Mirror symmetry and enumerative geometry of cluster varieties.

(Pierrick Bousseau, University of Georgia)

Today & tomorrow: Mirror sym for cluster varieties

Two mirror constructions ↗ Fock-Goncharov duality (combinatorial)

↘ Gross-Siebert mirror symmetry (Enumerative geometry: $g=0$ Gromov-Witten invariants) ^ Log

Main result:
a comparison between these two approaches
(jt work with H. Argüz 2206.10584)

Last lecture: Donaldson-Thomas invariants of quivers with potentials.

Main result: a correspondence with $g=0$ log GW invariants.

Plan for today: 1/ Cluster varieties & Fock-Goncharov duality,

2/ Gross-Siebert mirror construction (sketch)

3/ Main result: comparison between 1/ and 2/.

1/ Cluster varieties.

Start with a seed s :
- finite rank free abelian group $N \simeq \mathbb{Z}^n$
- basis $(e_i)_{i \in I}$ of N
- Skew-sym form $\omega : N \times N \to \mathbb{Z}$

Seed s, Fix e_k $\longmapsto$ New seed : $\mu_k(s)$ mutation of s at e_k:

same N, ω

New basis $e_i' = \begin{cases} e_i + \max(\omega(e_i, e_k), 0) e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k \end{cases}$

$$s \to \mu_k(s) \to \mu_\ell(\mu_k(s)) \to \cdots$$

Cluster varieties: union of tori $\underbrace{(\mathbb{C}^*)^n}_{\text{one copy for each mutated seed.}}$

$I \subset \bar{I}$ | Only allow mutations at unfrozen indices.

"Unfrozen indices"

$M = \mathrm{Hom}(N, \mathbb{Z})$ $\quad v_i = \iota_{e_i}\omega = \omega(e_i, -) \in M$

$s \quad e_i \quad N \quad \omega$

$v_i \quad M$

$\mathcal{X}$-cluster variety $\mathcal{X} = \bigcup \underbrace{\mathrm{Spec}\, \mathbb{C}[N]}_{\substack{\text{IS} \\ (\mathbb{C}^*)^n}}$ torus with character lattice N

Cluster birational transformations

$$z^n \mapsto z^n (1 + z^{e_k})^{-\underbrace{(v_k, n)}}$$

$z^n \quad n \in N$

$(-,-)$ duality pairing between M and N

$\mathcal{A}$-cluster variety $\mathcal{A} = \bigcup \underbrace{\mathrm{Spec}\, \mathbb{C}[M]}$

$(\mathbb{C}^*)^n$ torus with character lattice M

$z^m \quad m \in M$

$$z^m \mapsto z^m (1 + z^{v_k})^{-(e_k, m)}$$

Fock-Goncharov

$\mathcal{X}$ and $\mathcal{A}$ are both Calabi-Yau varieties
Admit holomorphic volume form.

$(\mathbb{C}^*)^n \qquad \Omega = \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$

Log Calabi-Yau varieties. (X, D)

X: smooth proj variety $/\mathbb{C}$

D: normal crossing divisor anticanonical: $\underbrace{K_X + D = 0.}$

$U = X \setminus D$ ← Non-compact Calabi-Yau varieties.

<u>Ex:</u> X_Σ smooth proj <u>toric</u> variety, defined by a fan Σ

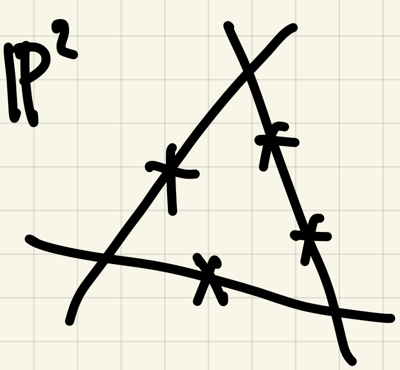
D_Σ toric boundary divisor / ← Is anticanonical

(X_Σ, D_Σ) is a log CY pair

$U = X_\Sigma \setminus D_\Sigma = (\mathbb{C}^*)^n$

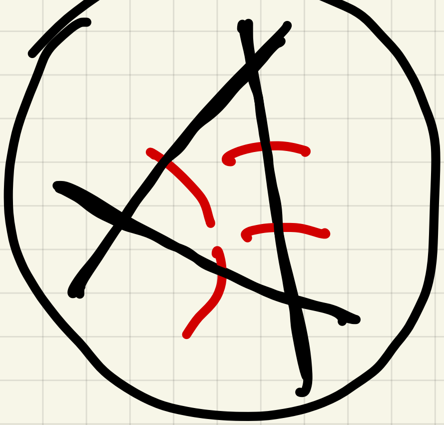
<u>Ex:</u> Start with (X_Σ, D_Σ) toric.

