Relative Koszul coresolutions and relative Betti numbers

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$\Bbbk$: a field

A : a f.d. $\Bbbk$-algebra (basic)

$\operatorname{mod} A$:= the cat. of (right) A-modules

$D := \operatorname{Hom}_{\Bbbk}(-, \Bbbk)$: the usual $\Bbbk$-duality of A

$G \in \operatorname{mod} A$ (basic)

i.e. $G = \bigoplus_{I \in \mathbb{I}} V_I$, V_I : ind , $I \neq J$ in $\mathbb{I} \Rightarrow V_I \not\cong V_J$.

G : a generator $\Longleftrightarrow A_A \underset{\sim}{\lessdot_{\oplus}} G$

a cogenerator $\Longleftrightarrow (DA)_A \underset{\sim}{\lessdot_{\oplus}} G$.

$\Lambda := \operatorname{End}_A(G) \quad \curvearrowright \quad {}_{\Lambda}G_A$: a bimodule

$\forall M \in \operatorname{mod} A$,

$\operatorname{add} M := \operatorname{full}\{ X \in \operatorname{mod} A \mid X \overset{\oplus}{\underset{\sim}{\subset}} M^{(n)},\ \exists n \geq 1 \}$

$\mathcal{J} := \operatorname{add} G.$

usually

Q : a finite quiver

$\rho \trianglelefteq \Bbbk Q$ (the path alg of Q)

(Q, ρ) : a <u>bound quiver</u>

$A := \Bbbk(Q, \rho) := \Bbbk Q/\rho$

<u>Exm</u> Q :
$$\begin{tikzcd} 3 \arrow[r, "a_3"] & 4 \\ 1 \arrow[u, "b_1"] \arrow[r, "a_1"'] & 2 \arrow[u, "b_2"'] \end{tikzcd}$$
, $\rho = \langle b_2 a_1 - a_3 b_1 \rangle$

1. Introduction

[CGRST] Chachólski et. al. gave "an effective" way to compute relative Betti numbers for modules M over the incidence algebra $\Bbbk\mathbb{P}$ of a poset $\mathbb{P}$ (upper semilattice) by using the so-called Koszul complexe.

In more detail,

(1) if $\cdots \longrightarrow \bigoplus_{a\in\mathbb{P}} P_a^{\beta^1_M(a)} \longrightarrow \bigoplus_{a\in\mathbb{P}} P_a^{\beta^0_M(a)} \longrightarrow M \to 0$,

(P_a is the proj. indecomposable $k\mathbb{P}$-mod $\leftrightarrow a \in \mathbb{P}$)
is a minimal proj. resolution of M, then

$\beta^i_M(a)$: the i-th <u>Betti number</u> for M at a.

(2) Let $K_a(M)_\bullet$ be a <u>Koszul complex</u> of M
at $a \in \mathbb{P}$ that is a complex of vector spaces:

$$K_a(M)_0 := M(a),\ \forall n \geq 1.\ K_a(M)_n := \bigoplus_{S \in \mathcal{U}(a)_n} M(\wedge S),$$

where $\mathcal{U}(a) := \{b \in \mathbb{P} \mid b < a,\ [b,a] = \{b,a\}\}$

$\mathcal{U}(a)_n := \{S \subseteq \mathcal{U}(a) \mid \#S = n,\ S \text{ bounded below}\}$

$$\partial_n : \bigoplus_{S \in \mathcal{U}(a)_n} M(\wedge S) \to \bigoplus_{T \in \mathcal{U}(a)_{n-1}} M(\wedge T) := \left[\chi(T \subseteq S)(-1)^{n(S,T)} M(\wedge S \leq \wedge T) \right]_{T,S}$$

$$\chi(T \subseteq S) := \begin{cases} 1 & (T \subseteq S) \\ 0 & \text{o/w} \end{cases}, \quad S = \{s_0 < s_1 < \cdots < s_{n-1}\},\ S \setminus T = s_i \Leftrightarrow n(S,T) = i.$$

