Structure

1st lecture

1. The starting point
2. Symmetric representation theory
3. Motivation

2nd lecture

4. Orbits (classification)

3rd lecture

5. Orbit closures
6. Outlook

What we did yesterday

Idea Consider quiver representations for classical groups

Let

(Q, σ) symmetric quiver

$\uparrow$ arrow-reversing involution on $Q_0 \cup Q_1$

with $\sigma(Q_0) = Q_0$, $\sigma(Q_1) = Q_1$

$I \subseteq kQ$, $\sigma(I) = I$, $A := kQ/I$

$V = \bigoplus_{i \in Q_0} V_i$, $\underline{d} = (\dim V_i)_{i \in Q_0}$

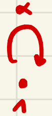
$\varepsilon \in \{\pm 1\}$,

$\langle , \rangle : V \times V \longrightarrow k$ non-deg. ε-form

s.th. $\langle , \rangle|_{V_i \times V_j} = 0$ unless $i = \sigma(j)$

Example $n \in \mathbb{N}$

$Q =$



$\sigma(1) = 1$ $\sigma(\alpha) = \alpha$

$I = (\alpha^n)$, $A = kQ/I$

$V = \mathbb{C}^n$ $\underline{d} = (n)$

$\varepsilon = -1$,

$$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$$
$$(v, w) \longmapsto v^T F w$$

where

$$F = \begin{cases} J_n = \begin{pmatrix} 0 & & 1 \\ & \diagup & \\ 1 & & 0 \end{pmatrix} & \varepsilon = 1 \\ \left(\begin{array}{c|c} 0 & J_{n/2} \\ \hline -J_{n/2} & 0 \end{array}\right) & \varepsilon = -1 \end{cases}$$

$$\langle M_\alpha(v), w\rangle + \langle v, M_{\sigma(\alpha)}(w)\rangle = 0$$

$$(M = -M^*)$$

$$\downarrow$$

rep. variety $R_{\underline{d}} A \supseteq R^{\varepsilon}_{\underline{d}} A$

$\circlearrowright$ change of basis $\circlearrowright$

$G_{\underline{d}} \supseteq G^{\varepsilon}_{\underline{d}}$ $\quad (g = (g^{-1})^*)$

Question

Orbits $\quad G_{\underline{d}} M \overset{?}{\longleftrightarrow} G^{\varepsilon}_{\underline{d}} M$

$$G_{\underline{d}} M \cap R^{\varepsilon}_{\underline{d}} A = G^{\varepsilon}_{\underline{d}} M \quad ?$$

Orbit closures $\quad \overline{G_{\underline{d}} M} \overset{?}{\longleftrightarrow} \overline{G^{\varepsilon}_{\underline{d}} M}$

$$\overline{G_{\underline{d}} M} \cap R^{\varepsilon}_{\underline{d}} A = \overline{G^{\varepsilon}_{\underline{d}} M} \quad ?$$

$$\mathbb{C}^{n\times n}$$

$$\cup|$$

nilp. cone $\downarrow$

$$\mathcal{N} = R_{\underline{d}} A \supseteq R^{\varepsilon}_{\underline{d}} A = \mathcal{N} \cap \mathfrak{o}_n$$

$\circlearrowright$ conjugation $g.N = gNg^{-1}$ $\circlearrowright$

$$GL_n = G_{\underline{d}} \supseteq G^{\varepsilon}_{\underline{d}} = O_n$$

nilpotent Jordan canonical form $\xrightarrow[\text{restriction}]{\text{induced via}}$ "ε-partitions"

$\hat{=}$ partitions

box dropping algorithm $\xrightarrow[\text{restriction}]{\text{induced via}}$ same algorithm

4. Orbits (classification)

Let's understand ε-representations

(up to $G_{\underline{d}}^{\varepsilon}$ - change of basis)

$M \in R(A, V)$ symmetric

$$\overset{\text{Def}}{\iff} \langle M_\alpha(v), w\rangle + \langle v, M_{\sigma(\alpha)}(w)\rangle = 0$$

$$\forall \alpha : i \to j \quad \forall v \in V_i \quad \forall w \in V_{\sigma(j)}$$

