Mirror symmetry and enumerative geometry of cluster varieties.

(Pierrick Bousseau, University of Georgia)

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$$\begin{tikzcd} \mathcal{X} \text{ cluster variety} \arrow[r, squiggly, "\text{FG duality}"'] & \mathcal{A} \text{ cluster variety} \end{tikzcd}$$

$\mathcal{X} = X \setminus D$

$$\begin{tikzcd} (X, D) \text{ log CY pair} \arrow[d, "\text{Blow-up}"'] \arrow[r, "\text{Gross-Siebert mirror symmetry construction}"'] & X^\vee \arrow[d] \\ (X_\Sigma, D_\Sigma) \text{ Toric pair} & \mathrm{Spf}\, \mathbb{C}[[NE(X)]] \end{tikzcd}$$

Monoid of effective curve classes on X

$\sum c_\beta t^\beta$

β ← Curve classes

Enumerative geometry ↑

rational curves of classes β.

(Gromov-Witten)

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook] \arrow[d] & \mathcal{A}_{prin} \arrow[d] \\ 1 \in & \mathrm{Spec}\, \mathbb{C}[N_{uf}] \end{tikzcd}$$

$\mathrm{Spec}\, \mathbb{C}[N_{uf}] \cong (\mathbb{C}^*)^n$ ← t_i <u>coordinates</u>

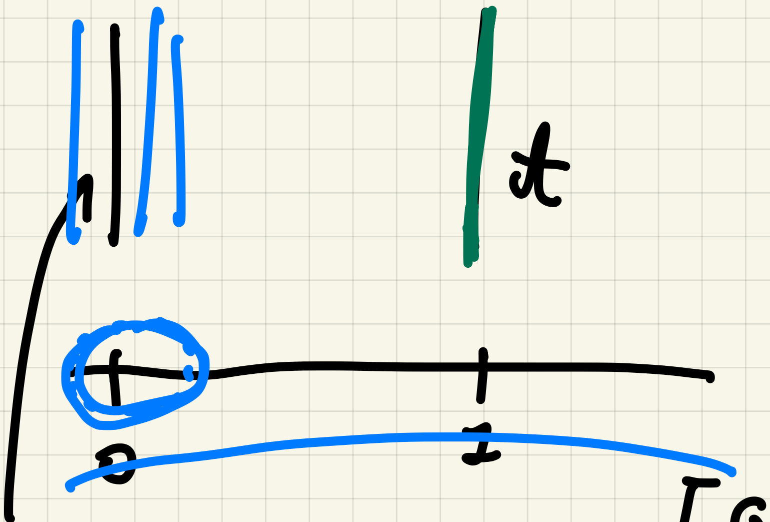
$1 + t_i z^{v_i} = 0$ → $\mathcal{A}_{prin}$

Blow-up $\underline{1 + z}^{v_i} = 0$ → $\mathcal{A}$

$$\begin{tikzcd}
\overline{\mathcal{A}}_{\text{prin}} \arrow[d] & & \operatorname{Spec} \mathbb{C}[M] \arrow[d] \\
\operatorname{Spec} \mathbb{C}[N_{uf}^{\oplus}] \simeq \mathbb{A}^n & & \ni 0
\end{tikzcd}$$

N

$M = \operatorname{Hom}(N, \mathbb{Z})$



$\operatorname{Spec} \mathbb{C}[M] \simeq (\mathbb{C}^*)^n$

$\boxed{\widehat{\overline{\mathcal{A}}}_{\text{prin}}}$ = formal completion of this family along the fiber over $0 \in \mathbb{A}^n$

[Gross-Hacking-Keel-Kontsevich:

Without any assumptions, construct canonical bases of "θ-functions" on $\widehat{\overline{\mathcal{A}}}_{\text{prin}}$.

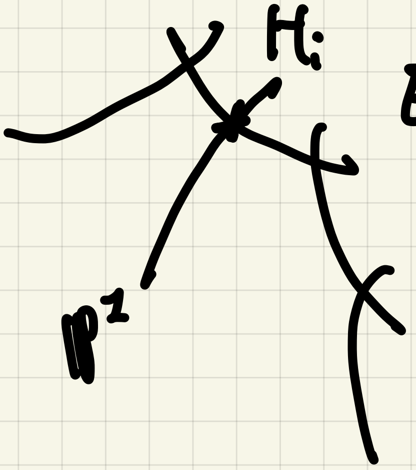
$$\begin{tikzcd}
X^\vee \arrow[d, dashed] & & \widehat{\overline{\mathcal{A}}}_{\text{prin}} \arrow[d] \\
\{\operatorname{Spf} \mathbb{C}[[NE(X)]]\} \arrow[rr, leftrightarrow] & & \boxed{\operatorname{Spf} \mathbb{C}[[N_{uf}^{\oplus}]]}
\end{tikzcd}$$

$N_{uf} \subset N$

$$\begin{tikzcd} X \arrow[d] \\ X_\Sigma \arrow[u, bend right] \end{tikzcd}$$

$$\underbrace{N_1(X)}_{\text{group of all curve classes}} \simeq N_1(X_\Sigma) \oplus \bigoplus_{i\in I} \mathbb{Z}E_i$$

↑ Ex curve.