($\prec$: another linear order on $\mathbb{P}$)

Then they proved that

$$\beta^d_M(a) = \dim H_d(K_a(M)_\bullet).$$

<u>Problem</u> For a minimal $\mathcal{I}$-resolution

$$\cdots \longrightarrow \bigoplus_{I\in\mathbb{I}} V_I^{\beta^1_M(I)} \longrightarrow \bigoplus_{I\in\mathbb{I}} V_I^{\beta^0_M(I)} \longrightarrow M \longrightarrow 0 \quad \text{of } M,$$

define a complex $K_I(M)_\bullet$ s.t.

$$\beta^d_M(I) = \dim H_d(K_I(M)_\bullet).$$

We will define $K_I(M)_\bullet$ below and call it the <u>Koszul complex</u> of M. (P.11)

2. Preliminaries

${}_A(-,\cdot) := \mathrm{Hom}_A(-,\cdot)$

Dfn (right min. $\mathcal{J}$-approximation) $M \in \operatorname{mod} A$.

(1) A <u>right $\mathcal{J}$-approximation</u> of M is a mor $f : X \to M$ with $X \in \mathcal{J}$ s.t.

$\forall Z \in \mathcal{J}$, ${}_A(Z,f) : {}_A(Z,X) \twoheadrightarrow {}_A(Z,M)$.

$$\begin{tikzcd} & Z \arrow[dl, dotted, "\exists"'] \arrow[d, "g"] \\ X \arrow[r, "f"'] & M \end{tikzcd}$$

$Z \in \mathcal{J}$, $X \in \mathcal{J}$

(G : a generator $\rightsquigarrow$ f : an epi)

$$\begin{tikzcd} Z = A^n \arrow[d, "g"] \\ M \end{tikzcd}$$

(2) $f : X \to M$ in $\operatorname{mod} A$ is <u>right minimal</u> if

$0 \neq {}^{\forall}X' \subseteq_{\oplus} X$, $X' \not\subseteq \operatorname{Ker} f$.

(3) $f : X \to M$ is a <u>minimal right $\mathcal{J}$-appx.</u> if it is a right min. right $\mathcal{J}$-appx.

Dually, left (min.) $\mathcal{J}$-appx $g: M \to Y$ is def^d.

<u>Dfn</u> (min. $\mathcal{J}$-resolution) $M \in \operatorname{mod} A$.

An ~~exact sequence~~ chain complex

$$\cdots \xrightarrow{f_{r+1}} X_r \xrightarrow{f_r} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0 \quad (1)$$

is called a <u>minimal $\mathcal{J}$-resolution</u> of M if

f_i restricts to a minimal right $\mathcal{J}$-appx

$X_i \to \operatorname{Ker} f_{i-1}$, $\forall i \geq 0$. (unique up to iso)

- $\forall i \geq 0$, $\exists! \left(\beta_M^i(I)\right) \in \mathbb{Z}_{\geq 0}^{\mathcal{I}}$, $X_i \cong \bigoplus_{I \in \mathcal{I}} V_I^{\beta_M^i(I)}$

$\beta_M^i(I)$: <u>$\mathcal{J}$-relative i-th Betti number of</u> M at I

- ${}_A(Z,-)$ sends the ~~ex seq~~ chain complex (1) to an ex. seq $\forall Z \in \mathcal{J}$.

- Dually a min. $\mathcal{J}$-coresolution

$$0 \xrightarrow{g^{-1}} M \xrightarrow{g^0} Y^0 \xrightarrow{g^1} Y^1 \xrightarrow{g^2} \cdots \xrightarrow{g^r} Y^r \xrightarrow{g^{r+1}} \cdots \qquad (2)$$

and the $\mathcal{J}$-relative i-th co-Betti numbers $\bar{\beta}^i_M(I)$ $(I \in \mathbb{I})$ is defined. $\left(Y^i \cong \bigoplus_{I \in \mathbb{I}} V_I^{\bar{\beta}^i_M(I)} \right)$

- ${}_A(-,Z)$ sends the ~~ex. seq~~ ch complex (2) to an ex seq $\forall Z \in \mathcal{J}$.