Example $Q = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{\omega}{\bullet} \xrightarrow{\sigma(\beta)} \underset{\sigma(2)}{\bullet} \xrightarrow{\sigma(\alpha)} \underset{\sigma(1)}{\bullet}$

$$M_1 = k \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{-1} k \xrightarrow{-1} k$$

orthogonal, not symplectic

$$M_2 = k^2 \xrightarrow{id} k^2 \xrightarrow{id} k^2 \xrightarrow{-id} k^2 \xrightarrow{-id} k^2$$

both

$$M_3 = k \xrightarrow{1} k \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} k^2 \xrightarrow{(0\ -1)} k \xrightarrow{-1} k$$

both

$\langle , \rangle$ given by

$$F = \begin{cases} \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} & \varepsilon = 1 \\[2ex] \left(\begin{array}{c|c} & \begin{matrix} & & 1 \\ & ⋰ & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & ⋰ & \\ -1 & & \end{matrix} & \end{array}\right) & \varepsilon = -1 \end{cases}$$

These conditions are not so nice to work with. Let's look for better ones!

Definition Let $\nabla: \operatorname{rep} A \longrightarrow \operatorname{rep} A$

be the contravariant functor defined

- on objects via

$$\nabla((M_i)_i, (M_\alpha)_\alpha)$$

$$= \big((M^*_{\sigma(i)})_i , (-M^*_{\sigma(\alpha)} : \overset{\overset{\text{dual}}{\downarrow}}{M^*_{\sigma(j)}} \longrightarrow M^*_{\sigma(i)})_\alpha \big)$$

- on homs via

$$\nabla((f_i)_i) = (f^*_{\sigma(i)})_i$$

This is a duality of categories!

Example

$$\begin{tikzcd} M = M_1 \arrow[r, "M_\alpha"'] & M_2 \arrow[r, "M_{\sigma(\alpha)}"'] & M_3 \end{tikzcd}$$

$$\begin{tikzcd} \nabla M = M_3^* \arrow[r, "-M^*_{\sigma(\alpha)}"'] & M_2^* \arrow[r, "-M^*_\alpha"'] & M_1^* \end{tikzcd}$$

Lemma

(1) M ε-representation wrt $\langle , \rangle$

$\iff \exists$ isomorphism $\varphi: M \longrightarrow \nabla M$
sth. $\nabla\varphi = \varepsilon\varphi$

(2) Let $M, N \in R^{\varepsilon}_{\leq} A$ wrt φ.

$f: M \xrightarrow{\simeq} N$ fulfills $\nabla f \circ \varphi \circ f = \varphi$

iff $\langle f(v), f(w) \rangle = \langle v, w \rangle$ $\forall v, w$

Sketch of proof

(1) $\varphi(m)(m') = \langle m, m' \rangle$

i.e. $\varphi(m) = \langle m, - \rangle$

$\nabla\varphi(m)(m') = \langle m', m \rangle$

identify $\nabla\nabla M \cong M$ via evaluation map

(2) $(\nabla f \circ \varphi \circ f(m))(m') = \nabla f\, (\varphi \circ f(m))\, (m')$

$= \varphi \circ f(m)\, (f(m')) = \langle f(m), f(m') \rangle$

$\square$

Detailed proof of (1)

Set
$$\begin{aligned} \psi_i : M_i &\longrightarrow M^*_{\sigma(i)} = \mathrm{Hom}(M_{\sigma(i)}, k) \\ m &\longmapsto \psi_i(m) : M_{\sigma(i)} \longrightarrow k \\ &\qquad\qquad m' \longmapsto \langle m, m' \rangle \end{aligned}$$

$$\Rightarrow \psi_i(m)(m') = \langle m, m' \rangle$$

Then

$$\begin{aligned} \nabla\psi_i = \psi^*_{\sigma(i)} : \overset{\overset{M_i}{\cong}}{M_i^{**}} &\longrightarrow M^*_{\sigma(i)} \\ ev_m &\longmapsto ev_m \circ \psi_{\sigma(i)} \end{aligned}$$

$$\Rightarrow \nabla\psi_i(m)(m') = ev_m \circ \psi_{\sigma(i)}(m') = \langle m', m \rangle$$

$$\Rightarrow \nabla\psi = \varepsilon\psi$$

"$\Leftarrow$" Assume $\psi : M \to \nabla M$ iso, $\nabla\psi = \varepsilon\psi$.
$(\psi_i : M_i \to M^*_{\sigma(i)})_i$

