What we did so far ...

<u>Idea</u> Consider quiver representations for classical groups

(Q,σ) symmetric quiver

$A := kQ/I$ symmetric quiver algebra

$V, \underline{d}, \varepsilon, \langle \cdot , \cdot \rangle$

condition

$\langle M_\alpha(v), w\rangle + \langle v, M_{\sigma(\alpha)}(w)\rangle$

$\forall \alpha : i \to j \ \forall v \in V_i \ \forall w \in V_{\sigma(j)}$

$(\Leftrightarrow M = -M^*)$

rep. variety	$R_{\underline{d}} A$	$\supseteq$	$R_{\underline{d}}^{\varepsilon} A$
	$\circlearrowleft$	change of basis	$\circlearrowleft$
	$G_{\underline{d}}$	$\supseteq$	$G_{\underline{d}}^{\varepsilon}$ $(g = (g^{-1})^*)$
	representations		ε-representations

Yesterday: Orbits and their classification!

* $\nabla : \operatorname{rep} A \to \operatorname{rep} A$ duality

* M ε-representation wrt $\langle\, ,\rangle$

  $\iff$ $\exists$ isomorphism $\varphi : M \longrightarrow \nabla M$
  s.th. $\nabla\varphi = \varepsilon\varphi$

Theorem [DW, BC]

Let $M, N \in R^{\varepsilon}_{\underline{d}} A$ wrt φ.

$$G_{\underline{d}} M = G_{\underline{d}} N \iff G^{\varepsilon}_{\underline{d}} M = G^{\varepsilon}_{\underline{d}} N$$

$$\boxed{\nabla \tau = \tau^{-} \nabla}$$

Example

Theorem [DW]

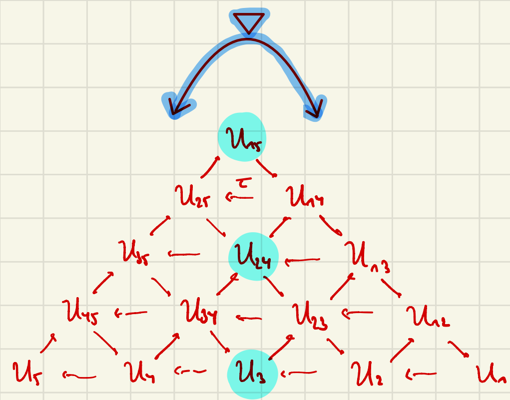
Let M be an indecomposable ε-rep.

One of the three cases appears:

(1) $M = L$ indec. rep "indecomposable"

(2) $M = L \oplus \nabla L$, L indec rep, $L \not\cong \nabla L$ "split"

(3) $M = L \oplus \nabla L$, L indec rep, $L \cong \nabla L$ "ramified"



5. Orbit closures

Let $M \in R^{\varepsilon}_{\underline{d}} A$

$$\overline{G^{\varepsilon}_{\underline{d}} M} = \bigcup G^{\varepsilon}_{\underline{d}} N$$

s.th. $\dim G^{\varepsilon}_{\underline{d}} N < \dim G^{\varepsilon}_{\underline{d}} M$ if $N \not\cong M$

<u>Question</u>

Decide whether

$$\left(G^{\varepsilon}_{\underline{d}} N \subseteq \overline{G^{\varepsilon}_{\underline{d}} M} \right) =: M \leq^{\varepsilon}_{deg} N$$

" ε- degeneration-order "

$$\begin{tikzcd} \leq_{deg} \arrow[r, Rightarrow, bend right, "?"'] & \leq^{\varepsilon}_{deg} \arrow[l, Rightarrow, bend right, "\checkmark"'] \end{tikzcd}$$

Many results on degenerations were obtained in the 70s - 90s.

We sketch them now and discuss $\leq^{\varepsilon}_{deg}$ afterwards.

Definition

Let $M, N \in R_d A$.

- $M \le_{hom} N \;:\Longleftrightarrow\; \dim_k \mathrm{Hom}(U,M) \le \dim_k \mathrm{Hom}(U,N) \;\; \forall U$

  "hom-order" [ADF, R]

- $\le_{ext}$ transitive closure of

$$\left(\begin{array}{c} M \le N \\ :\Longleftrightarrow \\ \exists\; 0 \to U \hookrightarrow M \to V \to 0 \\ \text{s.th. } N \cong U \oplus V \end{array} \right)$$

  "ext-order" [B]

Note $\le_{deg}$, $\le_{hom}$, $\le_{ext}$ are partial orders.