Ex divison = $\mathbb{P}^1$- bundle over hypersurface H_i

E_i = class of $\mathbb{P}^1$-fiber

$$NE(X) \subset \left(NE(X_\Sigma) \oplus \bigoplus_{i\in I} \mathbb{Z}E_i \right)$$

$$\cup$$

$$NE(X_\Sigma) \oplus \underbrace{N_{uf}^{\oplus}}_{\text{IS}}$$

$$\begin{tikzcd} (-a_i E_i) \\ \mathbb{N}^I \ni (a_i) \arrow[u, mapsto] \end{tikzcd}$$

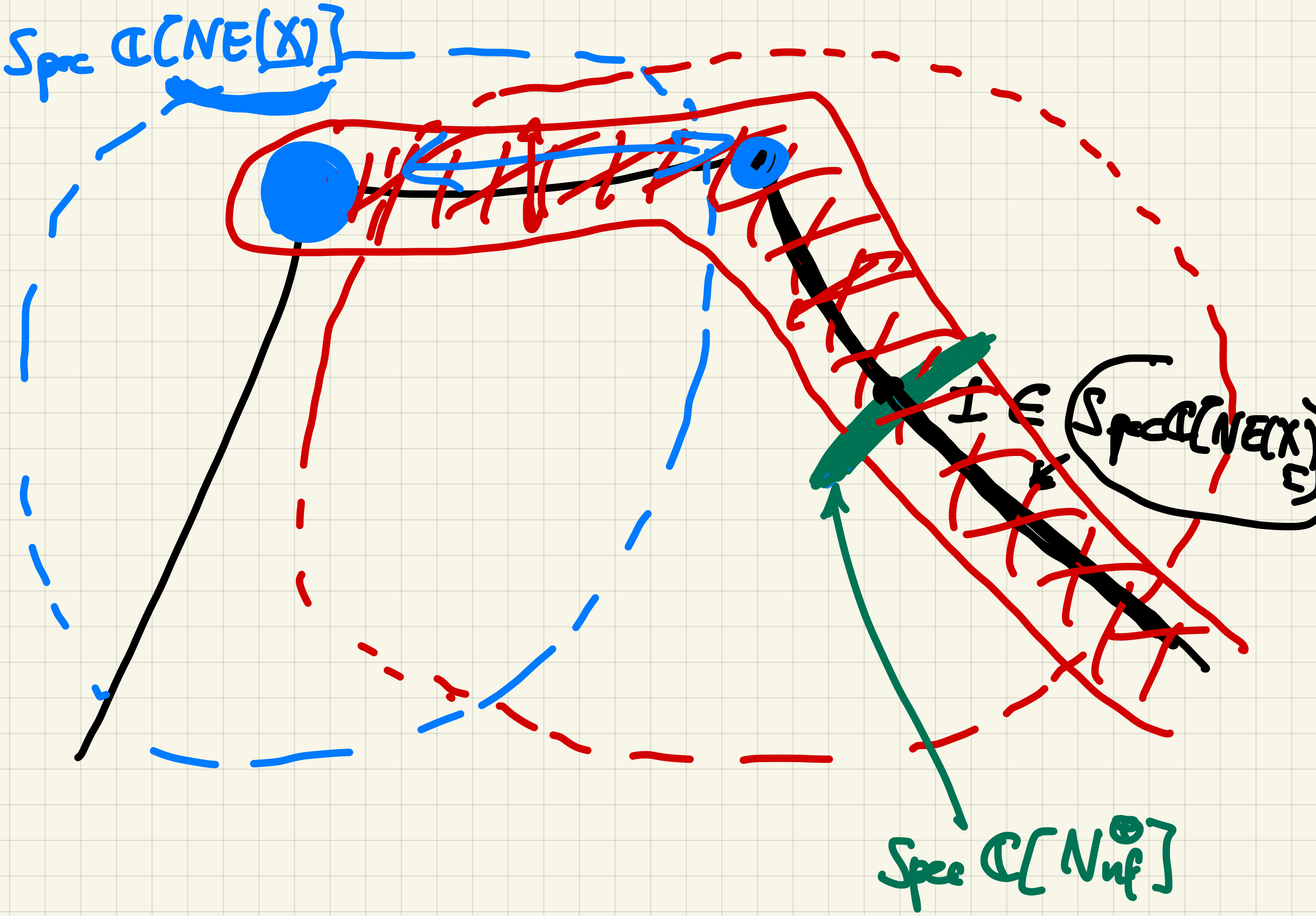
$$\operatorname{Spec} \mathbb{C}[NE(X)] \supset \operatorname{Spec} \mathbb{C}[NE(X_\Sigma) \oplus \bigoplus_i \mathbb{Z}E_i)$$

$$t_i$$

$$\cup$$

$$\operatorname{Spec} \mathbb{C}[NE(X_\Sigma) \oplus N_{uf}^{\oplus}]$$

Glue them: →



<u>Thm</u> (Argüz-B 2206.10584]

