Classifying subcat's of Noetherian algebras

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§1 Intro

A : Noeth ring
mod A : fin. gen. left A-mod

Problem Classify "good" subcat's of mod A

"good" ⇝ Serre, torsion, torsion free

{Serre subcat. of mod A} : poset by inclusion

Def $\mathcal{C} \subseteq \operatorname{mod} A$

(1) $\mathcal{C}$ is fac-closed
$:\Leftrightarrow$ $C \twoheadrightarrow X$ in mod A, $C \in \mathcal{C}$
$\Rightarrow X \in \mathcal{C}$

(2) $\mathcal{C}$ is sub-closed

$:\Leftrightarrow$ $X \hookrightarrow C$ in mod A $C \in \mathcal{C}$

$\Rightarrow X \in \mathcal{C}$

(3) $\mathcal{C}$ is ext-closed

$:\Leftrightarrow$ $0 \to X \to Y \to Z \to 0$ ex in mod A

$X, Z \in \mathcal{C}$ $\Rightarrow Y \in \mathcal{C}$

(4) $\mathcal{C}$ is called

- Serre subcat. if it is ext, fac, sub-closed
- torsion class if it is ext, fac-closed
- torsion free class if it is ext, sub-closed

serre A : the set of Serre subcat's of mod A

tors A : ——— torsion classes ———

torf A : ——— torsionfree classes ———

posets by inclusion

Case 1 : A is comm. ring

$$\mathrm{Spcl}(A) := \left\{ W \subseteq \mathrm{Spec}\,A \;\middle|\; \begin{array}{l} \mathfrak{p} \supseteq \mathfrak{q} \text{ in } \mathrm{Spec}\,A \\ \mathfrak{q} \in W \Rightarrow \mathfrak{p} \in W \end{array} \right\}$$

↑ specialization closed subset

Thm 1 Let A be a comm. Noeth. ring.

(a) [Gabriel '62]

$$\mathrm{serre}\,A \xrightarrow[\sim]{\mathrm{Supp}} \mathrm{Spcl}(A)$$

$$\mathrm{Supp}(\mathcal{C}) := \bigcup_{X \in \mathcal{C}} \mathrm{Supp}\,X,$$

$\mathcal{C} \underset{\text{serre}}{\subseteq} \mathrm{mod}\,A$ $\qquad \mathrm{Supp}\,X := \{\mathfrak{p} \in \mathrm{Spec}\,A \mid X_\mathfrak{p} \neq 0\}$

(b) [Takahashi '08]

$$\mathrm{torf}\,A \xrightarrow[\sim]{\mathrm{Ass}} \{\text{subset of } \mathrm{Spec}\,A\} \overset{\mathcal{P}(\mathrm{Spec}\,A)}{=}$$

(c) [Stanley-Wang '11]

$$\mathrm{tors}\,A = \mathrm{serre}\,A$$

$$
\begin{tikzcd}
\mathrm{serre}\,A = \mathrm{tors}\,A \arrow[r, hook, "(-)^{\perp}"] \arrow[d, "\mathrm{Supp}", "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{torf}\,A \arrow[d, "\mathrm{Ass}", "\simeq"'] \\
\mathrm{Spcl}(A) \arrow[r, hook, "(-)^{c}"'] & P(\mathrm{Spec}\,A)
\end{tikzcd}
$$

complement in Spec A

$$\mathcal{C}^{\perp} := \{ X \in \mathrm{mod}\,A \mid \mathrm{Hom}_A(C, X) = 0 \ \forall C \in \mathcal{C} \}$$

$\mathcal{C} \subseteq \mathrm{mod}\,A$

<u>Aim</u>] Generalize Thm1 for non-comm algebras

Case 2 : A is fin. dim. alg

<u>Rmk</u> A : Noeth. ring

- $\mathrm{tors}\,A \underset{{}^{\perp}(-)}{\overset{(-)^{\perp}}{\rightleftarrows}} \mathrm{torf}\,A$, ${}^{\perp}(-) \circ (-)^{\perp} = \mathrm{Id}_{\mathrm{tors}\,A}$