Theorems

IN GENERAL

$$\le_{ext} \overset{[B]}{\Longrightarrow} \le_{deg} \overset{[ADF,R]}{\Longrightarrow} \le_{hom}$$

DYNKIN $A = KQ$

$$\le_{ext} \overset{[B]}{\Longleftrightarrow} \le_{deg} \overset{[B]}{\Longleftrightarrow} \le_{hom}$$

REP-FINITE

$$\le_{deg} \overset{[Z]}{\Longleftarrow} \le_{hom}$$

This is fantastic! Homs and Exts are well-known via ART in many cases.

<u>Next goal</u> Find analogues for ε-degs.

The plan for the rest of the lecture:

(1) Find $\mathcal{E}xt$ and $\mathcal{H}om$-analogues

(2) The Dynkin case

(3) First look at the rep-finite case

(1) Find $\leq_{ext}$ and $\leq_{hom}$-analogues

Definition Let $M, N \in R^{\varepsilon}_{d} A$.

- $M \leq_{hom} N \;:\Longleftrightarrow\; \dim_k \mathrm{Hom}(U,M) \leq \dim_k \mathrm{Hom}(U,N) \;\forall U$

  "hom-order"

- $\leq^{\varepsilon}_{ext}$ transitive closure of

$$\left(\begin{array}{c} M \leq^{\varepsilon} N \\ :\Longleftrightarrow \\ \exists\; 0 \to U \hookrightarrow M \to V \to 0 \\ \text{s.th. } \cancel{N \cong U \oplus V} \\ N \cong U \oplus \nabla U \oplus U^{\perp}/U \end{array} \right)$$

↳ isotropic $\iota(U) \leq \iota(U)^{\perp}$ $\qquad$ ε-rep!

Note $\leq^{\varepsilon}_{deg}$, $\leq_{hom}$, $\leq^{\varepsilon}_{ext}$ are partial orders.

Lemma $M \leq^{\varepsilon}_{ext} N \Longrightarrow M \leq^{\varepsilon}_{deg} N \Longrightarrow M \leq_{hom} N$

Sketch of proof Assume $M \leq^{\varepsilon}_{ext} N$ s.th.

$\exists\; 0 \to U \hookrightarrow M \to V \to 0$

s.th. $N \cong U \oplus \nabla U \oplus U^{\perp}/U$, $\underbrace{\iota(U)}_{\cong U} \leq \underbrace{\iota(U)^{\perp}}_{\cong U^{\perp}}$

Then there is a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & Y \\
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & U^{\perp} \arrow[r] \arrow[d, "f"] & U^{\perp}/U \arrow[r] \arrow[d] \arrow[ur, phantom, "\cong" sloped] & 0 \\
0 \arrow[r] & U \arrow[r, hook] & M \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
 & & \nabla U \arrow[r, equal] \arrow[d] & \nabla U \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

type A $\Longrightarrow$ $\quad M \leq_{deg} U \oplus V \leq_{deg} U \oplus U^{\perp}/U \oplus \nabla U$

Write

$$M_\alpha = \begin{pmatrix} u_\alpha & \nu_\alpha & \xi_\alpha \\ 0 & Y_\alpha & \mu_\alpha \\ 0 & 0 & \nabla u_\alpha \end{pmatrix}$$

Set $\lambda(t) := \begin{pmatrix} t\,\mathrm{id}_u & 0 & 0 \\ 0 & \mathrm{id}_y & 0 \\ 0 & 0 & t^{-1}\mathrm{id}_{\nabla u} \end{pmatrix} \in G^{\varepsilon}_{\underline{d}}$ if $t \neq 0$

Then

$$\lambda(t).M = \begin{pmatrix} u_\alpha & t\nu_\alpha & t^2\xi_\alpha \\ 0 & Y_\alpha & t\mu_\alpha \\ 0 & 0 & \nabla u_\alpha \end{pmatrix} \xrightarrow[t\to 0]{} N$$

