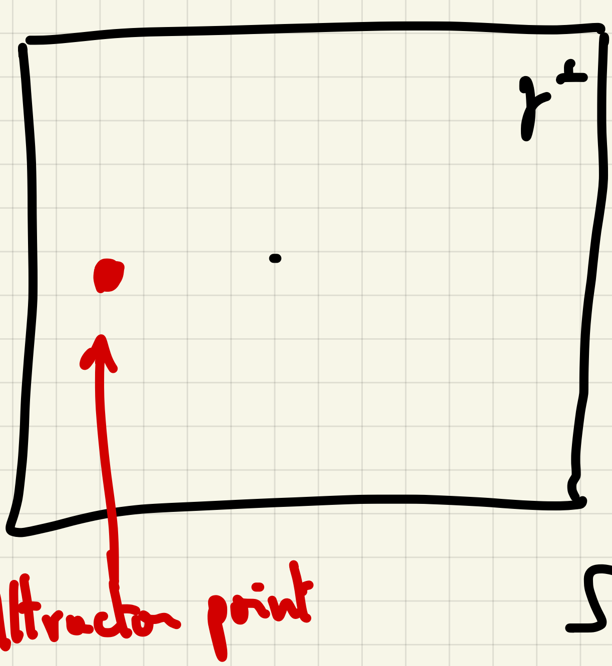
Quiver rep of dim γ $\quad$ $\gamma \in N$

$\cup$

$\in N^+$

Attractor DT invariant.

Fix γ: $\quad$ $\subset M_{IR}$



Attractor point:

$$\omega(\gamma, -) \in M \supset \gamma^\perp$$

$$\omega(\gamma,-) = \iota_\gamma \omega \in \gamma^\perp$$

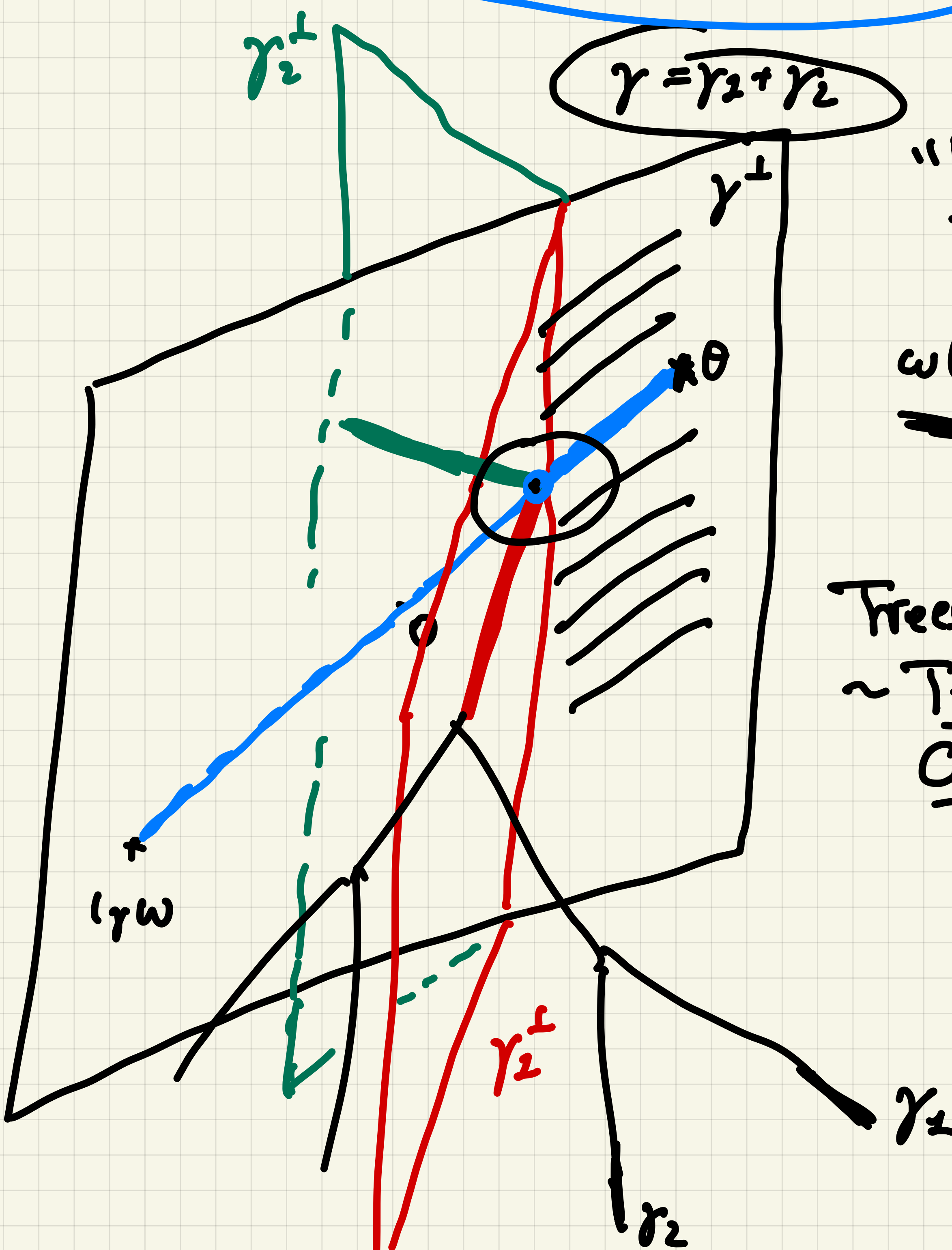
$$\omega(\gamma,\gamma) = 0$$

Attractor DT invariant:

$$\Omega_\gamma^* := \Omega_\gamma^{\iota_\gamma \omega}$$

$$\{\Omega^+_\gamma\}_\gamma \rightsquigarrow \{\Omega^\theta_\gamma\}_{\gamma,\theta}$$

$$\bar{\Omega}^\theta_\gamma = \sum_{\gamma=\gamma_1+\cdots+\gamma_n} \underbrace{F_\theta(\gamma_1,\dots,\gamma_n)}_{\in \mathbb{Q}} \, \bar{\Omega}^+_{\gamma_1} \cdots \bar{\Omega}^+_{\gamma_n}$$

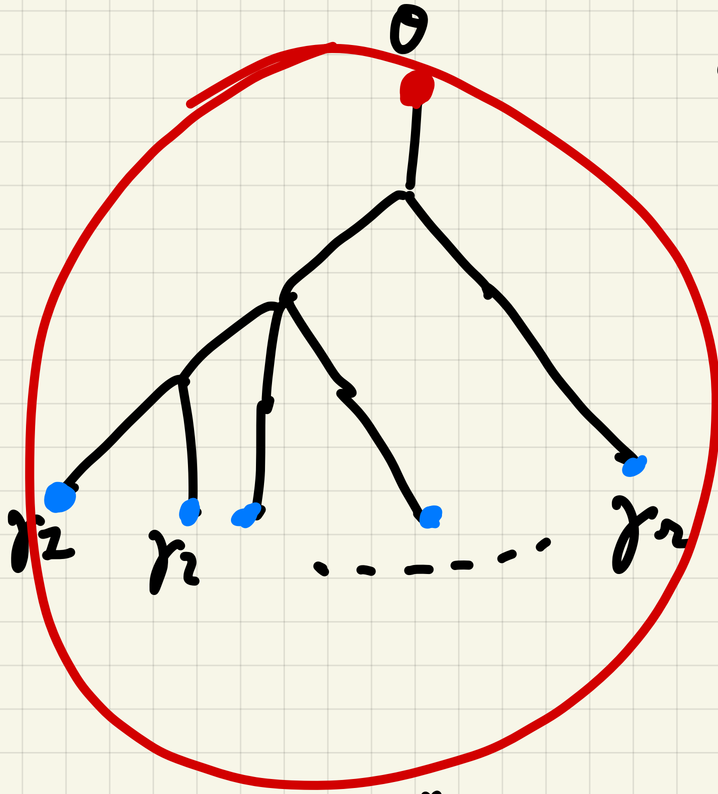


"Flow"