- Mirror family originally defined over 🔵

∧ extends over ▨

canonically

Extended family ϑ_w $\simeq \hat{\mathcal{A}}_{\mathrm{prin}}$

Remark: . Extension compatible with mirror symmetry

$$\begin{tikzcd} X \arrow[d, "\text{Blow-up}"] \\ X_\Sigma \end{tikzcd}$$

Moving in the Kähler moduli of X → Complex interpolation between mirror of X and the mirror of X_Σ

. Proof using a comparison:

$\mathfrak{D}_{(X,D)}$ "canonical scattering diagram" Enumerative geometry ⟷ $\mathfrak{D}_{GHKK}$

M. Gross
H. Argüz
"The higher dimensional tropical vertex"

Scattering diagram used by GHKK

↳ Construct "Θ-functions" on $\hat{\mathcal{A}}_{prin}$

Mirror symmetry

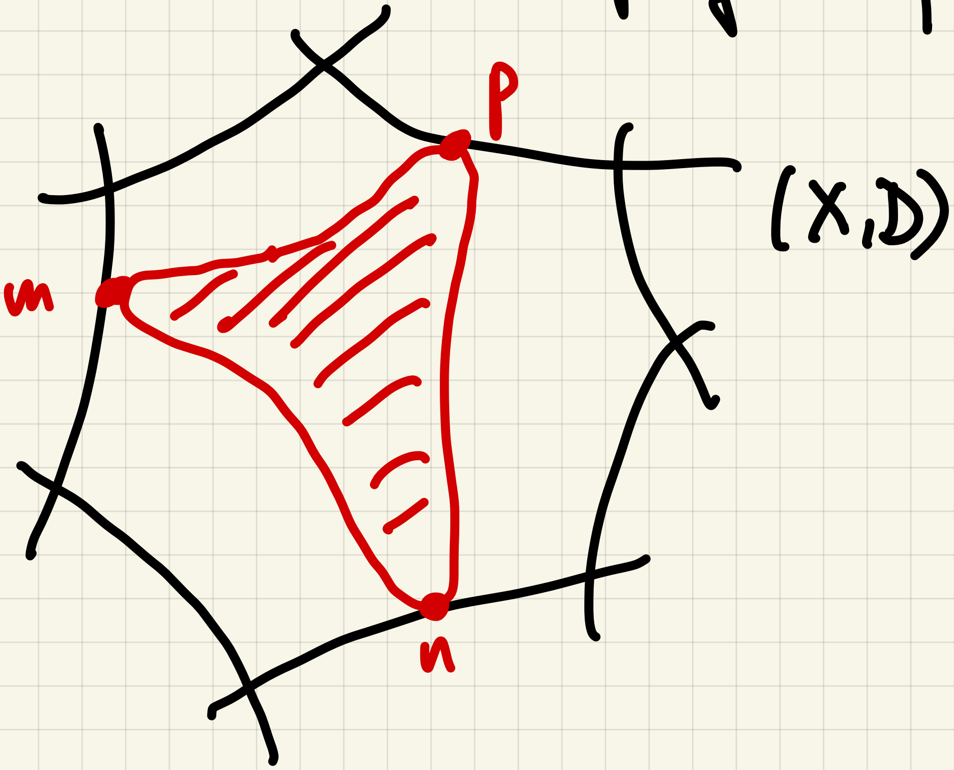
"Θ-functions" on $X^\vee$ ⟷

Interpretation in terms of $g=0$ GW invariant

Gross-Siebert

$$\Theta_m \Theta_n = \sum_p C_{mn}^p \Theta_p$$

Generating series of $g=0$ punctured GW invariants.



$\mathfrak{D}_{(X,D)}$ "canonical scattering diagram" [Gross-Siebert]

(X,D) log CY pair $\longmapsto$ B "tropicalization"
Maximal
dual intersection complex of (X,D)

$D = D_1 + \cdots + D_{i_r}$

$$\boxed{\begin{array}{c} D_{i_1} \cap \cdots \cap D_{i_k} = \phi \\ \text{codim } k \end{array}} \longmapsto \boxed{\mathbb{R}^k_{\geq 0}}$$

k-dim cone

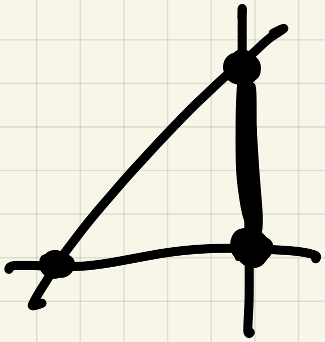
Glue

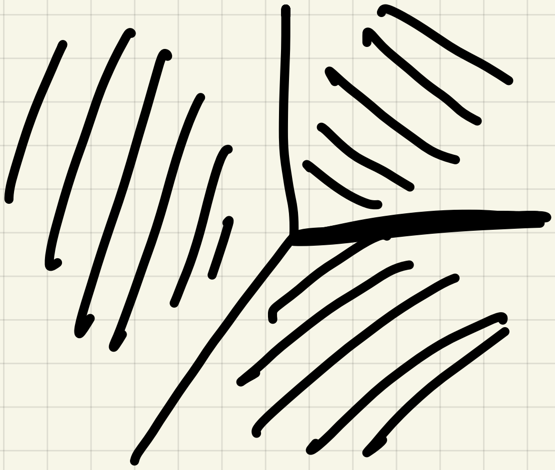
$$\hookrightarrow B = \bigcup_{\sigma \in P} \sigma$$