$\mathbb{P}^2$



Pick general hypersurface $H \subset D_\Sigma$ (codim 2 in X_Σ)

$X :=$ blow-up of X_Σ along H.



X D

D : strict transform of D_Σ

(X,D) is log CY.

$$U = X \setminus D \neq (\mathbb{C}^*)^n$$
$$\cup$$
$$(\mathbb{C}^*)^n$$

Gross-Hacking-Keel: cluster varieties can be obtained that way.

$\mathcal{A}$ Σ fan in N containing the rays $\mathbb{R}_{\geq 0} e_i$.
(smooth proj)

↳ X_Σ smooth proj toric variety

$D_{\Sigma,i}$

$$H_i = \{\underbrace{1+z^{v_i}} = 0\} \subset D_{\Sigma,i}$$

$$X = Bl_{\cup_i H_i}(X_\Sigma) \qquad D \uparrow \qquad U_{\mathcal{A}} = X \setminus D$$

Only for $i \in I$

← non-compact CY variety.

Similarly for $\mathcal{X}$

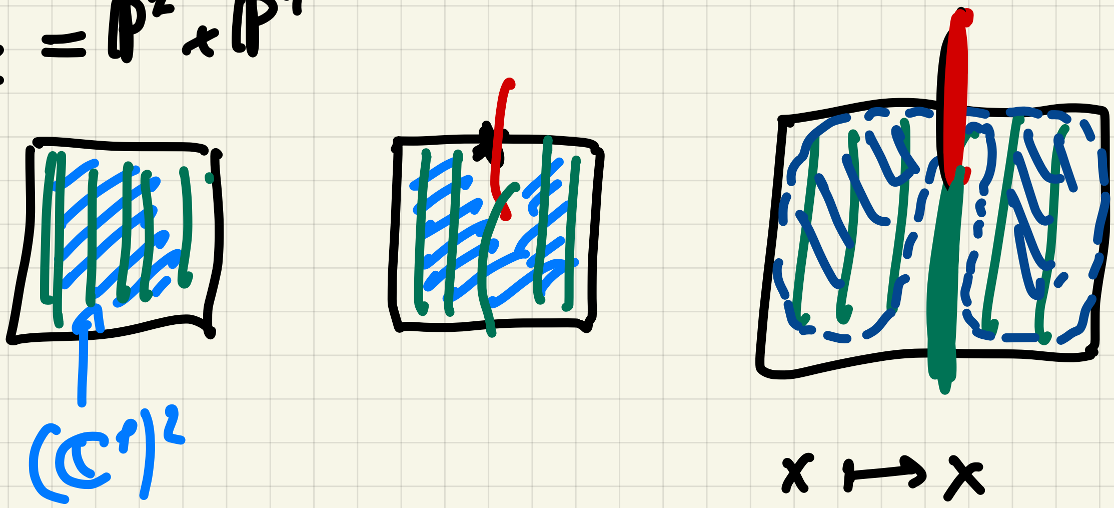
Σ M $\mathbb{R}_{\geq 0} v_i$

$$H_i = \{(1+z^{e_i})^{|v_i|} = 0\}$$

$$\rightsquigarrow (X', D') \qquad U_{\mathcal{X}} = X' \setminus D'$$

[Thm (GHK) Up to codim 2, $\underline{U_{\mathcal{A}}} \simeq \mathcal{A}$
$\underline{U_{\mathcal{X}}} \simeq \mathcal{X}$

$X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$



$x \mapsto x$
$y \mapsto y(1+x)$

Fock-Goncharov duality:

Cluster varieties come in pairs $\mathcal{A}$ and $\mathcal{X}$.
"dual"!

↳ "Mirror" ?

2/ Mirror symmetry construction.

Mirror symmetry: X CY variety $\rightsquigarrow$ $X^\vee$ mirror CY variety

Symplectic geometry of X $\longleftrightarrow$ Complex geometry of $X^\vee$

(X, D) log CY pair

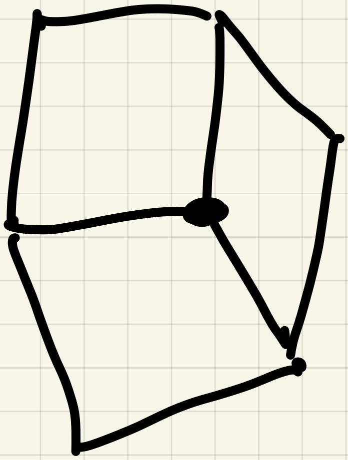
$U = X \setminus D$

Gross–Siebert: "general mirror construction"

<u>Maximal</u> (D contains a 0-dim stratum)

<u>Automatic</u> if (X_Σ, D_Σ) toric.