Set
$$\begin{aligned} \langle\ ,\ \rangle : V \times V &\longrightarrow k \\ (v, w) &\longmapsto \psi(v)(w) \end{aligned}$$

underlying vsp $V = \oplus M_i$

$$\langle v, w \rangle = \varepsilon \langle w, v \rangle = \varepsilon\, \nabla\psi(v)(w)$$

Let $\alpha : i \to j$, $v \in M_i$, $w \in M_{\sigma(j)}$

$$\begin{aligned} \langle M_\alpha(v), w \rangle &= \psi(M_\alpha(v))(w) \\ &\overset{①}{=} (\nabla M)_\alpha(\psi(v))(w) \\ &\overset{②}{=} -M^*_{\sigma(\alpha)}(\psi(v))(w) \\ &\overset{\text{dual}}{=} \psi(v) \circ (-M_{\sigma(\alpha)})(w) \\ &= \psi(v)(-M_{\sigma(\alpha)}(w)) \\ &= -\langle v, M_{\sigma(\alpha)}(w) \rangle \end{aligned}$$

① $\psi \circ M_\alpha = (\nabla M)_\alpha \circ \psi$

is true since ψ is hom. of reps.

②
$$\begin{aligned} (\nabla M)_\alpha = -M^*_{\sigma(\alpha)} : M^*_{\sigma(i)} &\longrightarrow M^*_{\sigma(j)} \\ \xi &\longmapsto -\xi \circ M_{\sigma(\alpha)} \end{aligned} \qquad \square$$

Theorem [DW, BC]

Let $M, N \in R^{\Sigma}_{\underline{d}}$ st wrt ψ.

$$G_{\underline{d}} M = G_{\underline{d}} N \Longleftrightarrow G^{\Sigma}_{\underline{d}} M = G^{\Sigma}_{\underline{d}} N$$

idea [BCI]

"$\Leftarrow$" ✓

"$\Rightarrow$" Let $G_{\underline{d}} M = G_{\underline{d}} N$, $\theta : M \xrightarrow{\sim} N$

Claim $\exists \rho \in \operatorname{Aut} M$ s.th. $\nabla(\theta \circ \rho) \circ \psi \circ (\theta \circ \rho) = \psi$

Observation: ψ, $\nabla\theta \circ \psi \circ \theta$ $\Sigma\nabla$-inv., invertible

$$\Big(\nabla\psi = \Sigma\psi \;, \quad \begin{aligned} \nabla(\nabla\theta \circ \psi \circ \theta) &= \nabla\theta \circ \nabla\psi \circ \nabla\nabla\theta \\ &= \nabla\theta \circ \Sigma\psi \circ \theta \\ &= \Sigma\, \nabla\theta \circ \psi \circ \theta \end{aligned} \Big)$$

Idea: find ρ s.th. $\psi = \nabla\rho \circ (\nabla\theta \circ \psi \circ \theta) \circ \rho$

Consider right group action

$$g : \operatorname{Hom}(M, \nabla M)^{\Sigma\nabla} \times \operatorname{Aut} M \longrightarrow \operatorname{Hom}(M, \nabla M)^{\Sigma\nabla}$$
$$(\xi, \rho) \longmapsto \nabla\rho \circ \xi \circ \rho$$

right group action

Every $\pi \in \operatorname{Hom}(M, \nabla M)^{\Sigma\nabla}$ invertible has a dense orbit $\operatorname{Aut}M . \pi$.