3. Relative Koszul coresolution

Fix $I \in \mathbb{I}$. $\quad e_I G \Lambda : G \xrightarrow{\text{can}} V_I \hookrightarrow G$.

<u>Ntn</u> $M, N \in \operatorname{mod} A$.

- $\operatorname{rad}_A(M, N) :=$ the radical maps $M \to N$.
- $\operatorname{rad}_A(G, G) = \operatorname{rad} \Lambda$, Jacobson rad of Λ
- ${}_A(V_I, G) = \Lambda e_I$: a left Λ-module

  $\operatorname{rad}_A(V_I, G) = \operatorname{rad}_A(G, G) \cdot {}_A(V_I, G)$

  ${}_I S := {}_A(V_I, G) / \operatorname{rad}_A(V_I, G)$: simple lt Λ-mod
- Similarly, ${}_A(G, V_I) = e_I \Lambda, \ldots, S_I$: simple rt Λ-mod

Dfn An $\mathcal{J}$-relative Koszul coresolution $K^\bullet(V_I)$ of V_I is a sequence in $\mathcal{J}$

$$0 \to \underset{(0)}{V_I} \xrightarrow{\eta} \underset{(1)}{X^1} \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots \quad \text{(uniq up to iso)}$$

st.

exact — min left $\mathcal{J}$-coresol

$$\begin{tikzcd}
0 \arrow[r] & V_I \arrow[r, "\eta"] \arrow[dr, "\mu_0"'] & X^1 \arrow[r, "d^1"] \arrow[dr] & X^2 \arrow[r, "d^2"] & X^3 \arrow[r, "d^3"] & \cdots \\
 & & E_I \arrow[u, "f_0"'] \arrow[dr] & C_1 \arrow[u] \arrow[r] & 0 & \\
 & 0 \arrow[ur] & & \mho^1 V_I \arrow[r] & 0 & \\
 & & & 0 \arrow[uu] & &
\end{tikzcd}$$

$\begin{cases} \mu_0 : \text{a source map} \\ f_0 : \text{min left } \mathcal{J}\text{-appx} \end{cases}$

$\begin{cases} V_I : \text{non-inj} \Rightarrow K^\bullet(V_I) : \text{exc} \\ V_I : \text{inj} \Rightarrow \operatorname{Ker} \eta = \operatorname{soc} V_I \end{cases}$

- $K_I(M)_\bullet := {}_A(K^\bullet(V_I), M)$: the $\mathcal{J}$-relative Koszul complex of M

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Lem Let $M \in \operatorname{mod} A$.

(1)
$$\cdots \xrightarrow{f_{r+1}} X_r \xrightarrow{f_r} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{f_{-1}} 0 \qquad (*1)$$

is a min right $\mathcal{G}$-resolution, ~~G : a generator~~

$$\Rightarrow \cdots \longrightarrow (G, X_1) \xrightarrow{(G, f_1)} (G, X_0) \xrightarrow{(G, f_0)} {}_A(G, M) \to 0$$

is a min proj resol. of ${}_A(G, M)$ in $\operatorname{mod} \Lambda$.

(2)
$$0 \xrightarrow{g^{-1}} M \xrightarrow{g^0} Y^0 \xrightarrow{g^1} Y^1 \xrightarrow{g^2} \cdots \xrightarrow{g^r} Y^r \xrightarrow{g^{r+1}} \cdots \qquad (*2)$$

is a min left $\mathcal{G}$-coresol. ~~G : a cogenerator~~

$$\Rightarrow \cdots \longrightarrow (Y^1, G) \xrightarrow{(g^1, G)} {}_A(Y^0, G) \xrightarrow{(g^0, G)} {}_A(M, G) \to 0$$

is a min proj resol. of ${}_A(M, G)$ in $\operatorname{mod} \Lambda^{op}$.