$\square$

Example

$$Q = \underset{i}{\bullet} \xrightarrow{\alpha} \underset{j}{\bullet} \xrightarrow{\sigma(\alpha)} \underset{\sigma(i)}{\bullet}$$

$$\underline{d} = (2, 2, 2)$$

$$M = \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \mathbb{C}^2$$

$$= \begin{array}{ccccc} \bullet & \xrightarrow{1} & \bullet & \xrightarrow{1} & \bullet \\ \bullet & \xrightarrow{-1} & \bullet & \xrightarrow{-1} & \bullet \end{array}$$

$$= \begin{array}{ccccc} \bullet & \xrightarrow{1} & \bullet & \xrightarrow{1} & \bullet \\ \bullet & \xrightarrow{-1} & \bullet & \xrightarrow{-1} & \bullet \end{array}$$

$$\cong U_{13}^2$$

$$U = 0 \to \mathbb{C} \xrightarrow{-1} \mathbb{C}$$

$$= \qquad \bullet \xrightarrow{-1} \bullet$$

Diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U \arrow[r] \arrow[d, equal] & U^{\perp} \arrow[r, two heads] \arrow[d, hook] & U^{\perp}/U \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & U \arrow[r] & M \arrow[r, two heads] \arrow[d, two heads] & X \arrow[r] \arrow[d, two heads] & 0 \\
 & & \nabla U \arrow[r, equal] \arrow[d] & \nabla U \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

$S_1 \oplus S_3$

$$M \leq_{\mathrm{deg}}^{\varepsilon} U \oplus \nabla U \oplus U^{\perp}/U$$

$$\cong U_{23} \oplus U_{12} \oplus S_1 \oplus S_3 = \begin{array}{ccccc} \bullet & \xrightarrow{1} & \bullet & & \bullet \\ \bullet & & \bullet & \xrightarrow{-1} & \bullet \end{array}$$

Type A

IN GENERAL

Types B, C, D

$$\leq_{ext} \Rightarrow \leq_{deg} \Rightarrow \leq_{hom}$$

$$\leq^{\varepsilon}_{ext} \Rightarrow \leq^{\varepsilon}_{deg} \Rightarrow \leq_{hom}$$

DYNKIN $A = KQ$

$$\leq_{ext} \Leftarrow \leq_{deg} \Leftarrow \leq_{hom}$$

Next step

A rep-finite

REP-FINITE

$$\leq_{deg} \Leftarrow \leq_{hom}$$

(2) The Dynkin case

Let Q be symmetric

$A = kQ$ be rep-finite.

$\overset{\text{DW}}{\Longrightarrow}$ Q Dynkin quiver of type A.

<u>Theorem</u> [BC, two versions]

Let $M, N \in R_{\underline{d}}^{\varepsilon} A$.

Then

$$M \leq_{ext}^{\varepsilon} N \Longleftrightarrow M \leq_{deg}^{\varepsilon} N \Longleftrightarrow M \leq_{hom} N$$

$$\Updownarrow$$

$$M \leq_{deg} N$$

!

<u>About the proof</u>

It is constructive! By induction ($\dim M$)

we obtain a sequence of ε-repr and

1-psgs going from one orbit to the other.

Let $M \leq_{deg} N$.

$\exists\, L \in N$ s.th. $\dim \mathrm{Ext}^{\wedge}(L, N) = 0$ (rep-directed)

$\overset{[B]}{\Longrightarrow}$ $L \hookrightarrow M$

<u>Case 1</u> $L \simeq \nabla L$:

only uses results of Bongartz in type A (Cancellation)

<u>Case 2</u> $L \not\simeq \nabla L$

Let's look at an example first!

The crucial steps are:

- $L \hookrightarrow M$ isotropically

  split type easy

  non-split type complicated

  (Prop 6.12 long version)

- explicit description of $Y = L^{\perp}/L$

  via ARQ combinatorics

- Show $Y \leq_{hom} X$

  (dims of homs between indecs,

  hom dims with ∇,

  dim formulas in rectangles in ARQ,

  description of Y, --)

$\square$

Write to Giovanni or me whenever

you have a question!

Type A | **Types B, C, D**

IN GENERAL

$\le_{ext} \Rightarrow \le_{deg} \Rightarrow \le_{hom}$

$\le^{\mathcal{E}}_{ext} \Longrightarrow \le^{\mathcal{E}}_{deg} \Longrightarrow \le_{hom}$

DYNKIN $A = KQ$

$\le_{ext} \Leftarrow \le_{deg} \Leftarrow \le_{hom}$

$\le^{\mathcal{E}}_{ext} \Longleftrightarrow \le^{\mathcal{E}}_{deg} \Longleftrightarrow \le_{hom}$

