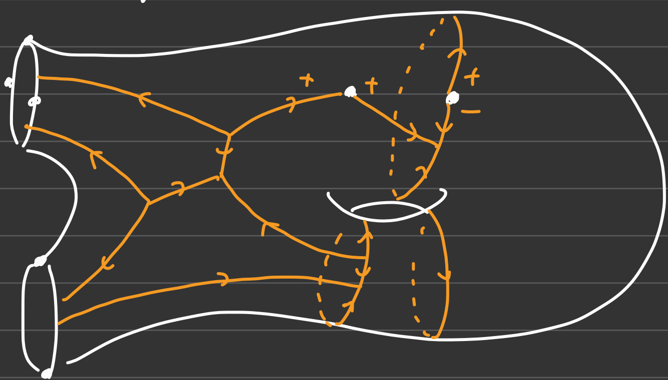
2022.5.17

@ Mods seminar

# Unbounded $\mathfrak{sl}_3$-laminations & their shear coordinates

joint work w/ Shunsuke Kano (Tohoku Univ.)

signed webs



$\updownarrow$

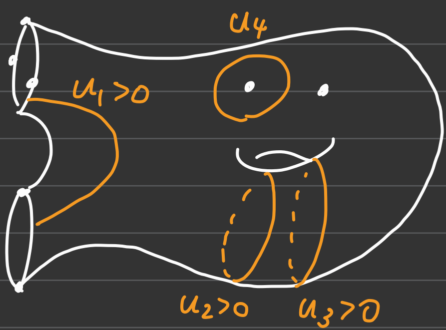
spiralling diagrams

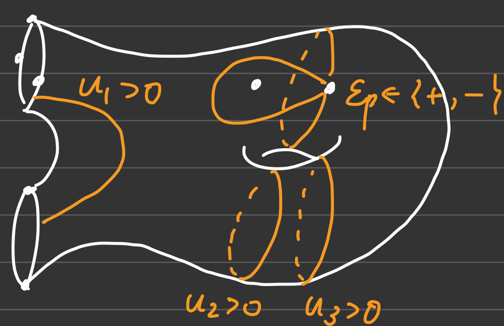
## §1. Introduction : $sl_2$-laminations

$\Sigma = (\Sigma, \mathbb{M})$ : a marked surface

$\rightsquigarrow (\mathcal{A}_{sl_2,\Sigma}, \mathcal{X}_{sl_2,\Sigma})$ : $sl_2$-cluster ensemble

( Fock-Goncharov'06 )




$\mathcal{L}^a_{sl_2}(\Sigma, \mathbb{Q})$ $\qquad$ $\mathcal{L}^x_{sl_2}(\Sigma, \mathbb{Q})$

$\cong$ intersection coord. $\qquad$ $\cong$ shear coord.

$\mathcal{A}_{sl_2,\Sigma}(\mathbb{Q}^T)$ $\qquad$ $\mathcal{X}^{uf}_{sl_2,\Sigma}(\mathbb{Q}^T)$

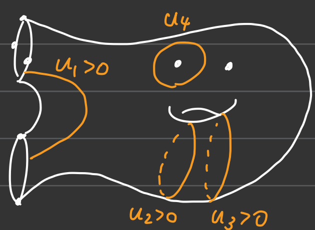
... They give topological models for the tropical points.

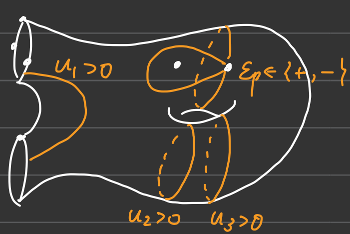
Two remarkable roles:

① The integral points $V_{\mathfrak{sl}_2,\Sigma}(\mathbb{Z}^T) \subset V_{\mathfrak{sl}_2,\Sigma}(\mathbb{Q}^T)$

$(V = \mathcal{A}, \mathcal{X})$

parametrizes a basis of the function ring on the dual side:




$$
\begin{tikzcd}
\mathcal{A}_{\mathfrak{sl}_2,\Sigma}(\mathbb{Z}^T) \arrow[dr, "\mathbb{I}_{\mathcal{A}}"] & \mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{Z}^T) \arrow[dl, "\mathbb{I}_{\mathcal{X}}"'] \\
\mathcal{O}(\mathcal{A}_{\mathfrak{sl}_2,\Sigma}) & \mathcal{O}(\mathcal{X}_{\mathfrak{sl}_2,\Sigma})
\end{tikzcd}
$$