Cone complex

Cones

Ex: $(X,D) = (X_\Sigma, D_\Sigma)$ toric $\rightarrow B =$ fan of (X_Σ, D_Σ)





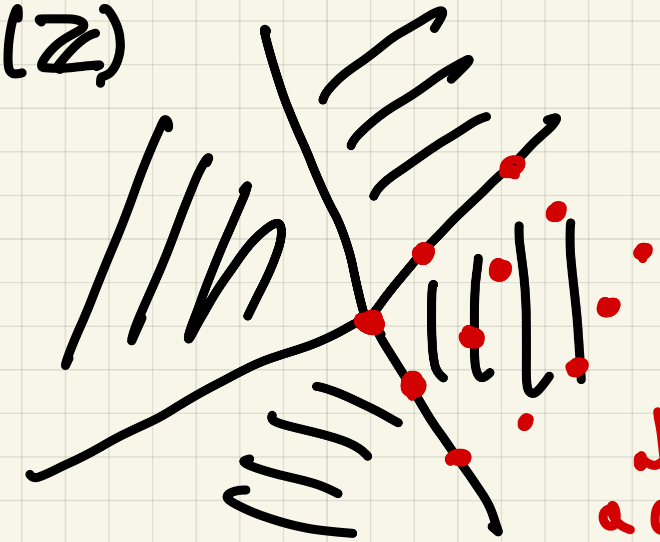
(X,D) non-toric $\rightsquigarrow B$ "generalization of the fan"

Abstract cone complex.

$\mathfrak{D}_{(X,D)}$ "canonical scattering diagram"

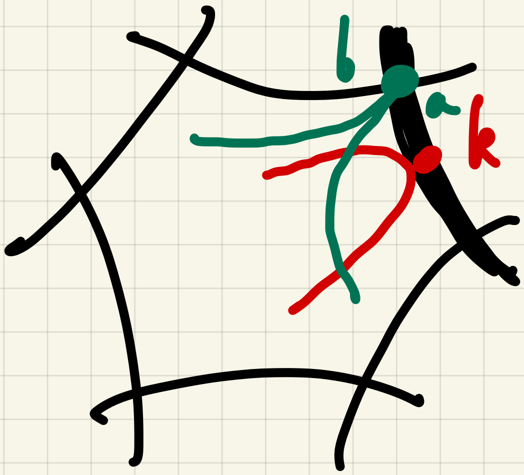
Object living in $\boxed{B}$

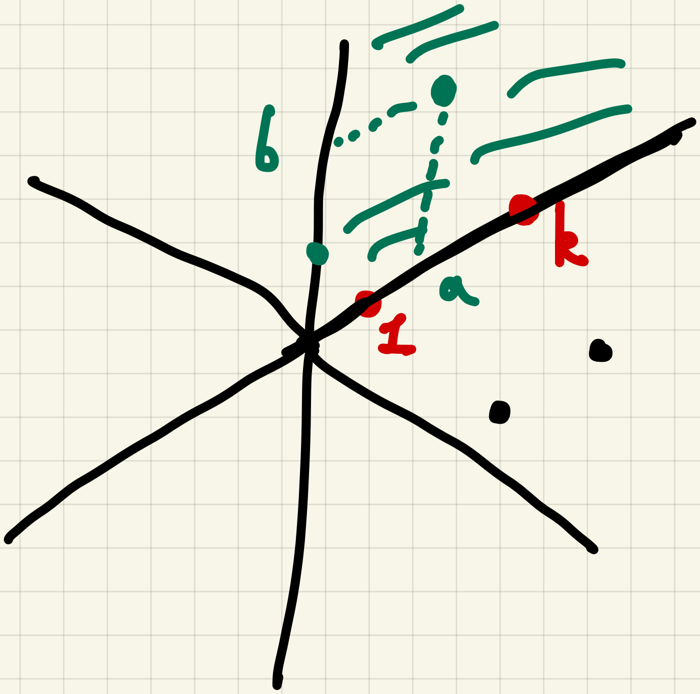
$B(\mathbb{Z})$



$m \in B(\mathbb{Z})$

$\rightsquigarrow$ Prescribing a tangency condition at a contact point between a curve in X and D.