- A : fin. dim. alg $\Rightarrow (-)^{\perp} \circ {}^{\perp}(-) = \mathrm{Id}_{\mathrm{torf}\,A}$

Thm2 Let A be a fin. dim. alg

$\mathrm{sim}\,A := \{\text{simple } A\text{-mod}\}/\simeq$

(a) $\mathrm{serre}\,A \xrightarrow{\sim} \mathcal{P}(\mathrm{sim}\,A)$

$\mathcal{C} \longmapsto \mathcal{C} \cap \mathrm{sim}\,A$

(b) [Adachi-Iyama-Reiten '14]

$\text{f-tors}\,A := \{\text{"functorially finite" torsion class}\}$

$\subseteq \mathrm{tors}\,A$

$$\begin{array}{ccc} \text{f-tors}\,A & \xleftrightarrow{1\text{-}1} & \{\text{basic supp. } \tau\text{-tilting } A\text{-mod}\}/\simeq \\ & & \cup \\ & & \{\text{tilting}\} \end{array}$$

(c) [Demonet-Iyama-Jasso '18]

$\#\mathrm{tors}\,A < \infty \iff \#\,\text{f-tors}\,A < \infty$

$\iff \text{f-tors}\,A = \mathrm{tors}\,A$

Aim2 Generalize Thm2 for Noeth. alg

(a) (c) [Iyama-Kimura]

(b) by [Kimura '20]

§2 Noeth. alg

R : comm. Noeth. ring

<u>Def</u> Λ : ring.

Λ : Noetherian R-algebra

$:\Longleftrightarrow$ $\exists$ ring hom $\phi : R \to \Lambda$ s.t.

$\phi(R) \subseteq Z(\Lambda)$ and $\Lambda_R \in \operatorname{mod} R$

<u>Example</u> (1) $\Lambda = R$

(2) R : field Λ : fin. dim. R-alg.

$$\{\text{Noeth alg}\} \supseteq \left\{\begin{matrix}\text{comm. Noeth. ring} \\ \text{fin. dim. alg}\end{matrix}\right\}$$

(3) Path algebra RQ for a finite acyclic quiver

Let Λ be a Noeth. R-alg.

Def $\mathfrak{p} \in \operatorname{Spec} R$

- $k_{\mathfrak{p}} := R_{\mathfrak{p}} / \mathfrak{p} R_{\mathfrak{p}}$ $\leftarrow$ field
- $\Lambda_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R \Lambda$ $\leftarrow$ Noeth. $R_{\mathfrak{p}}$-alg
- $k_{\mathfrak{p}} \Lambda := k_{\mathfrak{p}} \otimes_R \Lambda$ $\leftarrow$ fin. dim. $k_{\mathfrak{p}}$-alg

$$\left(\begin{array}{l} \simeq \Lambda_{\mathfrak{p}} / \mathfrak{p} \Lambda_{\mathfrak{p}} \twoheadleftarrow \Lambda_{\mathfrak{p}} \\ \Rightarrow \operatorname{mod} k_{\mathfrak{p}} \Lambda \subseteq \operatorname{mod} \Lambda_{\mathfrak{p}} \end{array} \right)$$

Idea Study $\operatorname{tors} \Lambda$ by using $\operatorname{tors} k_{\mathfrak{p}} \Lambda$

For $\mathfrak{p} \in \operatorname{Spec} R$. construct maps

$$\operatorname{tors} \Lambda \xrightarrow{(-)_{\mathfrak{p}}} \operatorname{tors} \Lambda_{\mathfrak{p}} \xrightarrow{(-) \cap \operatorname{mod} k_{\mathfrak{p}} \Lambda} \operatorname{tors} k_{\mathfrak{p}} \Lambda$$

Def $\mathcal{C} \subseteq \operatorname{mod} \Lambda$ $\quad \mathfrak{p} \in \operatorname{Spec} R$

$$\mathcal{C}_{\mathfrak{p}} := \{ X_{\mathfrak{p}} \mid X \in \mathcal{C} \} \subseteq \operatorname{mod} \Lambda_{\mathfrak{p}}$$