(cluster alg.) $\subset$

$\mathcal{O}(\mathcal{A}_{\mathfrak{sl}_2,\Sigma}) \cong \mathcal{O}(\mathcal{A}^{\times}_{SL_2,\Sigma})$ [I.-Oya-Shen'22]

$\mathcal{O}(\mathcal{X}_{\mathfrak{sl}_2,\Sigma}) \cong \mathcal{O}(\mathcal{P}_{PGL_2,\Sigma})$ [Shen'20]

(if $\Sigma$ is unpunctured)

$\exists$ quantum versions via $\mathfrak{sl}_2$-skein theory.

This is one of the motivating examples of

(quantum) Fock–Goncharov duality map.

[ classical Fock–Goncharov'06 ($\mathcal{A}/\mathcal{X}$)

Musiker–Schiffler–Williams'13 ($\mathcal{A}$), D. Thurston'14 ($\mathcal{A}$), ...

quantum Bonahon–Wong'10 ($\mathcal{X}_q$), Muller'13 ($\mathcal{A}_q$),

D. Thurston'14 ($\mathcal{A}_q$), Allegretti–Kim'17 ($\mathcal{X}_q$), ... ]

② The real completion $\mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{R}^T) \supset \mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{Q}^T)$

captures the large-scale geometry of $\mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{R}_{>0})$ ::

$\exists$ Fock–Goncharov compactification

$$\overline{\mathcal{X}_{\mathfrak{sl}_2,\Sigma}} := \mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{R}_{>0}) \cup \mathbb{S}\mathcal{X}_{\mathfrak{sl}_2,\Sigma}(\mathbb{R}^T)$$

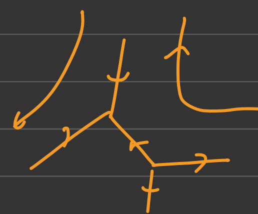
cf. Thurston compactification $\overline{\mathcal{T}(\Sigma)} = \mathcal{T}(\Sigma) \cup \mathbb{S}\mathcal{ML}(\Sigma)$

**Aim** Find a topological model of $\mathcal{L}^{x}_{\mathfrak{sl}_3,\Sigma}(\mathbb{Q}^T)$.

**Recent developments**:

- Douglas–Sun '20 + H.K. Kim '21 :

  topological model of $\mathcal{A}_{\mathfrak{sl}_3,\Sigma}(\mathbb{Q}^T)$

  by Kuperberg's $\mathfrak{sl}_3$-webs.

- H.K. Kim '21 : construction of (quantum) duality map

  $$\mathbb{I}^{(q)}_{\mathcal{X}} : \mathcal{A}_{\mathfrak{sl}_3,\Sigma}(\mathbb{Z}^T) \longrightarrow \mathcal{O}_{(q)}(\mathcal{X}_{\mathfrak{sl}_3,\Sigma})$$

  via a skein model of $\mathcal{O}_q(\mathcal{X}_{\mathfrak{sl}_3,\Sigma})$.

- I.–Yuasa '21 : skein model of $\mathcal{O}_q(\mathcal{A}_{\mathfrak{sl}_3,\Sigma})$

  ($\Sigma$: unpunctured)

The last piece to be filled :

appropriate control of $\mathfrak{sl}_3$-webs at punctures.

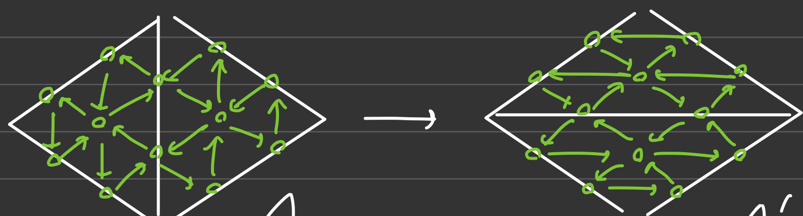
**Theorem** (I.–Kano '22)

$$\mathcal{L}^{x}_{sl_3}(\Sigma, \mathbb{Q}) := \{\text{non-elliptic, } \underline{\text{signed}}\ \mathbb{Q}_{>0}\text{-weighted } sl_3\text{-webs on } \Sigma\} / \sim$$

1) For each ideal triangulation $\Delta$ of $\Sigma$,

$\exists\ x^{uf}_{\Delta} : \mathcal{L}^{x}_{sl_3}(\Sigma, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^{I_{uf}(\Delta)}$ "shear coord."

s.t. $x^{uf}_{\Delta'} \circ (x^{uf}_{\Delta})^{-1}$ are tropical cluster transformations.