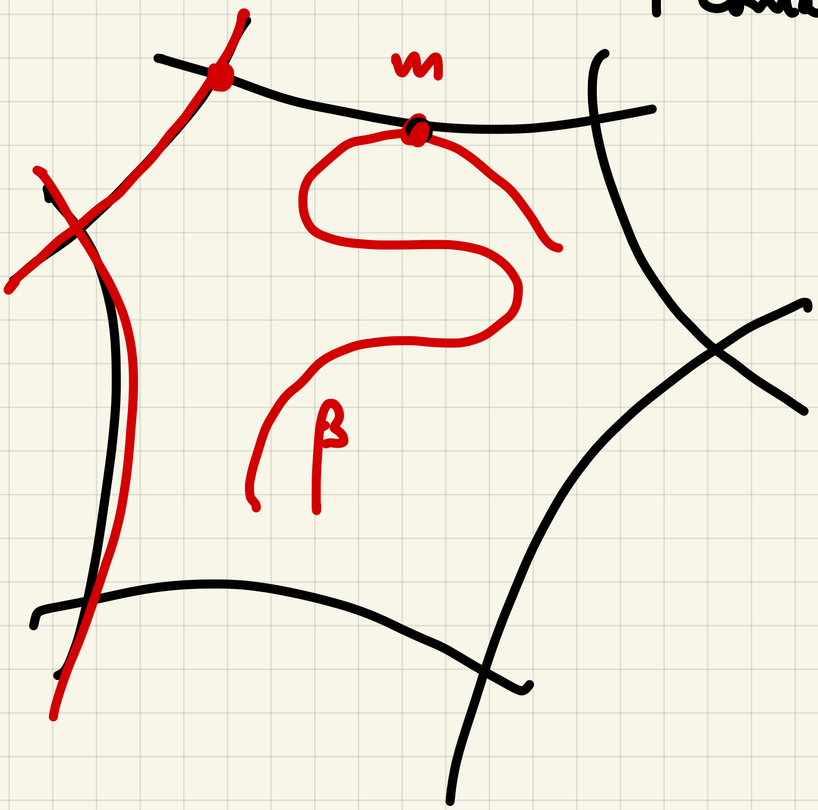




$m \in B(\mathbb{Z})$

β curve class

$$\mathcal{M}^\circ_{m,\beta} = \left\{ \begin{array}{l} \underline{f : (\mathbb{P}^1, \infty) \longrightarrow (X, D)} \\ \text{s.t. } f^{-1}(D) = \{\underset{\nearrow}{\infty}\} \\ + \text{ contact order } m \end{array} \right\}$$



(X, D)

↑

Non-compact in general

$$\begin{tikzcd}
\mathcal{M}^{\circ}_{m,\beta} \arrow[r, phantom, "\subset"] & \mathcal{M}_{m,\beta} \arrow[d, equal] \\
& \left\{ f : (C, \infty) \longrightarrow (X, D) \right\}
\end{tikzcd}$$

"Nice compactification"

$\uparrow$ Nodal of genus 0

+ Extra data
Log structures

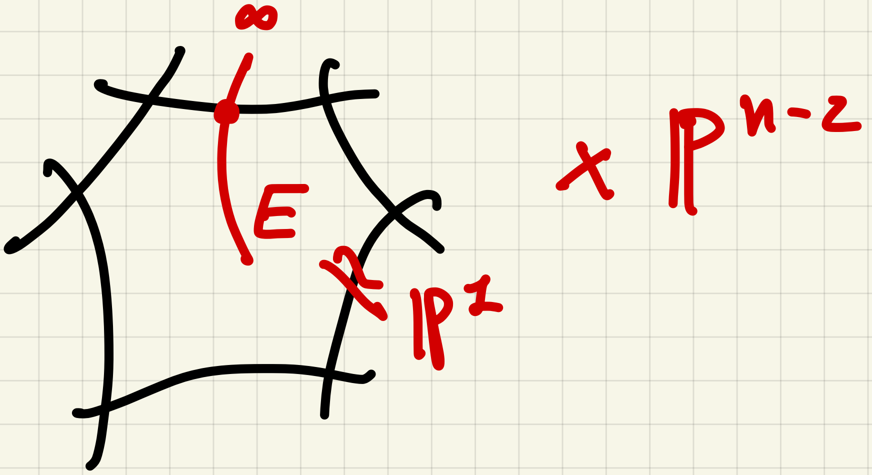
Uses "<u>Log geometry</u>"

$\underline{\mathcal{M}_{m,\beta}}$: <u>proper</u> Deligne-Mumford stack

"virtually <u>smooth</u> of dim $\boxed{n-2}$"

$n = \dim X$

$n = 2$



$$\begin{array}{c|c} \mathcal{M}_{m,\beta} & \longrightarrow \\ \dim n-2 & \end{array}$$

"<u>Canonical</u> <u>scattering</u> <u>diagram</u>" $\mathcal{D}_{(X,D)}$ on B

$\mathcal{M}_{m,\beta}$ = compactification of $\mathcal{M}^{\circ}_{m,\beta}$

"Virtually" $\mathcal{M}_{m,\beta} \setminus \mathcal{M}^{\circ}_{m,\beta}$ is a normal crossing divisor