$$\Big(\text{show } \dim \operatorname{Aut}M\,\pi = \dim \operatorname{Aut}M - \dim \operatorname{stab}_{\operatorname{Aut}M} \pi = \dim \operatorname{Hom}(M, \nabla M)^{\Sigma\nabla} \Big)$$

$$\dim \operatorname{stab}_{\operatorname{Aut}M} \pi = \dim \operatorname{Aut}(M, \pi) = \dim \operatorname{Hom}(M, \nabla M)^{\Sigma\nabla}$$

$$\operatorname{Hom}(M, \nabla M) = \operatorname{Hom}(M, \nabla M)^{\nabla} \oplus \operatorname{Hom}(M, \nabla M)^{-\nabla} \Big)$$

$\operatorname{Hom}(M, \nabla M)^{\Sigma\nabla}$ is irred. (as k-vsp)

$\Rightarrow$ two such orbits meet!

$\Rightarrow$ $\psi = (\nabla\theta \circ \psi \circ \theta) . \rho$ $\square$

Let's find the indec. Σ-reps!

Theorem [DW]

Let M be an indecomposable Σ-rep.

One of the three cases appears:

(1) $M = L$ indec. rep "indecomposable"
$(L \simeq \nabla L)$

(2) $M = L \oplus \nabla L$, L indec rep, $L \not\simeq \nabla L$
"split"

(3) $M = L \oplus \nabla L$, L indec rep, $L \simeq \nabla L$
"ramified"

idea

Helpful lemma [DW, Lemma 2.8]

M Σ-rep, $L \trianglelefteq M$ s.th. $\langle , \rangle|_L$ non-deg.

$\Rightarrow$ $M \simeq L \oplus L^{\perp}$ (Σ-rep decomp.)

$(L^{\perp} = \{ m \in M \mid \langle m, l \rangle = 0 \ \forall l \in L \})$

proof straightforward (show that $L^{\perp}$ is a rep)

$(\alpha : i \to j$, $u \in L_i^{\perp}$ show $M_\alpha(u) \in L_j^{\perp}$,

$0 \overset{!}{=} \langle M_\alpha(u), w \rangle \quad \forall\, w \in L_{\sigma(j)}$

via $0 = \langle M_\alpha(u), w \rangle + \underbrace{\langle u, \underbrace{M_{\sigma(\alpha)}(w)}_{\in L_{\sigma(i)}} \rangle}_{=0}$)

Let $L \trianglelefteq M$,

$$L \xhookrightarrow{} M \xrightarrow{\psi} \nabla M \xrightarrow{\nabla\iota} \nabla L$$

(1) $\nabla_i \omega_i$ iso $\Rightarrow$ L Σ-rep wrt $\langle , \rangle|_L$

Lemma $\Rightarrow$ $M = L \oplus L^{\perp}$ $\overset{M \text{ indec}}{\Rightarrow}$ $M = L$

(2) $\nabla_i \omega_i$ not iso

Construct iso $\nabla_j \psi_j : L \oplus \nabla L \to \nabla(L \oplus \nabla L)$

$\Rightarrow$ $L \oplus \nabla L$ Σ-rep

Lemma $\Rightarrow$ $M = L \oplus \nabla L \oplus \underbrace{(L \oplus \nabla L)^{\perp}}_{=0 \text{ (indec)}}$ □

Example

$$Q = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{\omega}{\bullet} \xrightarrow{\sigma(\beta)} \underset{\sigma(2)}{\bullet} \xrightarrow{\sigma(\alpha)} \underset{\sigma(1)}{\bullet}$$

$$M_1 = 0 \to k \xrightarrow{1} k \xrightarrow{-1} k \to 0$$ indec

orthogonal, not symplectic

$$M_2 = k^2 \xrightarrow{id} k^2 \xrightarrow{id} k^2 \xrightarrow{-id} k^2 \xrightarrow{-id} k^2$$ ramified

both

$$M_3 = k \xrightarrow{1} k \xrightarrow{\binom{1}{0}} k^2 \xrightarrow{(0\ ,1)} k \xrightarrow{-1} k$$ split

both

Note: If $\varepsilon = -1$, every indec ε-rep ramified or split

Very brief Auslander-Reiten Theory

History While proving the first Brauer-Thrall conjecture, Maurice Auslander and Idun Reiten developed the notion of "almost split sequences".