$\Delta$ $\qquad\qquad\qquad\qquad$ $\Delta'$

$\rightsquigarrow \exists$ canonical isom. $\mathcal{L}^{x}_{sl_3}(\Sigma, \mathbb{Q}) \xrightarrow{\sim} \mathcal{X}^{uf}_{sl_3,\Sigma}(\mathbb{Q}^{T})$

2) We also introduce $\mathcal{L}^{P}_{sl_3}(\Sigma, \mathbb{Q}) \cong \mathcal{X}_{sl_3,\Sigma}(\mathbb{Q}^{T})$.

+ "pinnings" (cf. [GS'19])

We have:

- amalgamation maps $\mathcal{L}^{p}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q}) \longrightarrow \mathcal{L}^{p}_{\mathfrak{sl}_3}(\Sigma', \mathbb{Q})$


- ensemble map $\tilde{p} : \mathcal{L}^{a}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q}) \longrightarrow \mathcal{L}^{p}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q})$

3) Construct a quantum duality map

$$\mathbb{I}^{q}_{\mathfrak{X}} : \mathcal{L}^{p}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z}) \hookrightarrow \mathcal{S}^{q}_{\mathfrak{sl}_3, \Sigma}[\partial^{-1}] \subset \mathcal{O}_q(\mathcal{A}_{\mathfrak{sl}_3, \Sigma})$$

[I.-Yuasa'21]

for unpunctured $\Sigma$.

<u>Remark</u> $\mathcal{S}^{1}_{\mathfrak{sl}_3, \Sigma}[\partial^{-1}] = \mathcal{A}_{\mathfrak{sl}_3, \Sigma} = \mathcal{U}_{\mathfrak{sl}_3, \Sigma} = \mathcal{O}(A^{\times}_{SL_3, \Sigma})$.

[I.-Oya-Shen'22]

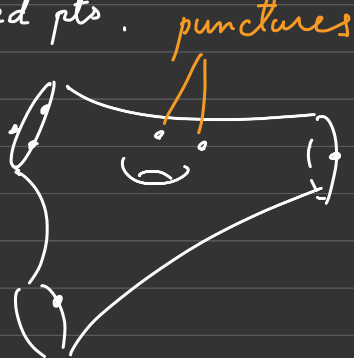
$\mathbb{I}^{1}_{\mathfrak{X}}(\mathcal{L}^{p}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z}))$ gives a $\mathbb{Z}$-basis of these alg's.

## §2. Unbounded $\mathfrak{sl}_3$-laminations

They will be defined as certain equiv classes of

<u>$\mathbb{Q}_{>0}$-weighted</u>, <u>signed</u> $\mathfrak{sl}_3$-webs on a marked surface.

A marked surface $(\Sigma, \mathbb{M})$ is a compact ori. surface $\Sigma$ equipped w/ a fin. set $\mathbb{M} \subset \Sigma$ of "marked pts".




- $\mathbb{M} = \mathbb{M}_\circ \cup \mathbb{M}_\partial$

  punctures special pts

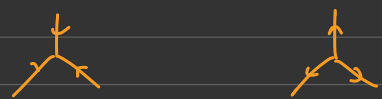
- $\Sigma^* := \Sigma \setminus \mathbb{M}_\circ$ : punctured surface

- $\partial^*\Sigma := \partial\Sigma \setminus \mathbb{M}_\partial$ : punctured boundary

When no confusion can occur, we write $\Sigma = (\Sigma, \mathbb{M})$.

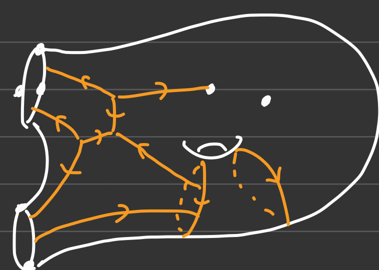
• An <u>oriented</u> <u>uni-trivalent graph</u> consists of:

- a fin. graph only w/ 1- or 3-val. vertices (we also allow loop components)
- an ori. of each edge s.t. any 3-val. vertex is a <u>sink</u> or a <u>source</u>



<u>Def</u>

An <u>$\mathfrak{sl}_3$-web</u> on $\Sigma$ is an imm. ori. uni-tri. graph $W$ s.t.