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"] \arrow[d] & (X,D) \\ \mathcal{M}_{m,\beta} & \end{tikzcd}$$

B = dual $\cap$ of (X,D)

$\mathcal{M}_{m,\beta}$ ← dual $\cap$ of moduli space

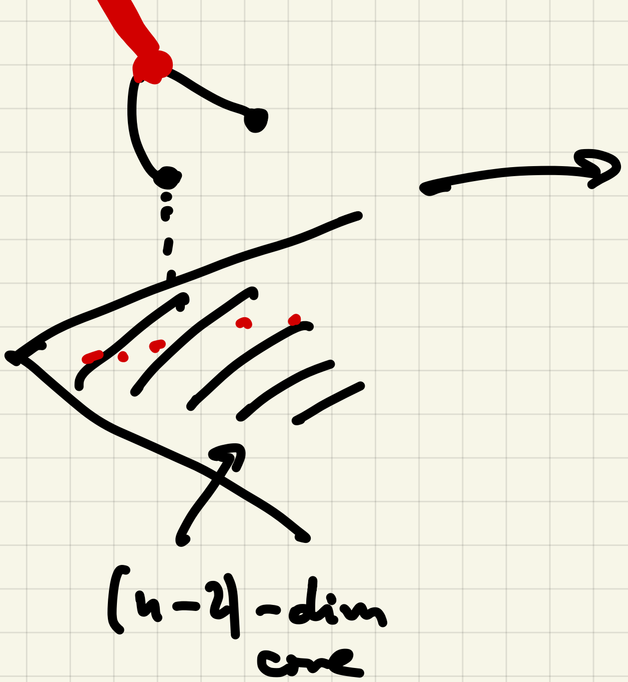
x 0-dim stratum of $\mathcal{M}_{m,\beta}$ "Most degenerate curves"

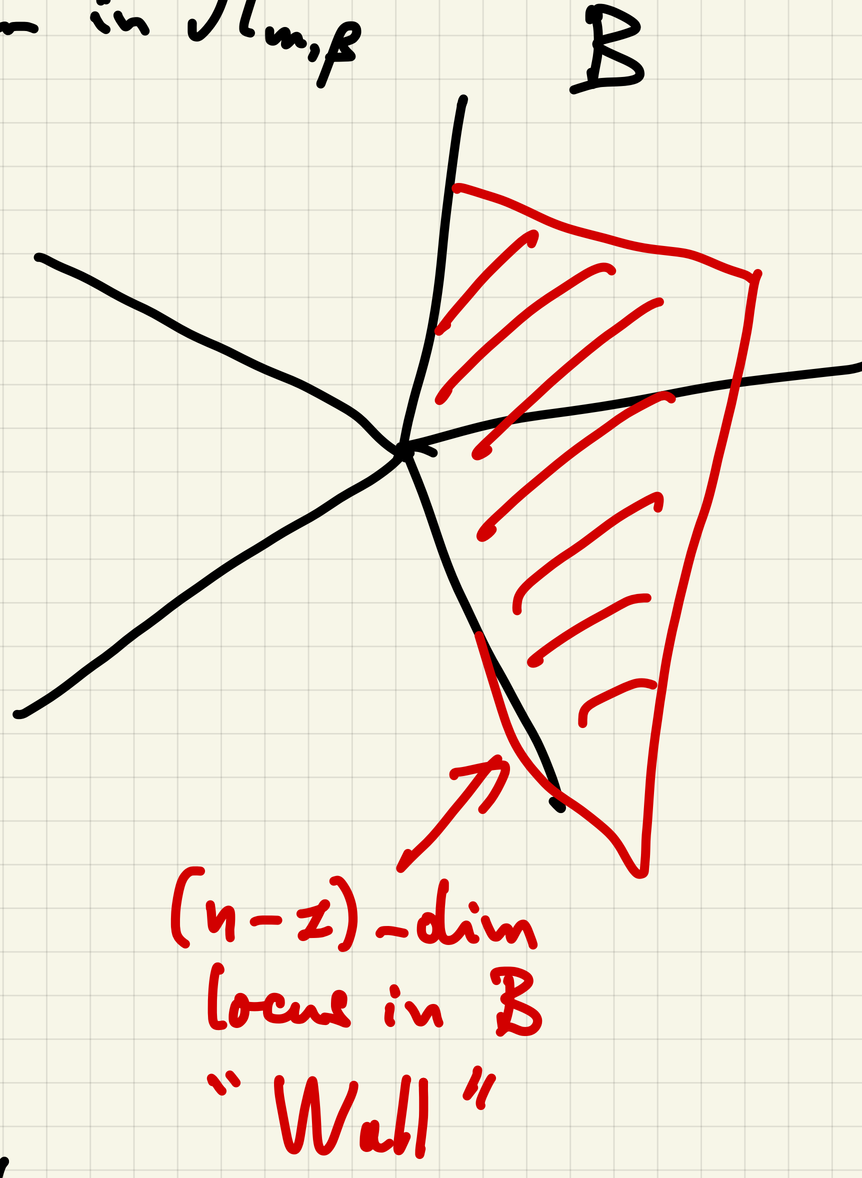
$$\begin{tikzcd} x \arrow[dd] & & \\ (C_x, p_x) \arrow[r] \arrow[d] & (X,D) & \\ \text{Family of graphs} \arrow[d] \arrow[rr] & & B \\ (n-2)\text{-dim cone} & & \end{tikzcd}$$

Relation with B?