This led to the so-called ART.

[ASS, A, AR]

Main idea

Given a quiver algebra $A = kQ/I$, develop the Auslander-Reiten quiver $\Gamma(A)$:

vertices = iso classes of indecs /iso

arrows ≙ basis of space of irred. morph.

Techniques

- Knitting [ASS]

  (webpage Crawley-Boevey [Applet])
- Certain techniques for string algs [CB]
- Covering techniques [Ga]

⇒ if we are able to calculate the ARQ, we know all iso classes of indecs ☺

Example

$$Q = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{\omega}{\bullet} \xrightarrow{\sigma(\beta)} \underset{\sigma(2)}{\bullet} \xrightarrow{\sigma(\alpha)} \underset{\sigma(1)}{\bullet}$$

$M_1 = 0 \rightarrow k \xrightarrow{1} k \xrightarrow{-1} k \rightarrow 0$ indec

orthogonal, not symplectic

$M_2 = k^2 \xrightarrow{id} k^2 \xrightarrow{id} k^2 \xrightarrow{-id} k^2 \xrightarrow{-id} k^2$ ramified

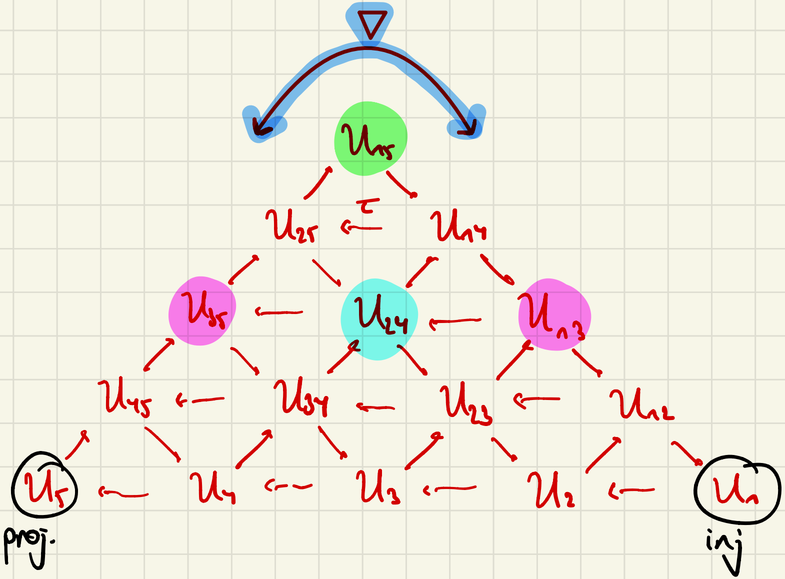
both

$M_3 = k \xrightarrow{1} k \xrightarrow{\binom{1}{0}} k^2 \xrightarrow{(0\ 1)} k \xrightarrow{-1} k$ split

both

Note: If $\varepsilon = -1$, every indec ε-rep ramified or split

The ARQ



How are the ARQ and ∇ connected ?

Symmetric Auslander-Reiten Theory

Lemma The functor ∇

- is exact $\quad \mathrm{Hom}(V,W) = \mathrm{Hom}(\nabla W, \nabla V)$
  $\quad \mathrm{Ext}^1(V,W) = \mathrm{Ext}^1(\nabla W, \nabla V)$
- sends proj. reps to inj. reps
  inj. reps to proj. reps
- preserves almost split sequences (and inverts their homs)

$$\nabla \tau = \tau^- \nabla$$

almost split

$$\begin{tikzcd}
& & L_1 \arrow[dr] & & {} \arrow[r, "\tau^-"] & {} & & \nabla L_1 \arrow[dr] & & \\
{} & M \arrow[l, "\tau"'] \arrow[ur] \arrow[dr] & & N \arrow[ll, "\tau"'] & & & \nabla N \arrow[ur] \arrow[dr] \arrow[rr, "\tau^-"'] & & \nabla M \arrow[r, "\tau^-"'] & {} \\
& & L_n \arrow[ur] & & {} & {} \arrow[l, "\tau"'] & & \nabla L_n \arrow[ur] & &
\end{tikzcd}$$

$$\nabla \tau M = \tau^- \nabla M$$
$$\nabla \tau N = \nabla M = \tau^- \nabla N$$