Thm Let $M \in \operatorname{mod} A$.

(1) ~~G: a cogenerator~~

$\cancel{\Rightarrow}$ ${}_A(K^{\bullet}(V_I), G)$ turns out to be a ~~min~~ proj resol.

$$\cdots \longrightarrow (X^2, G) \longrightarrow (X^1, G) \longrightarrow {}_A(V_I, G) \longrightarrow {}_I S \longrightarrow 0$$

of the simple ${}_I S$.

(2) ~~G: a generator and a cogenerator~~

$$\cancel{\Rightarrow} \quad \beta^i_M(I) = \dim H_i(K_I(M)_\bullet), \quad \forall i \geq 0.$$

This gives an answer to <u>Problem</u>.

$\underline{\text{Cor}}$ Let $M \in \operatorname{mod} A$. Then TFAE:

(1) $M \in \mathscr{C}$

(2) $H_1(K_I(M)_\bullet) = 0$, $\forall I \in \mathbb{I}$.

(3) $\dim M = \sum_{I \in \mathbb{I}} \dim H_0(K_I(M)_\bullet) \dim V_I$

(4) $M \cong \bigoplus_{I \in \mathbb{I}} V_I^{\dim H_0(K_I(M)_\bullet)}$

$\underline{\text{Pf}}$ Each of (1) ~ (4) is eq to saying that f_0 in (*1) is an iso. $\square$

<u>Rmk</u> To compute $H_0(K_I(M).)$ in (3), enough to know

$$\begin{tikzcd} V_I \arrow[rr, tail, "\eta"] \arrow[dr, "\mu_0"'] & \arrow[d, phantom, "\circlearrowright"] & X^1 \\ & E_I \arrow[ur, "f_0"'] & \end{tikzcd}$$

$$\beta^0_M(I) = \dim_A(V_I, M) - \dim \mathrm{Im}_A(\eta, M).$$

☺ Enough to know

- $\mu_0 : V_I \longrightarrow E_I$ a source map (Computed)
- $f_0 : E_I \longrightarrow X^1$ min left $\mathcal{J}$-appx $\forall I \in \mathbb{I}$.

$\hookleftarrow$ (computable for interval case easier.)

Application $\mathbb{P}$: a finite poset

$A := k\mathbb{P}$ (incidence alg), $\mathbb{I} :=$ the set of intervals of $\mathbb{P}$

V_I : interval module $\leftrightarrow I$, $G := \bigoplus_{I \in \mathbb{I}} V_I$: gen + cogen, $\mathcal{J} = \operatorname{add} G$.

$\mathcal{J} \ni M$: interval decomposable.

Cor Let $M \in \operatorname{mod} A$. Then (TFAE):

(1) M : interval decomposable

(2) $H_1(K_I(M)_\bullet) = 0$, $\forall I \in \mathbb{I}$.

(3) $\dim M = \sum_{\substack{V_I \rightarrowtail M \twoheadrightarrow V_I \\ I \in \mathbb{I}}} \dim H_0(K_I(M)_\bullet) \dim V_I$

(4) $M \cong \bigoplus_{\text{as above}} V_I^{\dim H_0(K_I(M)_\bullet)}$

4. Applications and examples

$\mathbb{P}$: a finite poset

$A = \Bbbk\mathbb{P}$: the incidence algebra of $\mathbb{P}$.

$\mathbb{I} :=$ the set of <u>intervals</u> of $\mathbb{P}$ (connected & convex)

$\forall I \in \mathbb{I}$, V_I : the interval module defined by I.

