Symmetric quiver representations

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Introduction

Idea Introduce quiver representation theory for classical Lie groups

Research still is in beginner's shoes.

But it might lead to results in numerous directions:

$\rightsquigarrow$ (degenerate) flag varieties

$\rightsquigarrow$ certain algebraic group actions on affine varieties

$\rightsquigarrow$ Cluster algebras ?

Let's introduce the theory, examples and first results.

Structure

1st lecture

1. The starting point
2. Symmetric representation theory
3. Motivation

2nd lecture

4. Orbits (classification)

3rd lecture

5. Orbit closures
6. Outlook

1. The starting point

Let

- $k=\bar{k}$ field of odd characteristic.
- $Q=(Q_0, Q_1, s, t)$ finite quiver
  - ⇝ Q_0 finite set of vertices
  - ⇝ Q_1 finite set of arrows

$$\alpha \in Q_1$$

$$s(\alpha) \xrightarrow{\alpha} t(\alpha)$$

Examples

(1) $Q = \underset{1}{\cdot} \longrightarrow \underset{2}{\cdot} \longrightarrow \underset{3}{\cdot} \longrightarrow \underset{4}{\cdot} \longrightarrow \underset{5}{\cdot}$

(2) $Q = \cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot \circlearrowleft \gamma$

Let

- kQ path algebra of Q
  $\cup$
- I admissible ideal
  $\underbrace{\text{admissible}}_{}$ ⇝ $\exists s \quad R_Q^s \subseteq I \subseteq R_Q^2$
  (R_Q arrow ideal)
- $A := kQ/I$ quotient algebra

  finite-dim, associative, basic, with 1.

"Relations"

Note $I = (\beta_1, \dots, \beta_n)$

s.th. $\beta_i = \sum_j \lambda_j \beta_j^{(i)}$,

$s(\beta_j^{(i)}) = s(\beta_l^{(i)})$, $t(\beta_j^{(i)}) = t(\beta_l^{(i)})$ $\forall j, l$

Example

(2) $I = (\gamma^2)$ $\quad A := kQ/I$ findim

The representation category of A

Definition

$\operatorname{rep} A$ = abelian category of fd A-reps

- Objects: $M = \left((M_i)_{i \in Q_0}, (M_\alpha)_{\alpha \in Q_1} \right)$

  ↳ M_i fd k-vsp

  ↳ $M_\alpha : M_{s(\alpha)} \longrightarrow M_{t(\alpha)}$ k-linear

  sth. $\sum_\gamma \lambda_\gamma \beta_\gamma \in I$

  $\Longrightarrow \sum \lambda_\gamma M_{\beta_\gamma} = 0$

  $\beta = \alpha_1 \circ \dots \circ \alpha_\ell$

  $\Longrightarrow M_\beta = M_{\alpha_1} \circ \dots \circ M_{\alpha_\ell}$

  (relations are fulfilled)

Example

Q: $K^2 \xrightarrow{\mathrm{id}} K^2 \xrightarrow{(1\ 0)} K \xrightarrow{5} K \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} K^3$

$\tilde{Q}$: $K \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} K^2 \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} K^3 \circlearrowleft \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}^2 = 0$$

- Morphisms: A-rep homs

$$(f_i : M_i \longrightarrow M_i')_i : M \longrightarrow N$$

sth. $\forall \alpha \in Q_1$:

$$\begin{tikzcd} M_{s(\alpha)} \arrow[r, "M_\alpha"] \arrow[d, "f_{s(\alpha)}"] \arrow[dr, phantom, "\circlearrowleft"] & M_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \\ N_{s(\alpha)} \arrow[r] & N_{t(\alpha)} \end{tikzcd}$$

Example

$$\begin{tikzcd} K \arrow[r, "M_\alpha"] \arrow[d, "f_1"] \arrow[dr, phantom, "\circlearrowleft"] & K^2 \arrow[r, "M_\beta"] \arrow[d, "f_2"] \arrow[dr, phantom, "\circlearrowleft"] & K^3 \arrow[loop right, "M_\gamma"] \arrow[d, "f_3"] \\ K \arrow[r, "N_\alpha"'] & K^2 \arrow[r, "N_\beta"'] & K^3 \arrow[loop right, "N_\gamma"] \end{tikzcd}$$

$f_2 \circ M_\alpha = N_\alpha \circ f_1$, $\quad f_3 \circ M_\gamma = N_\gamma \circ f_3$

$f_3 M_\beta = N_\beta \circ f_2$,

The representation variety

Let $V = \bigoplus_{i\in Q_0} V_i$ graded k-vsp

$\underline{d} = (d_i)_{i\in Q_0}$, $d_i = \dim V_i$

dimension vector

Definition

$$\bigoplus_{\alpha\in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)})$$

$\cup\!|$ closed

$$R_{\underline{d}}A := R(A,V)$$

$\circlearrowleft$ change of basis $\quad (gM)_\alpha = g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)}$

$$G_{\underline{d}} := G(V) := \prod_{i\in Q_0} GL(V_i)$$

Common goals:

(1) Understand the orbits

$$G_{\underline{d}} M := \{gM \mid g\in G_{\underline{d}}\}$$

($G_{\underline{d}}$-orbits in $R_{\underline{d}}A \overset{1:1}{\longleftrightarrow}$ iso classes in rep A of dim vector $\underline{d}$)

We call both M

(2) Understand their Zariski closures

$$\overline{G_{\underline{d}}M} \subseteq R_{\underline{d}}A$$

Many results known, in part. if

A rep-finite (#indecs/iso $< \infty$)

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GENERALIZED QUIVERS ASSOCIATED TO REDUCTIVE GROUPS

BY

HARM DERKSEN (Ann Arbor, MI) and JERZY WEYMAN (Boston, MA)

0. Introduction. The representation theory of quivers has played an important role in the representation theory of Artin algebras for more than twenty years. It can be viewed as a formalization of a natural class of linear algebra problems. However if viewed in such a way, this theory has the drawback that it deals only with representations of general linear groups.

History / State of the art / Literature

Kruglyak 1979 „Representations of free involutive quivers
Roiter 1979 „Bocses with involution“
Sergeichuk 1979 „Representations of simple involutive quivers“
1983 „Representations of orschemes“ '
1988 "Tame collections of linear maps, symmetric, skew-symmetric and bilinear forms“

Magyar-Weyman-Zelevinsky 1998 „Symplectic multiple flag varieties of finite type“

(*) Derksen, Weyman 2002 „Generalized quivers associated to reductive groups“ [DW]

Shmelkin 2006 „Signed quivers, symmetric quivers and root systems“

(*) B.-Cerulli Irelli-Esposito 2019 „Parabolic orbits of 2-nilpotent elements for classical groups“ [BCE]
(*) B.-Cerulli Irelli 2021 „On degenerations and extensions of symplectic and orthogonal quiver representations" [BC]
(*) B.- Cerulli Irelli 2022 " Symmetric degenerations are not in general induced by Type A degenerations“ [BC2]

2. Symmetric representation theory

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver

together with an arrow-reversing involution σ on $Q_0 \cup Q_1$ s.th.

$$\sigma(Q_0) = Q_0 \;, \quad \sigma(Q_1) = Q_1$$

Example

$$Q = \overset{\bullet}{1} \xrightarrow{\alpha} \overset{\bullet}{2} \xrightarrow{\beta} \overset{\bullet}{3} \xrightarrow{\sigma(\beta)} \overset{\bullet}{\sigma(2)} \xrightarrow{\sigma(\alpha)} \overset{\bullet}{\sigma(1)} \qquad \sigma(3) = 3$$

$$\tilde{Q} = \overset{\bullet}{1} \xrightarrow{\alpha_1} \overset{\bullet}{2} \xrightarrow{\alpha_2} \overset{\bullet}{\omega} \circlearrowleft^{\gamma} \xrightarrow{\sigma(\alpha_2)} \overset{\bullet}{\sigma(2)} \xrightarrow{\sigma(\alpha_1)} \overset{\bullet}{\sigma(1)} \qquad \sigma(\gamma) = \gamma, \; \sigma(\omega) = \omega$$

Let kQ path algebra

$\;\;\;\;\;\cup\!|$

$\;\;\;\;\;I$ admissible $\quad \sigma(I) = I\,)$

Then $\underset{kQ/I}{\overset{\shortparallel}{A}} \xrightarrow{\sigma} A^{op}$ algebra with isomorphism

Let $V = \bigoplus_{i \in Q_0} V_i \qquad \underline{d} = (d_i)_i$ dim vector

$G_{\underline{d}} \circlearrowright R_{\underline{d}}$ ★ via base change.

Let us fix some data:

- $\varepsilon \in \{\pm 1\}$
- $\langle\, , \rangle : V \times V \to k$ bilinear form s.th.
  - $\hookrightarrow$ $\langle\, , \rangle$ non-degenerate
  - $\hookrightarrow$ $\varepsilon \langle v, w \rangle = \langle w, v \rangle$ "ε-form"
  - $\hookrightarrow$ $\langle\, , \rangle|_{V_i \times V_j} = 0$ unless $i = \sigma(j)$)

Now we are able to define symmetric representations.