Graphs = dual $\cap$ graph of C_x

For each 0-dim stratum in $\mathcal{M}_{m,\beta}$



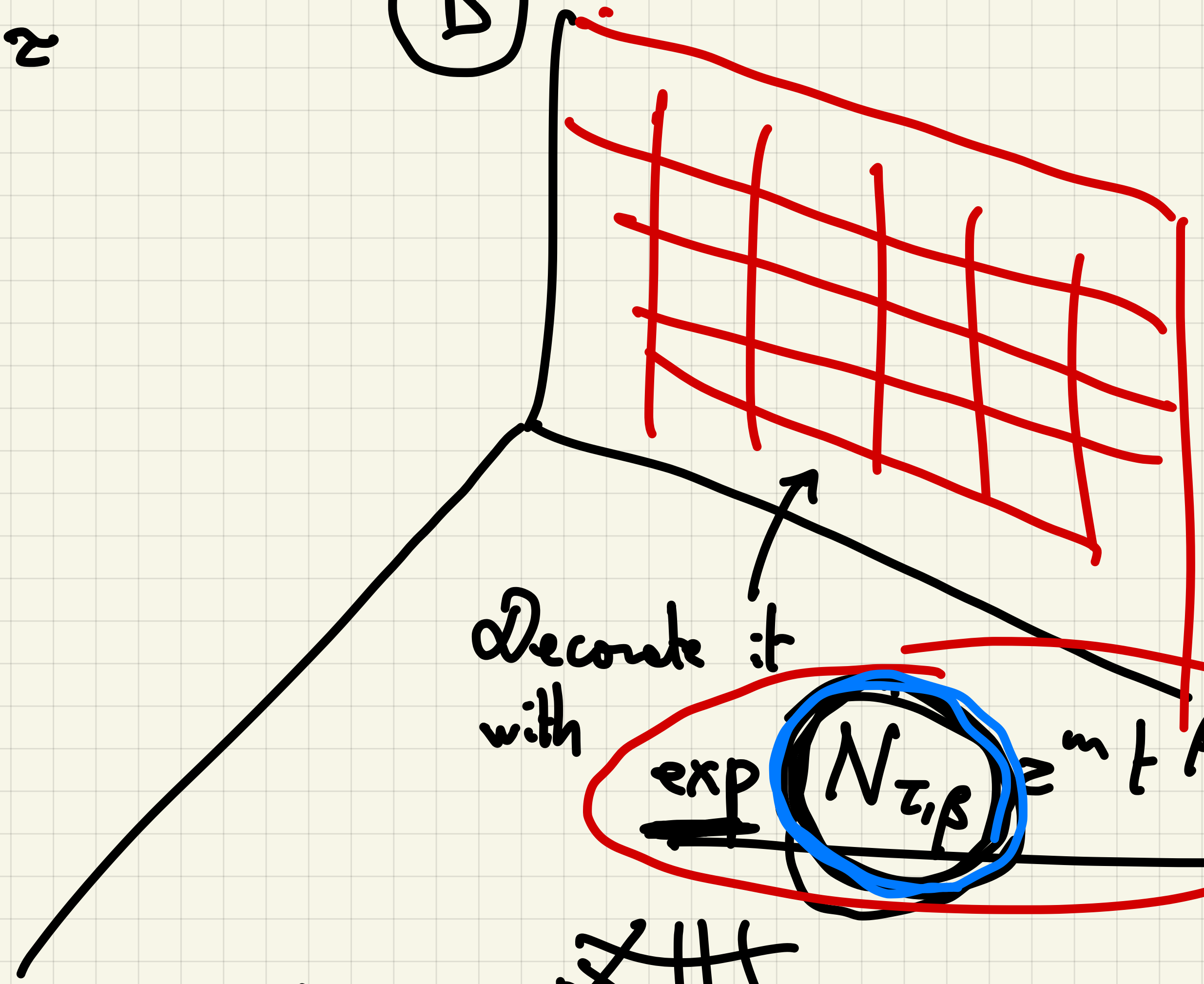


Fix τ "combinatorics"

↳ $\boxed{N_{\tau,\beta}}$ = Count of 0-dim strata in $\mathcal{M}_{m,\beta}$ with combinatorics τ and of class β