(X, D) obtained as blow-up.



$$\underbrace{(X, D)}_{} \overset{\text{Output}}{\rightsquigarrow} \text{Mirror family } X^\vee$$

$$X^\vee \to \operatorname{Spf} \mathbb{C}[[\underbrace{NE(X)}]]$$

$NE(X)$: monoid spanned by effective curve classes on X in $\left\{ \begin{matrix} \text{curve} \\ \text{classes} \end{matrix} \right\} / \begin{matrix} \text{Num} \\ \text{equivalence} \end{matrix}$

Symplectic Geometry

Kähler moduli space $\sim H^2(X) = \{\omega\}$ → Complex moduli

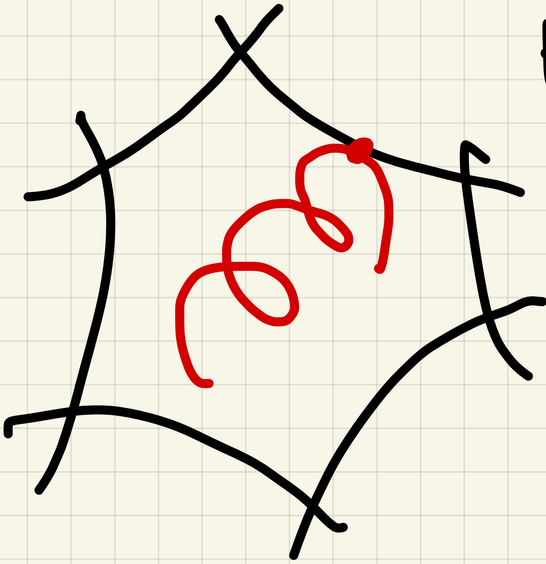
Space t^β $\quad \beta \in NE(X)$

$$\sum_\beta \cdots t^\beta$$

$\beta \quad \omega \mapsto t^{\beta \cdot \omega}$

Uses enumerative of (X, D)

Gromov-Witten counts of rational curves in X with tangency conditions along D.



3/ <u>Comparison.</u>

$$\begin{tikzcd} \mathcal{X} \arrow[r, leftrightarrow, "\text{FG duality}"] \arrow[d, equal] & \mathcal{A} \arrow[d, leftrightarrow, "?"] \\ X \setminus D & X^\vee \end{tikzcd}$$

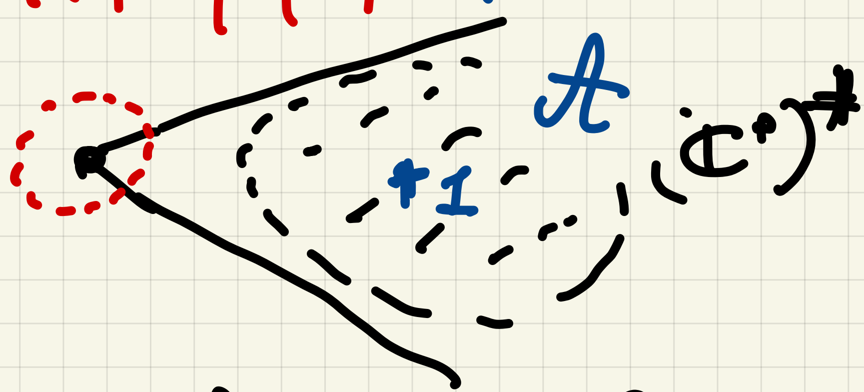
(X, D)

$$\begin{tikzcd} (X, D) \arrow[r, squiggly, "\substack{\text{Gross-Siebert} \\ \text{MS construction} \\ \text{GW\#}}"'] & X^\vee \arrow[d] \\ & \operatorname{Spf} \mathbb{C}[[NE(X)]] \end{tikzcd}$$

"Easy" case: if $\mathcal{X}$ is affine

$\rightarrow$ Mirror family

$$\begin{array}{c} X^\vee \\ \downarrow \pi \\ \operatorname{Spec} \mathbb{C}[NE(X)] \end{array}$$



<u>Thm</u> (Weel-Yu)

$\pi^{-1}(1) \simeq \mathcal{A}$

<u>GOAL</u>: What about the general case where $\mathcal{X}$ is not affine?

Mirror family

$$\begin{array}{c} X^\vee \\ \downarrow \\ \operatorname{Spf} \mathbb{C}[[NE(X)]] \end{array}$$

$\mathcal{A}$

Should are a family containing $\mathcal{A}$.