Type A $\quad$ **representations** $\qquad$ **symmetric representations** $\quad$ Types B, C, D

"ε-rep"

"symplectic" $\quad \varepsilon = -1$

"orthogonal" $\quad \varepsilon = 1$

$$R_{\underline{d}} A \qquad\qquad R^{\varepsilon}_{\underline{d}} A$$

$$\bigoplus_{\alpha \in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)}) \supseteq R(A,V) \supseteq R^{\langle,\rangle}(A,V) := \{ M \mid \langle M_\alpha(v), w\rangle = -\langle v, M_{\sigma(\alpha)}(w)\rangle \ \forall \alpha \}$$

$\forall \alpha: i \to j \ \forall v \in V_i \ \forall w \in V_{\sigma(j)}$

$\Rightarrow M = -M^*$

adj. wrt $\langle,\rangle$

change of basis

$$(g_i)_i \cdot (M_\alpha)_\alpha = (g_{t(\alpha)} M_\alpha g_{s(\alpha)}^{-1})_\alpha$$

$$\prod_{i \in Q_0} GL(V_i) =: G(V) \supseteq G^{\langle,\rangle}(V) := \{ g \mid g = (g^*)^{-1} \}$$

$$G_{\underline{d}} \qquad\qquad G^{\varepsilon}_{\underline{d}}$$

GOAL

Try to understand ($M \in R_{\underline{d}}^{\xi} A$)

MANY results known, in part. if A rep-finite (# indec / iso $< \infty$)

Orbits	$G_{\underline{d}}.M = \{gM \mid g \in G_{\underline{d}}\}$	$\overset{?}{\longleftrightarrow}$	$G_{\underline{d}}^{\xi}.M = \{gM \mid g \in G_{\underline{d}}^{\xi}\}$
	$\cap\!\vert$	interrelation	$\cap\!\vert$
Orbit closures	$\overline{G_{\underline{d}}M} = \bigcup_{\text{certain } N} G_{\underline{d}}.N$	$\underset{?}{\longleftrightarrow}$	$\overline{G_{\underline{d}}^{\xi} M}$

3. Motivation

IN GENERAL: **Type A** $\overset{\text{deduce}}{\underset{\text{knowledge}}{\rightsquigarrow}}$ **Types B, C, D**

In particular

(1) Algebraic group actions, e.g.

Type A			**Types B,C,D**
GL_n	$\supseteq$	G classical	
$\cup$		$\cup$	
$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in B$	$\supseteq$	$B \cap G$	
$\circlearrowright$	conjugation	$\circlearrowright$	
$\mathcal{N}^{(2)}$	$\supseteq$	$\mathcal{N}^{(2)} \cap \operatorname{Lie} G$	

$\mathcal{N}^{(2)} = \{N \mid N^2 = 0\}$

Classification

via rep theory — via symmetric rep theory

$$\bullet \to \bullet \to \bullet \cdots \to \bullet \to \overset{\alpha \circlearrowright}{\bullet} \to \bullet \to \bullet \cdots \to \bullet \to \bullet \qquad \alpha^2 = 0$$

orbits, closures, ... — not enough results known!

[BCE]

(2) Linear degenerations of flag varieties (Cerulli Irelli, Fang, Feigin, Fourier, Reineke 2017)

Type A

$Q = \underset{1}{\bullet} \to \underset{2}{\bullet} \to \cdots \to \underset{n}{\bullet}$

$\underline{d} = (n+1, n+1, \ldots, n+1)$

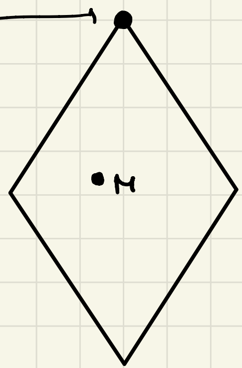
[CFFFR]

$V = \mathbb{C}^{n+1}$

$$\begin{array}{ccc} G_{\underline{d}} & & G_{\underline{d}} \\ \circlearrowright & & \circlearrowright \\ \pi : Y & \longrightarrow & R_{\underline{d}}\, kQ \end{array}$$

proj.
$G_{\underline{d}}$-equiv.

$Fl(V) \longmapsto$



orbit closure relations $\leq_{deg}$

$\pi^{-1}(M)$
quiver Grassm.
$\subseteq$ linear deg. of flag variety

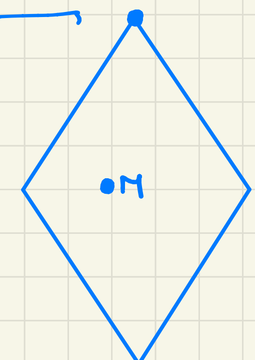
Results: Geometric properties
e.g. irred. locus, flat locus, ...
in terms of $\leq_{deg}$

Analogy: **Types B,C,D**

$\underline{d} = (2n, \ldots, 2n)$

$$\begin{array}{ccc} G_{\underline{d}}^{\varepsilon} & & G_{\underline{d}}^{\varepsilon} \\ \circlearrowright & & \circlearrowright \\ \pi : Y^{\varepsilon} & \longrightarrow & R_{\underline{d}}^{\varepsilon}\, kQ \end{array}$$

sympl. complete flag var $\longmapsto$



$\pi^{-1}(M)$
Lagrangian quiver Grassmannia

First step: Understand $G_{\underline{d}}^{\varepsilon}$-orbits + their closures in $R_{\underline{d}}^{\varepsilon}\, kQ$

Tomorrow

2nd lecture

4. Orbits (classification)

5. Orbit closures

6. Outlook