$G := \bigoplus_{I \in \mathbb{I}} V_I$ $\leftarrow$ gen & cogen. $\mathcal{J} := \operatorname{add} G$.
(interval decomposable)

$\begin{cases} \mathcal{J}\text{-resolution} \rightsquigarrow \text{interval resolution} \\ \mathcal{J}\text{-appx} \rightsquigarrow \text{interval appx} \end{cases}$

right [left] intv. appx $\longrightarrow$ RIA [LIA]

<u>Lem</u> $I \in \mathbb{I}$, $S \leq V_I \Rightarrow V_I/S,\ S \in \mathscr{J}$

<u>Prp</u> Let $f: X \to M$ (resp. $M \to X$) in $\operatorname{mod} A$. $(\circlearrowleft)$

(1) f: a RIA [LIA]

sometimes max $S_{int}(M)$ is enough

(2) $\forall I \in S_{int}(M) := \{ I \in \mathbb{I} \mid V_I \overset{\exists}{\rightarrowtail} M \}$,

$\forall g: V_I \rightarrowtail M$ in $\operatorname{mod} A$,

$g = fh$, $\exists h: V_I \to X$

$$\begin{tikzcd} & V_I \arrow[dl, dashed, "\exists h"'] \arrow[d, tail, "g"] \\ X \arrow[r, "f"'] & M \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

(resp. $\forall I \in F_{int}(M) := \{ I \in \mathbb{I} \mid M \overset{\exists}{\twoheadrightarrow} V_I \}$

$\forall g: M \twoheadrightarrow V_I$ in $\operatorname{mod} A$,

$g = hf$, $\exists h: X \to V_I$.)

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, two heads, "g"'] & X \arrow[dl, dashed, "\exists h"] \arrow[dl, phantom, "\circlearrowleft", near start] \\ V_I & \end{tikzcd}$$

<u>Cor</u> A min. RIA of $M \in \mathrm{mod}\, A$ is obtained as follows.

(1) $\forall I \in S_{int}(M)$, choose $\{f_I^{(1)}, \dots, f_I^{(d_I)}\} \subseteq \mathrm{Mon}(V_I, M)$ st. ← monos

$\langle \mathrm{Mon}(V_I, M) \rangle = \langle f_I^{(i)} \rangle_{i=1}^{d_I}$, and set $f_I := (f_I^{(i)})_{i=1}^{d_I} : V_I^{d_I} \to M$.

(2) Then by Prp, $(f_I)_{I \in S_{int}(M)} : \bigoplus_{I \in S_{int}(M)} V_I^{(d_I)} \to M$ is a <u>RIA</u>.

Set this to be $(g_i)_{i=1}^n : \bigoplus_{i=1}^n W_i \to M$ (W_i : intv. mod).

(3) $J_0 := \{1, \dots, n\} \ni {}^{\forall} j$, $J_j := \begin{cases} J_{j-1} \setminus \{j\} & ((g_i)_{i \in J_{j-1} \setminus \{j\}} : \mathrm{RIA}) \\ J_{j-1} & (\text{o/w}) \end{cases}$, <u>$J := J_n$</u>.

Then $(g_i)_{i \in J} : \bigoplus_{i \in J} W_i \to M$ is a <u>min. RIA</u>.

(The dual statement also holds.)

<u>Rmk</u> Let $f: X \to M$ be a min RIA of $M \in \mathrm{mod}\, A$.

Then by <u>Cor</u>,

$\exists\, X = \bigoplus_{i\in J} W_i$ (W_i : intv, $\forall i$) st. $\forall i \in J$, $f|_{W_i}$ is a mon. ----- (*)

Concerning this, <u>Prp</u> [Aoki-Escolar-Tada] states:

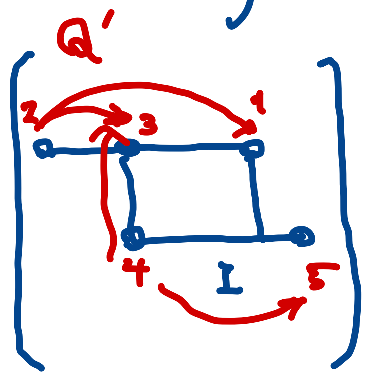
$f: X \to M$: a min RIA

$\Rightarrow$ (1) f : epi (true for all RIA ∵ G: gen),
(2) $\forall\, X = \bigoplus_{i\in J} W_i$ (W_i : indec), $\forall i \in J$, $f|_{W_i}$ is a mon,
(3) $\mathrm{supp}\, X = \mathrm{supp}\, M$ (by (1), (2))