The rep-finite case

Let A be Σ-rep-finite, i.e.

$$\# \text{ indec } \Sigma\text{-reps} / \text{iso} < \infty$$

Classify the orbits via

* Krull-Remak-Schmidt

* Symmetric ARQ

$\hookrightarrow$ indec Σ-reps $\rightsquigarrow$ combinatorics

Note

$A = kQ$ Σ-rep-finite $\overset{DW}{\Longrightarrow}$ Q Dynkin type A

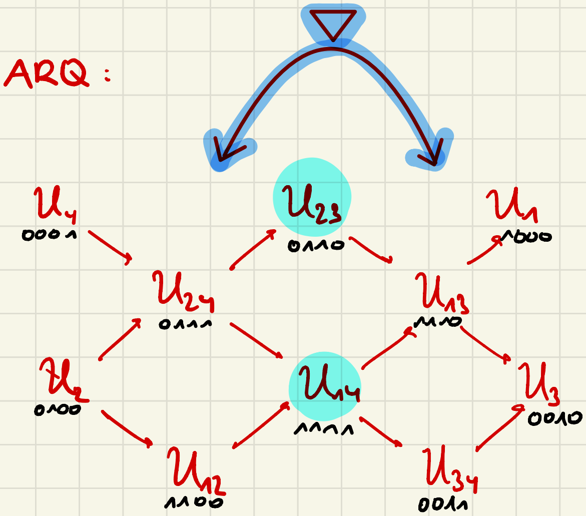
Let's look at some examples!

Example

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} & \underset{\sigma(2)}{\bullet} \arrow[l, "\beta"'] \arrow[r, "\sigma(\alpha)"] & \underset{\sigma(1)}{\bullet} \end{tikzcd}$$

$(\beta = \sigma(\beta))$

ARQ :



$$\left.\begin{array}{l} \mathcal{U}_4 \oplus \mathcal{U}_1 \\ \mathcal{U}_{24} \oplus \mathcal{U}_{13} \\ \mathcal{U}_2 \oplus \mathcal{U}_3 \\ \mathcal{U}_{12} \oplus \mathcal{U}_{34} \end{array}\right\} \text{split}$$

NO "indec." ε-reps
$\Rightarrow$ "split type"

$$\left.\begin{array}{l} \mathcal{U}_{23} \oplus \mathcal{U}_{23} \\ \mathcal{U}_{14} \oplus \mathcal{U}_{14} \end{array}\right\} \text{ramified}$$

$\varepsilon = -1$ The indecomposable ε-reps /iso are

$$\left.\begin{array}{l} \mathcal{U}_4 \oplus \mathcal{U}_1 \\ \mathcal{U}_{24} \oplus \mathcal{U}_{13} \\ \mathcal{U}_2 \oplus \mathcal{U}_3 \\ \mathcal{U}_{12} \oplus \mathcal{U}_{34} \end{array}\right\} \text{split}$$

$\Rightarrow$ "non-split type"

$$\left.\begin{array}{l} \mathcal{U}_{23} \\ \mathcal{U}_{14} \end{array}\right\} \text{indecomposable}$$