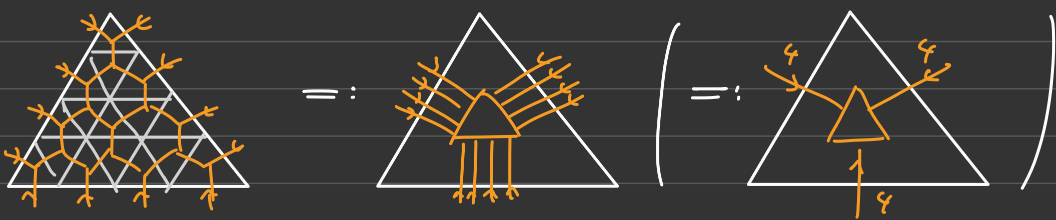


- 1-val. vertex $\longmapsto \mathbb{M}_\circ \cup \partial^* \Sigma$
- other part is embedded into $\mathrm{int}\,\Sigma^*$

- $W$ is non-elliptic $:\Leftrightarrow$ it has none of

- $W$ is founded $:\Leftrightarrow$ $W \cap M_0 = \phi$

<u>Example</u> (honeycomb web)




<u>Def</u> A <u>signed web</u> is a web $W$ on $\Sigma$ together w/ a sign $\in \{+,-\}$ assigned to each end of $W$ incident to a puncture.

<u>Rem</u> A founded web is automatically a signed web.

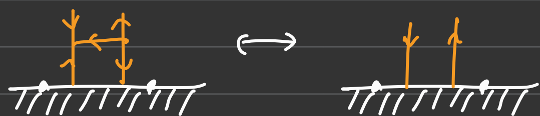
• The following patterns are not allowed :

($\varepsilon \in \{+,-\}$)

## (iii) Elementary moves of signed webs

E1) Loop parallel-move

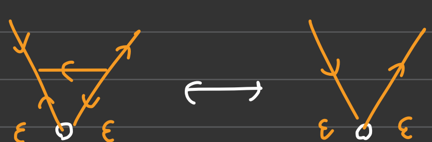
E2) Boundary H-move



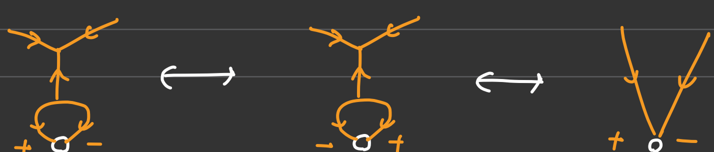
E3) Puncture H-moves

[ cf. skein rel.

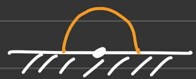
]

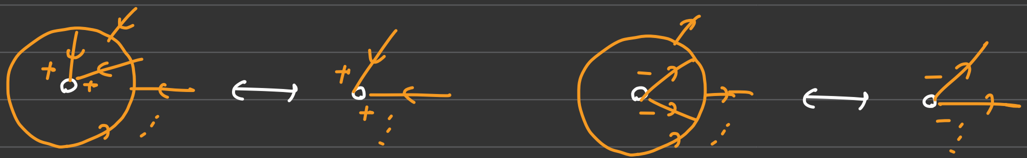


($\varepsilon \in \{+,-\}$)



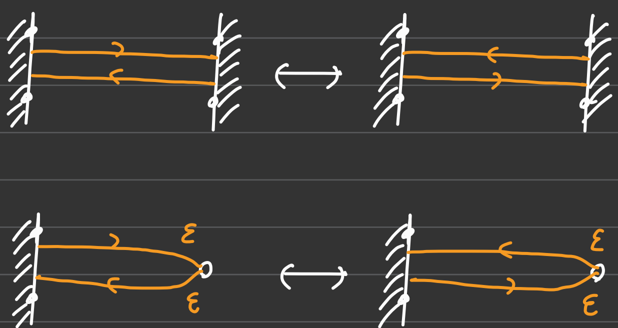
E4) Peripheral moves:

- remove / create  or 

- 



<u>Lemma</u> Parallel moves

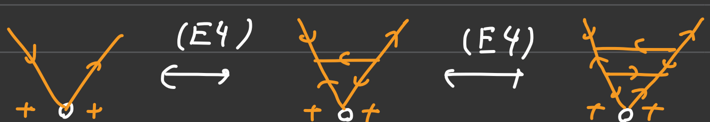






follow from (E2), (E3)

<u>Remark</u> non-ellipticity is not preserved.

e.g. 