- (2) is stronger than (*).
- However, ($\Leftarrow$) does not hold. (e.g. $(f, f|_{W_1}) : X \oplus W_1 \to M$)

5. Application to compression multiplicity

Thm [AENY] $\mathbb{P} = A_n \times A_2$ $\quad (A_n : 1 \to 2 \to \cdots \to n)$



$\exists\, \xi = (\xi_I)_{I \in \mathbb{I}}$: a compression system

$$\xi_I : Q' := \; 1 \leftarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \;\longrightarrow\; I \hookrightarrow \mathbb{P}$$

$\overline{\xi}_I : \Bbbk Q' \to \Bbbk\mathbb{P} = A$, a unique extension

restriction functor by $\overline{\xi}_I$

$$R_I : \mathrm{mod}\, A \to \mathrm{mod}\, \Bbbk Q'$$
$$M,\ V_I \mapsto R_I(M),\ R_I(V_I)$$

st. $\left(\exists\, r \geq 0 \text{ st. } \beta_M^i = 0 \text{ for } \forall i > r\right)$

$$\Rightarrow\ c_M^{\xi}(I) = \sum_{I \leq J} \left(\sum_{i=0}^{r} (-1)^i \beta_M^i(J) \right)$$

: multiplicity of $R_I(V_I)$ in the indecomp decomp. of $R_I(M)$

$$\left[:= d_{R_I(M)}(R_I(V_I)) \right]$$

$$\delta_M^{\xi}(I) = \sum_{i=0}^{r} (-1)^i \beta_M^i(I)$$: Möbius inversion of $c_M^{\xi} : \mathbb{I} \to \mathbb{R}$

$$\delta^{\xi}(M) = \Big[\!\Big[\bigoplus_{\substack{I \in \mathbb{I} \\ \delta_M^{\xi}(I) > 0}} V_I^{\delta_M^{\xi}(I)} \Big]\!\Big] - \Big[\!\Big[\bigoplus_{\substack{I \in \mathbb{I} \\ \delta_M^{\xi}(I) < 0}} V_I^{-\delta_M^{\xi}(I)} \Big]\!\Big]$$ interval replacement of M under ξ

$$M = \bigoplus_{I \in \mathbb{I}} V_I^{d_M(V_I)}$$

Cor

$$c_M^{\xi}(I) = \sum_{I \leq J} \Big(\sum_{i=0}^{r} (-1)^i \dim H_i(K_I(M)_\bullet) \Big)$$

$$\delta_M^{\xi}(I) = \sum_{i=0}^{r} (-1)^i \dim H_i(K_I(M)_\bullet)$$

Exm $\mathbb{P} := (A_3 \times A_2)$:

$$\begin{tikzcd} \circ \arrow[r] & \circ \arrow[r] & \circ \\ \circ \arrow[u] \arrow[r] & \circ \arrow[u] \arrow[r] & \circ \arrow[u] \end{tikzcd}$$

$A = k\mathbb{P}$

$I := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $M := \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\begin{tikzcd}[column sep=small]
 & & & & 0 \\
 & & & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \arrow[ur] & \\
0 \arrow[r] & V_I \arrow[rr, "\eta"] \arrow[dr, "\mu_0"'] & & \underbrace{\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{X^1} \arrow[u] \arrow[dr] & \\
 & & \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_I} \arrow[ur] \arrow[dr, "\varepsilon_0"] & & \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{C^1 = X^2} \arrow[dr] \\
 & 0 \arrow[ur] & & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \arrow[dr] & & 0 \\
 & & & & 0 &
\end{tikzcd}$$

$K^\bullet(V_I)$

$$K_M(I)_\bullet : 0 \longrightarrow \underbrace{{}_A(V_I, M)}_{\cong k} \longrightarrow \underbrace{(X^1, M)}_{=0} \longrightarrow \underbrace{(X^2, M)}_{=0} \longrightarrow 0$$