Lem3 $\mathcal{C} \mapsto \mathcal{C}_p$ induces poset morph's

$$\operatorname{tors}\Lambda \longrightarrow \operatorname{tors}\Lambda_p$$
$$\operatorname{serre}\Lambda \longrightarrow \operatorname{serre}\Lambda_p$$
$$\operatorname{torf}\Lambda \longrightarrow \operatorname{torf}\Lambda_p$$

- $\mathcal{C}$: ext-closed $\Rightarrow$ $\mathcal{C}_p$: ext-closed
  fac / fac
  sub / sub
  //

Lem4 $\mathcal{C} \subseteq \operatorname{mod}\Lambda_p$ $(p \in \operatorname{Spec}R)$

$\mathcal{C} \longmapsto \mathcal{C} \cap \operatorname{mod} k_p\Lambda$ induces poset morph's

$$\operatorname{tors}\Lambda_p \longrightarrow \operatorname{tors} k_p\Lambda$$
$$\operatorname{serre}\Lambda_p \longrightarrow \operatorname{serre} k_p\Lambda$$
$$\operatorname{torf}\Lambda_p \longrightarrow \operatorname{torf} k_p\Lambda$$

//

- $\phi^p : \operatorname{tors}\Lambda \longrightarrow \operatorname{tors}\Lambda_p \longrightarrow \operatorname{tors} k_p\Lambda$

$$\mathcal{C} \longmapsto \mathcal{C}_p \cap \operatorname{mod} k_p\Lambda$$

(same for Serre, torsion free)

Def

(1) $\Phi_t : \operatorname{tors}\Lambda \longrightarrow \prod_{p \in \operatorname{Spec} R} \operatorname{tors} k_p\Lambda$

$\mathcal{C} \longmapsto (\phi^p(\mathcal{C}))_{p \in \operatorname{Spec} R}$

(2) $\Phi_f : \operatorname{torf}\Lambda \longrightarrow \prod_p \operatorname{torf} k_p \Lambda$

(3) $\Phi_s : \operatorname{serre}\Lambda \longrightarrow \prod_p \operatorname{serre} k_p \Lambda$

$$\mathbb{T}_R(\Lambda) := \prod_{p \in \operatorname{Spec} R} \operatorname{tors} k_p \Lambda$$

$$\mathbb{F}_R(\Lambda) := \prod_p \operatorname{torf} k_p \Lambda$$

$$\mathbb{S}_R(\Lambda) := \prod_p \operatorname{serre} k_p \Lambda$$

$$\begin{array}{ccccc}
\operatorname{serre}\Lambda & \subseteq & \operatorname{tors}\Lambda & \xrightarrow{(-)^{\perp}} & \operatorname{torf}\Lambda \\
\downarrow{\scriptstyle \Phi_s} & & \downarrow{\scriptstyle \Phi_t} & & \downarrow{\scriptstyle \Phi_f} \\
\mathbb{S}_R(\Lambda) & \subseteq & \mathbb{T}_R(\Lambda) & \underset{\sim}{\xrightarrow{(-)^{\perp}}} & \mathbb{F}_R(\Lambda)
\end{array}$$

anti-isom

- $\mathbb{I}_R(\Lambda) \ni (\mathcal{X}^p)_p, (\mathcal{Y}^p)_p$

$(\mathcal{X}^p)_p \leq (\mathcal{Y}^p)_p :\Longleftrightarrow \mathcal{X}^p \subseteq \mathcal{Y}^p \quad \forall p$

$\Rightarrow \Phi_\bullet$ are poset morph's

<u>Thm5</u> [IK]

(a) $\Phi_f : \mathrm{torf}\,\Lambda \xrightarrow{\sim} \mathbb{F}_R(\Lambda)$ is isom