**Def** A <u>rational unfounded $\mathfrak{sl}_3$-lamination</u> on $\Sigma$ is a signed non-elliptic $\mathfrak{sl}_3$-web $W$ on $\Sigma$ equipped w/ a $\mathbb{Q}_{>0}$-weight on each comp.

It is considered modulo the moves (E1) — (E4) and:

5) Weighted isotopy


6) Weighted cabling


$$\begin{pmatrix} u \in \mathbb{Q}_{>0} \\ n \in \mathbb{N} \end{pmatrix}$$

$\lambda^3$

$$\mathcal{L}^x_{sl_3}(\Sigma, \mathbb{Q}) := \{ \text{rational unfounded } sl_3\text{-laminations on } \Sigma \}$$

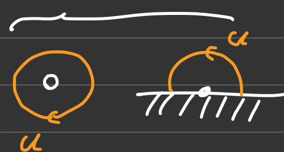
$$\cup$$

$$\mathcal{L}^x_{sl_3}(\Sigma, \mathbb{Z}) := \{ \text{weights are integral} \}$$

ⓘⓘⓘ Relation to Douglas–Sun–Kim:

$$\mathcal{L}^a_{sl_3}(\Sigma, \mathbb{Z}) := \underline{\{ \text{founded, integrals} \}} \times \overbrace{\mathbb{Z}^{2\#M}}$$

[H.K.Kim'21] [Douglas–Sun'20]




We have the geometric ensemble map

$$p : \mathcal{L}^a_{sl_3}(\Sigma, \mathbb{Z}) \longrightarrow \mathcal{L}^x_{sl_3}(\Sigma, \mathbb{Z})$$

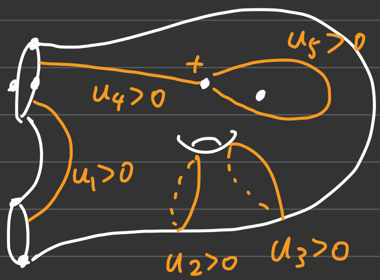
forgetting the $\mathbb{Z}^{2\#M}$-factor.

It will satisfy $\quad p^* x_i^\Delta = \sum \varepsilon_{ij}^\Delta a_j^\Delta$

shear coord. (Today) $\quad$ $\frac{1}{3} \times$ DS coord.

## §3. Shear coordinates

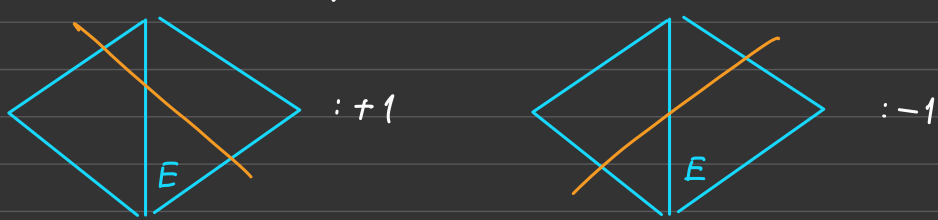
<u>Recall</u> : $\mathfrak{sl}_2$-case ( FG'07, "Dual Teich. & lamination spaces")




Here, a sign is assigned to <u>each puncture</u>.

<u>spiralling diagram</u> :


$\Delta$ : an ideal triangulation




$x_E^{\Delta}$ is the weighted sum of these contributions.

W : a non-elliptic signed web

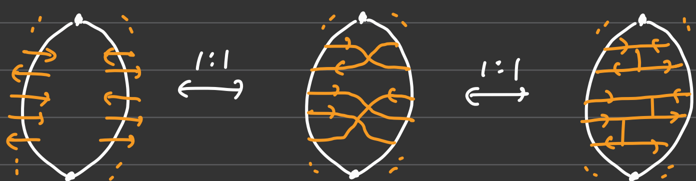
<u>spiralling diagram</u> :


It may produce $\infty$'ly many self-intersections.