REP-FINITE

$\le_{deg} \Leftarrow \le_{hom}$

Next step

13) First look at the rep-finite case

Example 1

$$Q = \begin{tikzcd} \underset{1=\sigma(1)}{\bullet} \arrow[loop right, "\alpha=\sigma(\alpha)"] \end{tikzcd}$$

$\underline{d} = (1)$

$$\leq_{deg} \iff \leq_{deg}^{\varepsilon}$$

Gerstenhaber [Ge] $\qquad$ Hesselink [He]

Example 2

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} \arrow[r, "\beta"] & \underset{\omega}{\bullet} \arrow[loop above, "\gamma=\sigma(\gamma)"] \arrow[r, "\sigma(\beta)"] & \underset{\sigma(2)}{\bullet} \arrow[r, "\sigma(\alpha)"] & \underset{\sigma(1)}{\bullet} \end{tikzcd}$$

$$\left.\begin{array}{l} \mathbb{C}Q \\ \cup\!| \\ I = (\gamma^2, \sigma(\beta)\circ\beta) \end{array}\right\} A := \mathbb{C}Q/I$$

$\underline{d} = (1, 2, 4, 2, 1)$

Let

$$M_x = \begin{tikzcd} \mathbb{C} \arrow[r, hook, "\iota_1"] & \mathbb{C}^2 \arrow[r, "\iota_2"] & \mathbb{C}^4 \arrow[loop above, "x"] \arrow[r, two heads, "p_2"] & \mathbb{C}^2 \arrow[r, two heads, "p_1"] & \mathbb{C} \end{tikzcd}$$

$$\iota_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \iota_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad p_1 = (0 \; -1)$$

Set
$$A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Then $M_A \leq_{deg} M_B$ (via $\leq_{hom}$)

$$\dim \overline{G_{\underline{d}}^{1} M_A} = \dim \overline{G_{\underline{d}}^{1} M_B}$$

$$\implies M_A \not\leq_{deg}^{\varepsilon} M_B$$

$$\boxed{\leq_{deg}^{\varepsilon} \quad \not\Leftrightarrow \quad \leq_{deg} \quad \iff \quad \leq_{hom}}$$

Type A		Types B, C, D
	IN GENERAL	
$\leq_{ext} \Rightarrow \leq_{deg} \Rightarrow \leq_{hom}$		$\leq^{\varepsilon}_{ext} \Longrightarrow \leq^{\varepsilon}_{deg} \Longrightarrow \leq_{hom}$
	DYNKIN $A = KQ$	
$\leq_{ext} \Leftarrow \leq_{deg} \Leftarrow \leq_{hom}$		$\leq^{\varepsilon}_{ext} \Longleftrightarrow \leq^{\varepsilon}_{deg} \Longleftrightarrow \leq_{hom}$
	REP-FINITE	
$\leq_{deg} \Longleftarrow \leq_{hom}$		$\exists$ example s.th. $\leq^{\varepsilon}_{deg} \not\Longleftrightarrow \leq_{hom}$

6. Outlook

Next goals

There is a lot to figure out!

- → rep-directed algebras
- → rep-finite cases
- → tame cases
- → examples, examples, examples

Note

There are results about symmetric moduli spaces: Franzen, Young [FY, Y]

Generalizations

[DW]: quiver approach for arbitrary reductive groups:

"generalized quiver with dimension vector"

[MW7] Fixed point group actions

$G \circlearrowleft X \qquad \rho : G \to G$

$\qquad\qquad\quad \Delta : X \to X$

with certain conditions

$\downarrow$

$G^{\rho} \circlearrowleft X^{\Delta} \qquad$ Q: connection between the actions?

Our actions fall into this setting.

About Cluster algebras ...

PhD project of Azzurra Ciliberti (Sapienza Università di Roma)

supervisor: Giovanni Cerulli Irelli

<u>Idea</u> Q Dynkin type A_{2n-1}

$\{\text{indec. } \varepsilon\text{-reps}\} \xleftrightarrow{\text{bij.}} \{\text{positive roots of root system of type } \underset{\varepsilon=1}{B_n} / \underset{\varepsilon=-1}{C_n}\}$

<u>1st task</u> give an explicit bijection via

Fomin / Zelevinsky's model of type B/C Cluster algebras (tricky)

$\rightarrow$ each indec ε-rep corresponds to a unique Cluster variable.

<u>2nd task</u> Calculate F-polynomial and g-vector of it.

$\rightsquigarrow$ try to find an explicit formula to express the Cluster variables.

⋮

どうもありがとうございます

THANK YOU

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