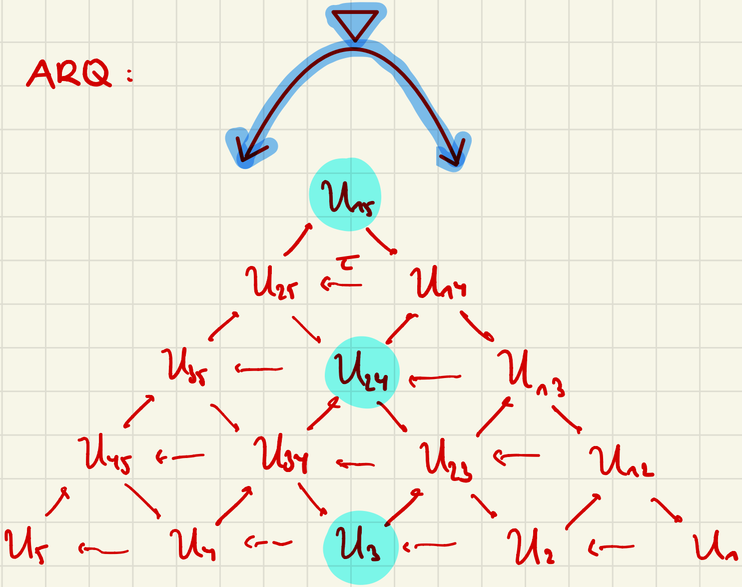
$\rightsquigarrow$ for every $\underline{d}$, the classification is pure combinatorics.

Example

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha"] & \underset{2}{\bullet} \arrow[r, "\beta"] & \underset{\omega}{\bullet} \arrow[r, "\sigma(\beta)"] & \underset{\sigma(2)}{\bullet} \arrow[r, "\sigma(\alpha)"] & \underset{\sigma(1)}{\bullet} \end{tikzcd}$$

$(\omega = \sigma(\omega))$

ARQ :



Note $(A_{odd}, 1)$, $(A_{even}, -1)$ non-split
$(A_{odd}, -1)$, $(A_{even}, 1)$ split

$\varepsilon = 1$ The indecomposable ε-reps/iso are

$\mathcal{U}_5 \oplus \mathcal{U}_1$
$\mathcal{U}_{45} \oplus \mathcal{U}_{12}$
$\mathcal{U}_{35} \oplus \mathcal{U}_{13}$
$\mathcal{U}_{25} \oplus \mathcal{U}_{14}$
$\mathcal{U}_4 \oplus \mathcal{U}_2$
$\mathcal{U}_{34} \oplus \mathcal{U}_{23}$ } split

$\mathcal{U}_{15}$
$\mathcal{U}_{24}$
$\mathcal{U}_3$ } indecomposable

"non-split type"

$\varepsilon = -1$ The indecomposable ε-reps /iso are

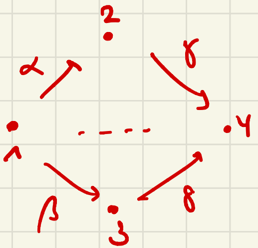
$\mathcal{U}_5 \oplus \mathcal{U}_1$
$\mathcal{U}_{45} \oplus \mathcal{U}_{12}$
$\mathcal{U}_{35} \oplus \mathcal{U}_{13}$
$\mathcal{U}_{25} \oplus \mathcal{U}_{14}$
$\mathcal{U}_4 \oplus \mathcal{U}_2$
$\mathcal{U}_{34} \oplus \mathcal{U}_{23}$ } split

$\mathcal{U}_{15} \oplus \mathcal{U}_{15}$
$\mathcal{U}_{24} \oplus \mathcal{U}_{24}$
$\mathcal{U}_3 \oplus \mathcal{U}_3$ } ramified

"split type"