<u>Q</u>. How to correctly resolve ?

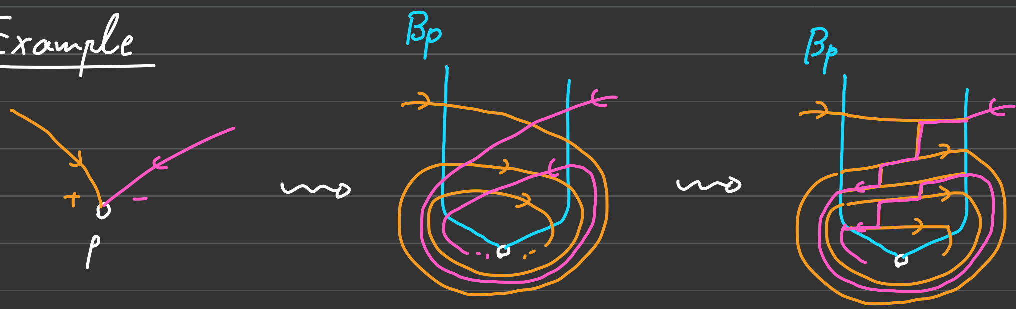
Recall the "ladder-web" construction.

[Frohman–Sikora '20] [DS'20]



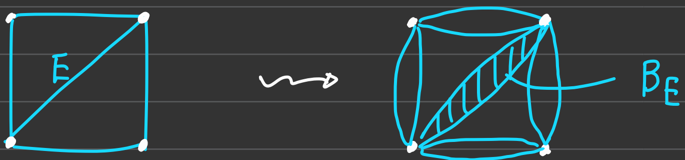

In the unbounded case, we allow $\infty$ but asymptotically periodic patterns toward punctures.

## Example

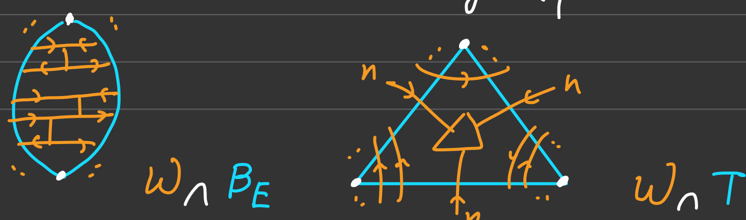



Resulting diagram $\mathcal{W}$ is called a spiralling diagram (independent of the half-bigon $B_p$).

- An ideal triangulation $\Delta$ $\rightsquigarrow$ split triangulation $\hat{\Delta}$