(b) Φ_t, Φ_s are poset embedding

(i.e. $\Phi_t(\mathcal{C}) \leq \Phi_t(\mathcal{C}') \Rightarrow \mathcal{C} \subseteq \mathcal{C}'$
$\mathcal{C}, \mathcal{C}' \in \mathrm{tors}\,\Lambda$)

//

(*) [$\forall p \in \mathrm{Spec}\,R$, Λ_p is Morita equiv to a local ring]

(e.g. $\Lambda = R$)

$\Rightarrow$ · $k_p\Lambda \simeq \Lambda_p / p\Lambda_p \overset{\text{Morita}}{\sim}$ local ring

· $\mathrm{tors}\, k_p\Lambda = \mathrm{torf}\, k_p\Lambda = \mathrm{serre}\, k_p\Lambda$

$= \{0, \mathrm{mod}\, k_p \Lambda\}$

- $s : \mathbb{T}_R(\Lambda) \xrightarrow{\sim} P(\mathrm{Spec}\, R)$

$$(\mathcal{C}^p)_p \longmapsto \{p \mid \mathcal{C}^p = \mathrm{mod}\, k_p \Lambda\}$$

<u>Cor b</u> Assume $(*)$

(a) $\mathrm{serre}\, \Lambda = \mathrm{tors}\, \Lambda$

(b) $\mathrm{Im}\, \Phi_t \underset{\sim}{\xrightarrow{s}} \mathrm{Spcl}(R)$

(c) $s \circ \Phi_t = \mathrm{Supp}$, $\quad s(-)^c \circ \Phi_f = \mathrm{Ass}$ //

$$\begin{tikzcd}
\mathrm{serre}\, \Lambda \arrow[r, equal] \arrow[d, "\Phi_s"', "\simeq"] & \mathrm{tors}\, \Lambda \arrow[r, hook, "(-)^{\perp}"] \arrow[d, "\Phi_t"', "\simeq"] & \mathrm{torf}\, \Lambda \arrow[d, "\Phi_f"', "\simeq"] & \\
\mathrm{Im}\, \Phi_s \arrow[r, equal] & \mathrm{Im}\, \Phi_t \arrow[r, hook, "(-)^{\perp}"] \arrow[d, "s"', "\simeq"] & \mathbb{F}_R(\Lambda) \arrow[d, "s(-)^c"', "\simeq"] \arrow[r, phantom, "\ni"] & (\mathcal{C}^p)_p \arrow[ddl, mapsto] \\
& \mathrm{Spcl}(R) \arrow[r, hook, "(-)^c"'] & P(\mathrm{Spec}\, R) \arrow[d, phantom, sloped, "\ni"] & \\
& & \{p \mid \mathcal{C}^p = 0\} &
\end{tikzcd}$$

<u>$\Lambda = R \Rightarrow$ Thm 7</u>

§3 Serre sub

Recall

$$\begin{array}{ccc} \operatorname{serre} k_p\Lambda & \xrightarrow[]{\sim} & P(\operatorname{sim} k_p\Lambda) \\ \mathcal{C} & \longmapsto & \mathcal{C} \cap \operatorname{sim} k_p\Lambda \end{array}$$

Thm2 (a)

Let $\operatorname{Sim}_R(\Lambda) := \bigcup_{p \in \operatorname{Spec} R} \operatorname{sim} k_p\Lambda$

$$\Rightarrow \begin{array}{ccc} \mathbb{S}_R(\Lambda) & \xrightarrow{\sim} & \operatorname{Sim}_R(\Lambda) \\ (\mathcal{C}^p)_p & \longmapsto & \bigcup_p (\mathcal{C}^p \cap \operatorname{sim} k_p\Lambda) \end{array}$$

$$\Rightarrow \operatorname{serre}\Lambda \xrightarrow{\Phi_s} \mathbb{S}_R(\Lambda) \xrightarrow[\sim]{\iota} \operatorname{Sim}_R(\Lambda)$$

$$\operatorname{Im}(\iota \circ \Phi_s) \underset{?}{\subseteq} \operatorname{Sim}_R(\Lambda)$$

Def

(1) $S \in \operatorname{sim} k_p\Lambda \quad T \in \operatorname{sim} k_q\Lambda$