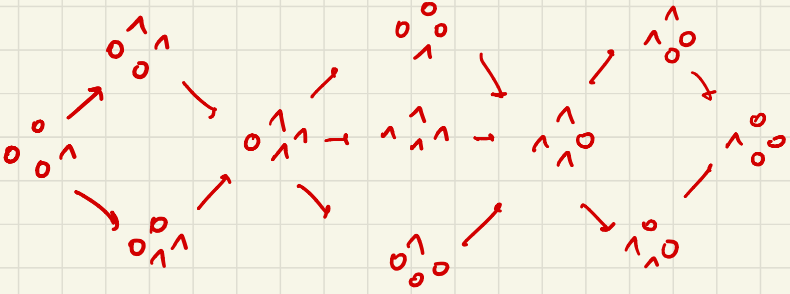
Example

$Q =$



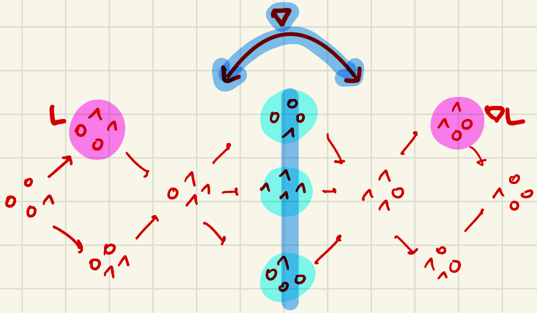
$I = (\gamma\alpha - \delta\beta)$

ARQ

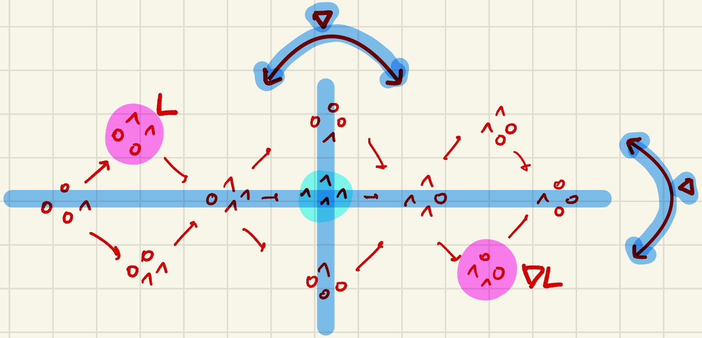


2 cases

(1) $\sigma(1)=4$, $\sigma(2)=2$, $\sigma(3)=3$
$\sigma(\alpha)=\gamma$, $\sigma(\beta)=\delta$



(2) $\sigma(1)=4$, $\sigma(2)=3$, $\sigma(\alpha)=\delta$, $\sigma(\beta)=\gamma$



$M = \nabla M$

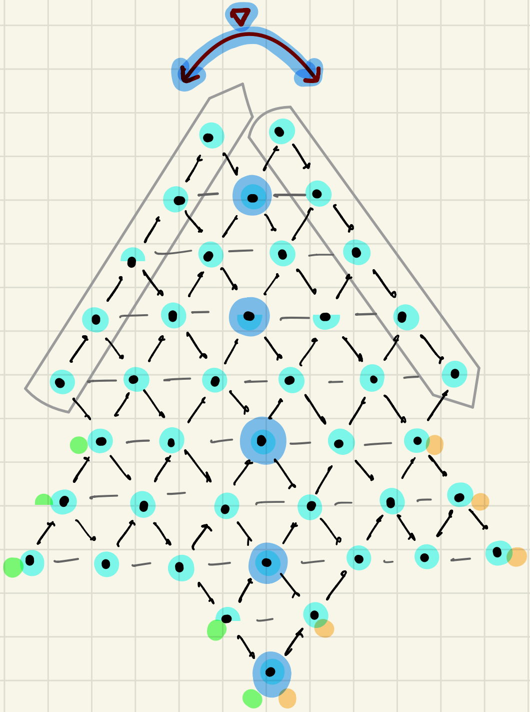
Example [BCE]

$$\begin{tikzcd} Q = & 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & \omega \arrow[loop above, "\gamma"] \arrow[r, "\sigma(\beta)"] & \sigma(2) \arrow[r, "\sigma(\alpha)"] & \sigma(1) \end{tikzcd}$$

$\sigma(\omega) = \omega$

$\sigma(\gamma) = \gamma$

ARQ



$\rightsquigarrow$ we can classify all indecomposable Σ-reps

Note This way, we can classify

B-orbits in $\mathcal{N}^{(2)}$ and

$B \rtimes G$-orbits in $\mathcal{N}^{(2)}_{ag}$, resp.

What about orbit closures?

Tomorrow

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2. Symmetric representation theory
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3rd lecture