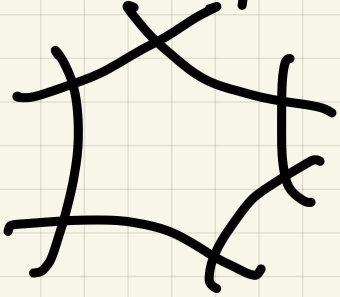
τ

B



Decorate it with $\exp\left(\boxed{N_{\tau,\beta}} z^m t^\beta\right)$

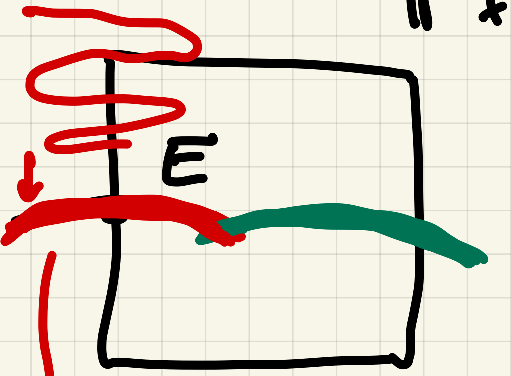
$$\begin{tikzcd}
\text{Cluster } (X,D) \arrow[d] \\
(X_\Sigma, D_\Sigma)
\end{tikzcd}$$

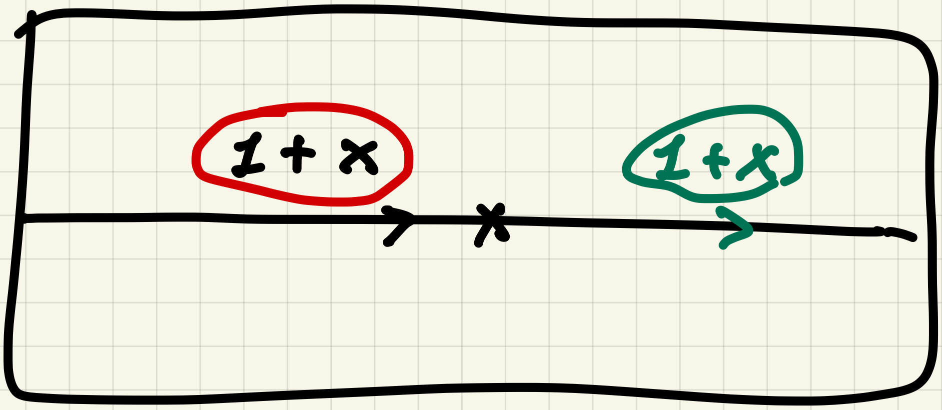


$$\Gamma : \underbrace{B}_{\text{Canonical scattering diagram}} \xrightarrow{\sim} \underbrace{M_{\mathbb{R}}}_{\text{GHKK. Scattering.}}$$

[Algorithmically defined from a set of explicit dim 2 $\mathbb{P}^1 \times \mathbb{P}^1$

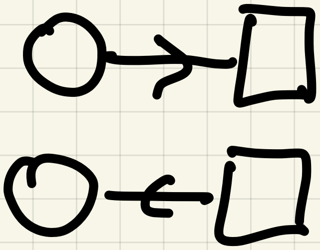
⌊ initial walls. ⌋

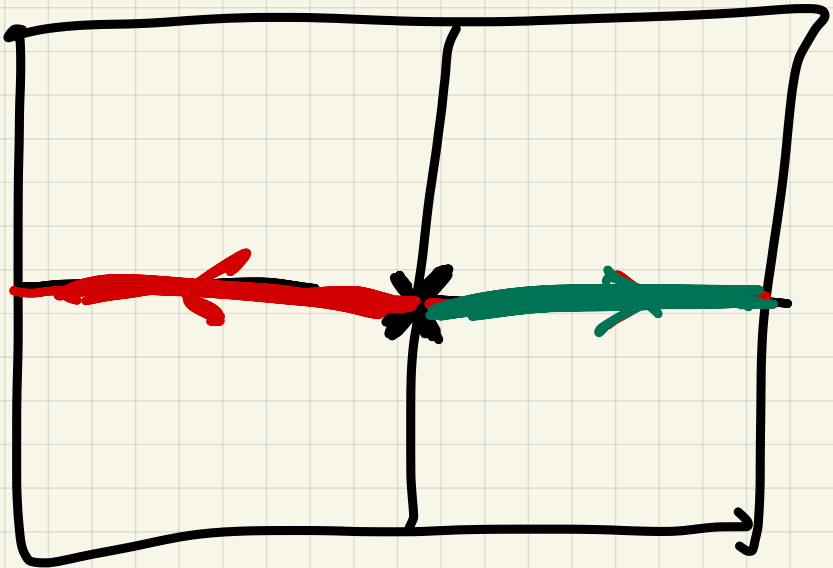




Cluster scattering

e_1, e_2
↑
Frozen





Canonical scattering diagram

[E]

$$N_{[E]} = 1$$

$$N_{k[E]} = \frac{(-1)^{k-1}}{k^2}$$

$$\exp\left(\sum_k k N_{k[E]} x^k\right)$$

$$= \exp\left(\sum_k k \frac{(-1)^{k-1}}{k^2} x^k\right) = \boxed{1+x}$$