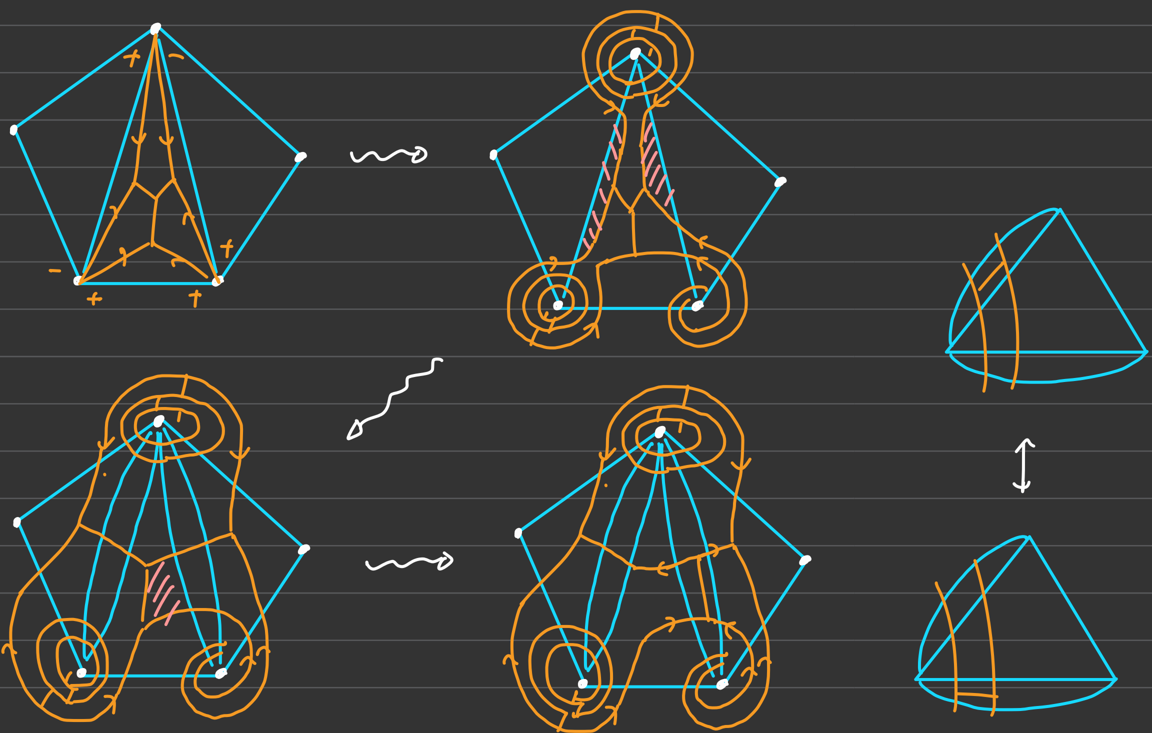



**Technical theorem** Any spiralling diagram $\mathcal{W}$ can be isotoped into a "good position" by a fin. seq. of (periodic) moves ①—③.



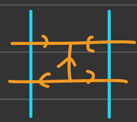
Moreover, such a good position is unique up to certain moves & strict isotopy rel. to $\hat{\Delta}$.

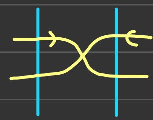

Example



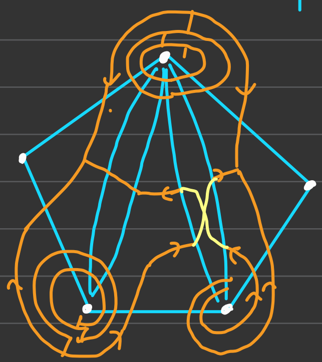
## (III) Definition of shear coordinates

- spiralling diagram $W$ $\rightsquigarrow$ <u>braid rep.</u> $W_{br}$

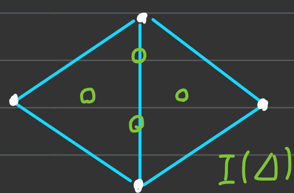




<u>Example</u>



Let us focus on a quadrilateral :

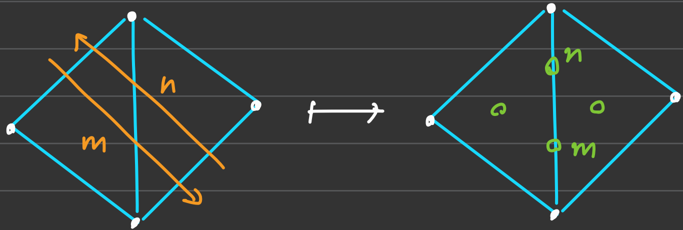


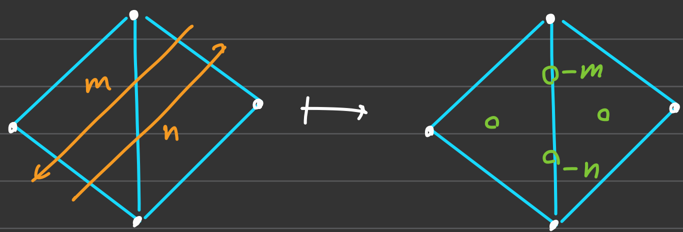

The shear coord's $x_i^{\Delta}(W)$ will be assigned to $i \in I_{uf}(\Delta)$.