$\because\ \dim H_i(K_M(I)_\bullet) = \delta_{i,0} \quad (i \geq 0)$

interval resolution of M :

$$0 \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \underset{V_I}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} \longrightarrow M \longrightarrow 0$$

$$\beta^i_M(I) = 0 \quad (i \geq 1), \qquad \beta^0_M(I) = 1$$

<u>Exm</u> $\mathbb{P} = A_5 \times A_2$, $A = k\mathbb{P}$, $I = \begin{bmatrix} 0&0&0&1&1 \\ 0&0&0&0&1 \end{bmatrix}$

$$M: \quad \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & k \arrow[r, "{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}"] & k^2 \arrow[r, "{(0,1)}"] & k \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & k \arrow[r, "1"] \arrow[u, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] & k \arrow[u] \end{tikzcd} \quad : \text{indecomp.} \quad \begin{bmatrix} 0&0&1&2&1 \\ 0&0&0&1&1 \end{bmatrix}$$

$K^\circ(V_I)$:

$$0 \to V_I \xrightarrow[\substack{\| \\ \left[\begin{smallmatrix} f_1 \\ f_2 \\ f_3 \end{smallmatrix}\right]}]{\substack{\mu_0 \\ \| \\ \eta}} \overbrace{\begin{bmatrix} 0&0&0&1&1 \\ 0&0&0&1&1 \end{bmatrix} \oplus \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix} \oplus \begin{bmatrix} 0&0&1&1&1 \\ 0&0&0&0&1 \end{bmatrix}}^{X^1 = E_I \in \mathcal{J}} \xrightarrow{\varepsilon_0} \overset{\substack{C^1 \\ \|}}{M} \to 0 \qquad (*)$$

embeddings

can epi

(ass)

also min intv resol of M

min. intv. coresol. of $M = C^1$:

$$0 \to M \xrightarrow[\substack{\| \\ \left[\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right]}]{\mu_1} \underbrace{\begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&0&0 \end{bmatrix} \oplus \begin{bmatrix} 0&0&0&1&0 \\ 0&0&0&1&1 \end{bmatrix} \oplus \begin{bmatrix} 0&0&1&1&1 \\ 0&0&0&1&1 \end{bmatrix}}_{X^2} \xrightarrow{d^2} \underbrace{\begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&1&1 \end{bmatrix}}_{X^3} \to 0$$

can epis

$\because$ $K^\circ(V_I)$:

$$\begin{tikzcd} 0 \arrow[r] & V_I \arrow[r, "\mu_0"] & X^1 \arrow[r, "d^1"] \arrow[dr, "\varepsilon_0"'] & X^2 \arrow[r, "d^2"] & X^3 \arrow[r] & 0 \\ & & & M = C^1 \arrow[u, "\mu_1"'] & & \end{tikzcd}$$

$\because$ $K_M(I)_\bullet$:

$$\begin{tikzcd} 0 \arrow[r] & {}_A(X^3, M) \arrow[r, "{(d^2, M)}"] \arrow[d, phantom, "\shortparallel"] & (X^2, M) \arrow[r, "{(d^1, M)}"] \arrow[d, phantom, "\shortparallel"] \arrow[dr] & (X^1, M) \arrow[r, "{(\mu_0, M)}"] \arrow[dr] & (V_I, M) \arrow[r] & 0 \\ & 0 & 0 & \mathrm{End}_A(M) \cong k \arrow[u] & 0 \arrow[u] & \\ & & 0 \arrow[ur] & & & \end{tikzcd}$$

ex

$\beta^0_M(I) = 0$

$\beta^1_M(I) = 1$

$\cdots = \beta^3_M(I) = 0 = \beta^2_M(I)$

$\because$ $\beta^i_M(I) = \delta_{i,1}$ $(i \geq 0)$ as $(*)$ suggests.

(min intv resol of M)