$S \leq T :\Longleftrightarrow p \supseteq q$ and

S is a subfactor of T
in $\operatorname{mod}\Lambda_p$

$$\left[\begin{tikzcd} \Lambda_p \arrow[r] \arrow[d, two heads] & \Lambda_q \arrow[d, two heads] \\ k_p\Lambda & k_q\Lambda \end{tikzcd} \Longrightarrow \begin{tikzcd} \operatorname{mod}\Lambda_p & \operatorname{mod}\Lambda_q \arrow[l] \\ \operatorname{mod}k_p\Lambda \arrow[u] & \operatorname{mod}k_q\Lambda \arrow[u] \\ S \arrow[u, phantom, "\in" rotate=90] & T \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}\right]$$

Then $(\operatorname{Sim}_R(\Lambda), \leq)$ is poset.

(2) $\mathcal{W} \subseteq \operatorname{Sim}_R(\Lambda)$ is a down-set
if $T \in \mathcal{W}$, $S \leq T \Rightarrow S \in \mathcal{W}$

<u>Thm 7</u> $\iota\circ\Phi_S : \operatorname{serre}\Lambda \longrightarrow \operatorname{Sim}_R(\Lambda)$
induces

$$\operatorname{serre}\Lambda \xrightarrow{\sim} \{\text{down-set of } \operatorname{Sim}_R(\Lambda)\}$$

$\Lambda = R \Rightarrow$ Thm 7 (a)

<u>Example</u> k : field

$R = k[\![x]\!] \supset (x) = \mathfrak{m}$

$\operatorname{Spec} R = \{0, \mathfrak{m}\}$ $K := R_0 = k(\!(x)\!)$

$\Lambda = \begin{pmatrix} R & R \\ \mathfrak{m} & R \end{pmatrix}$: Noeth R-alg

- $k_{\mathfrak{m}} \Lambda = \Lambda / \mathfrak{m}\Lambda = \begin{pmatrix} R/\mathfrak{m} & R/\mathfrak{m} \\ \mathfrak{m}/\mathfrak{m}^2 & R/\mathfrak{m} \end{pmatrix}$

$\simeq k(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2) / \langle \alpha\beta, \beta\alpha \rangle$

$\operatorname{sim} k_{\mathfrak{m}} \Lambda = \left\{ \underset{\substack{\| \\ S_1}}{\begin{pmatrix} k \\ 0 \end{pmatrix}}, \underset{\substack{\| \\ S_2}}{\begin{pmatrix} 0 \\ k \end{pmatrix}} \right\}$

- $k_0 \Lambda = \Lambda_0 = \begin{pmatrix} R_0 & R_0 \\ \mathfrak{m}_0 & R_0 \end{pmatrix} = \begin{pmatrix} K & K \\ K & K \end{pmatrix} = \operatorname{Mat}_2(K)$

$\operatorname{sim} k_0 \Lambda = \left\{ \underset{\substack{\| \\ T}}{\begin{pmatrix} K \\ K \end{pmatrix}} \right\}$

$K = k(\!(x)\!)$, so S_1, S_2 are subfactor of T in $\operatorname{mod} \Lambda$ i.e. $S_1 \leq T$, $S_2 \leq T$

$$\Rightarrow \quad \mathrm{Sim}_R(\Lambda) = \left\{ \begin{tikzcd}[column sep=small] & T \arrow[dl] \arrow[dr] & \\ S_1 & & S_2 \end{tikzcd} \right\}$$

$$\Rightarrow \quad \{\text{down-set of } \mathrm{Sim}_R(\Lambda)\}$$

$$= \left\{ \begin{tikzcd}[column sep=small] & \mathrm{Sim}_R(\Lambda) \arrow[d] & \\ & \{S_1 . S_2\} \arrow[dl] \arrow[dr] & \\ \{S_1\} \arrow[dr] & & \{S_2\} \arrow[dl] \\ & \phi & \end{tikzcd} \right\}$$

$$\simeq \text{serre } \Lambda$$