$\omega(\gamma_i, \gamma_j)$

Trees

$\sim$ Tropical Curves



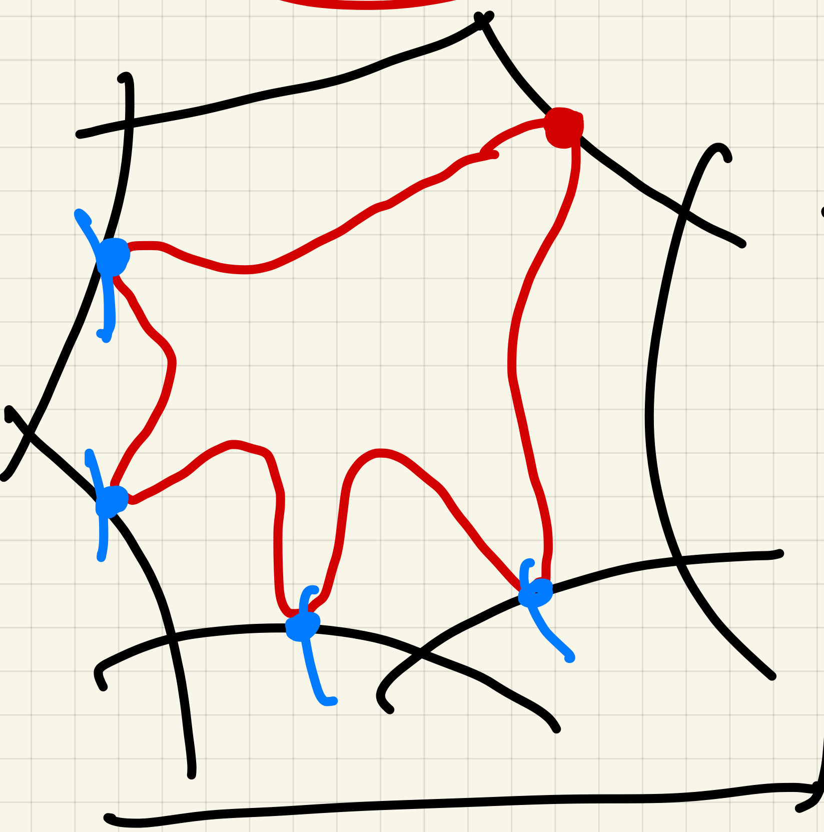
$\sum \gamma_i = \gamma$

<u>Thm</u> (Argüz-B) (2303.10811)

$F_\theta(\gamma_1, \dots, \gamma_r)$

= genus 0 (punctured) Gromov-Witten of a toric variety.

(with torus Spec $\mathbb{C}[N]$)



<u>Def</u>: (Q,W) has <u>trivial attractor DT invariants</u> if:

$$\Omega^*_{e_i} = 1$$

$$\Omega^*_\gamma = 0 \quad \text{if } \gamma \neq e_i$$

if $\gamma \notin \ker \omega$

Holds in many cases:

- If Q acyclic (so $W=0$) : ✓ (Bridgeland)
- If W non-degenerate
  Q has green-to-red sequence : ✓ (Lang Mou)

- Conj (Beaujard-Manschot-Pioline)
  Toric CY3-fold X $\exists\,(Q,W)$
  $D^b Coh(X) \simeq D^b Rep(Q,W)$
  $\to$ (Q,W) has trivial attractor DT invts.

Proved: $X = K_{\mathbb{P}^2}$ "local $\mathbb{P}^2$" [B-Descombes-Le Floch-Pioline]

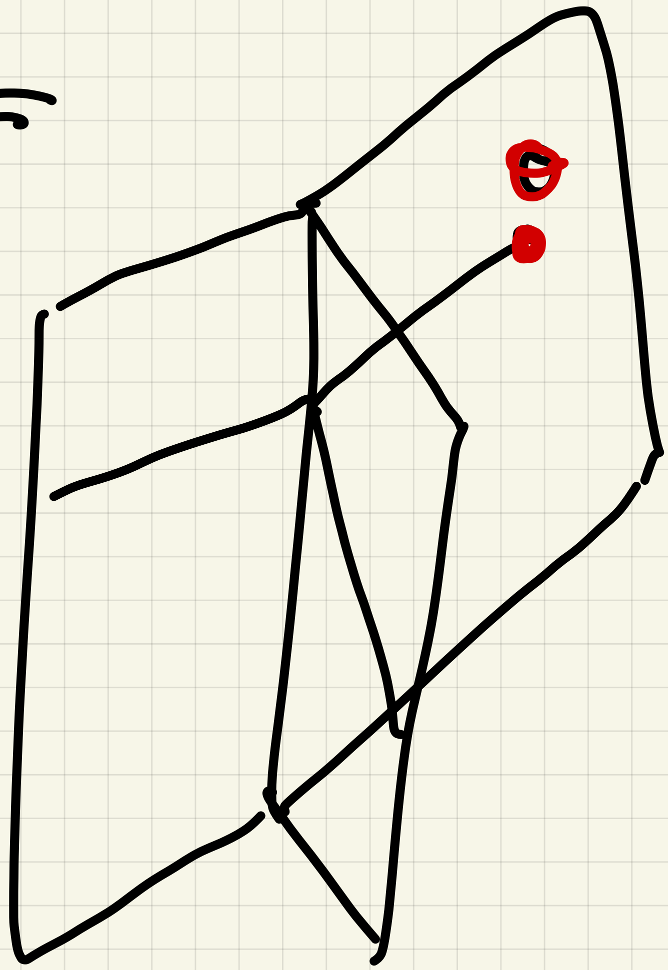
Thm (Argüz-B) (Q,W) trivial attractor DT invariants.
Then $\forall \gamma \in N \setminus N_{uf}$,

$$\overline{\Omega}_\gamma^\theta = \frac{1}{|\gamma|} \sum_{\tau \in T_\gamma^\theta} k_\tau \, N^{(X,D)}_{\tau, \beta_\gamma^\theta} \in \mathbb{Q}$$

$\in \mathbb{Q}$

genus 0 punctured GW invariants of a log CY compactification (X,D)

of the $\mathcal{X}$-cluster variety.



in $M_{\mathbb{R}}$ $\simeq$ B

tropicalization of (X, D)