$\mathcal{A}_{prin}$: $\mathcal{A}$ - cluster with principal coeff

$|$

$\mathcal{A}$

$\downarrow \qquad \downarrow$

$\operatorname{Spec}\mathbb{C}[N_{uf}] \ni 1$

$\mathcal{A} \qquad 1 + t_i z^{v_i} = 0$

$t_i = 1 \rightsquigarrow \mathcal{A}$

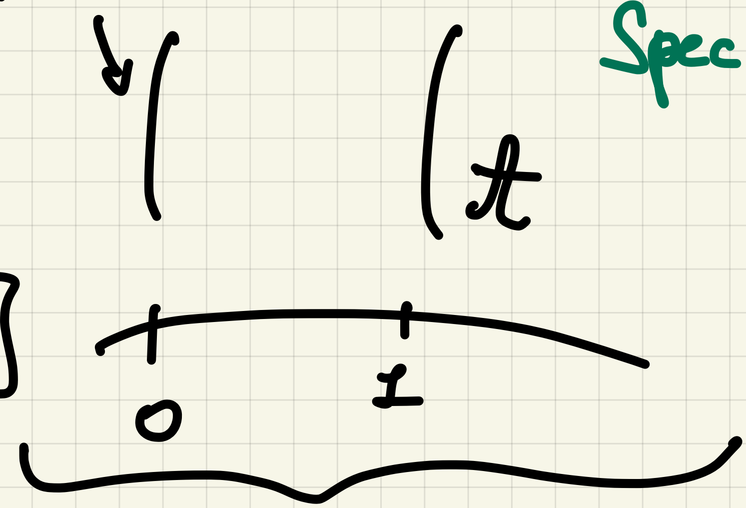
$t_i = 0 \rightsquigarrow$ Torus $\operatorname{Spec}\mathbb{C}[M]$

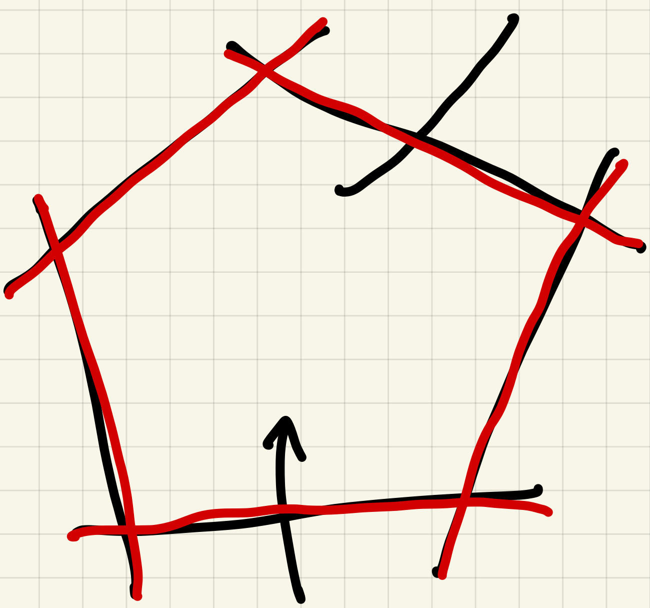
$\overline{\mathcal{A}}_{prin}$

$\downarrow$

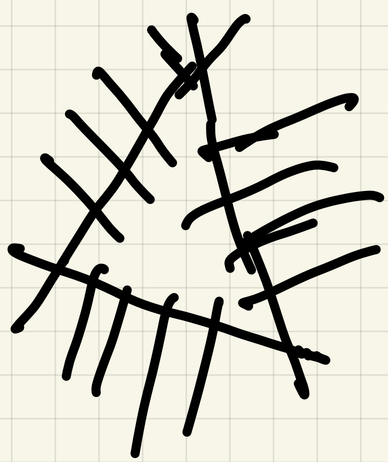
$\operatorname{Spec}\mathbb{C}[N_{uf}^{\oplus}]$

$\operatorname{Spec}\mathbb{C}[M]$





$(\mathbb{C}^*)^n$



$$
\begin{tikzcd}
(X,D) \arrow[rr] & {} & X^\vee \arrow[d] \\
\text{Geometry} \arrow[r] & \mathfrak{D}_{(X,D)} \arrow[u] \arrow[rd, leftrightarrow] & \mathrm{Spf}\, \mathbb{C}[[NE(X)]] \\
 & & \mathfrak{D}^{GHKK} \arrow[r, squiggly] & \mathcal{A}
\end{tikzcd}
$$

[canonical scattering diagram]