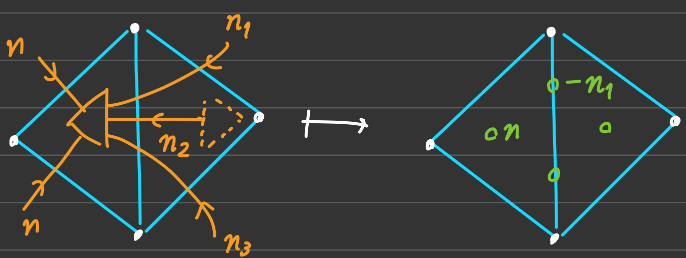
not on $\partial\Sigma$

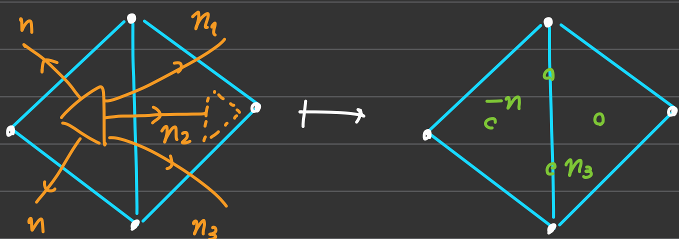
They are weighted sums of

following contributions (a) — (d) :

a) 


b) 


c) 


d) 


… up to symmetry.

- Recall that $\hat{L} \in \mathcal{L}^{x}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z})$ can be rep'ed by a signed web $W$ of weight 1.

Define $\boxed{x_\Delta(\hat{L}) := x_\Delta(W) = (x_i^\Delta(W))_{i \in I_{uf}(\Delta)}}$

- Extend to $\mathcal{L}^{x}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q})$ by $x_\Delta(\alpha \cdot \hat{L}) = \alpha \cdot x_\Delta(\hat{L})$

$(\forall \alpha \in \mathbb{Q}_{>0})$

<u>Theorem</u>

We have a well-defined bijection

$$x_\Delta : \mathcal{L}^{x}_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^{I_{uf}(\Delta)}.$$

proof)

- Reconstruction : "FG-like" gluing reconstruction

  $\rightsquigarrow$ surjectivity is easier.

- Injectivity : Unbounded version of

  "Fellow-traveler lemma" [DS'20]

## (III) Supplying the frozen coordinates (briefly)

$\mathbb{B} := \pi_0(\partial^* \Sigma) \ni$ "boundary intervals" $\quad (|\mathbb{B}| = |M_\partial|)$

$$\mathcal{L}^p_{sl_3}(\Sigma, \mathbb{Q}) := \mathcal{L}^x_{sl_3}(\Sigma, \mathbb{Q}) \times \underbrace{\bigoplus_{E \in \mathbb{B}} P^\vee_{\mathbb{Q}}}_{\text{"pinnings"}}$$

Here, $P^\vee = \mathbb{Z}\varpi_1^\vee \oplus \mathbb{Z}\varpi_2^\vee$ : coweight lattice

$$P^\vee_{\mathbb{Q}} := P^\vee \otimes \mathbb{Q}$$

cf. [GS'19]

$\mathcal{P}_{PGL_3, \Sigma} \longrightarrow \mathcal{X}_{PGL_3, \Sigma}$ dominant, $H^{\mathbb{B}}$–f'dl over the image

$$\overset{\text{Trop}}{\rightsquigarrow} \; 0 \longrightarrow \underbrace{H^{\mathbb{B}}(\mathbb{Q}^T)}_{\cong \bigoplus_{\mathbb{B}} P^\vee_{\mathbb{Q}}} \longrightarrow \mathcal{L}^p_{sl_3}(\Sigma, \mathbb{Q}) \longrightarrow \mathcal{L}^x_{sl_3}(\Sigma, \mathbb{Q}) \longrightarrow 0$$

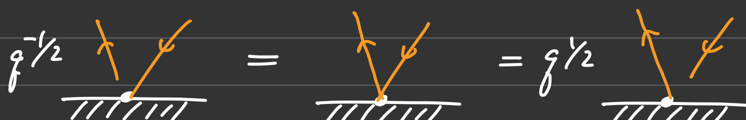
We can extend $x_\Delta : \mathcal{L}^p_{sl_3}(\Sigma, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^{I(\Delta)}$

## §4. Relation to the graphical basis [IY'21]

Assume $\mathbb{M}_\circ = \phi$ $\quad (\mathbb{M} = \mathbb{M}_\partial \subset \partial\Sigma)$

$\rightsquigarrow$ $\mathscr{S}^q_{sl_3,\Sigma}$ : "clasped" $sl_3$-skein alg.

, etc.



, etc.

"simultaneous"

**Theorem** (I.-Yuasa'21)

- $\mathrm{BWeb}_{sl_3,\Sigma} = \{$ flat trivalent graphs $\}$ gives a $\mathbb{Z}_q := \mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathscr{S}^q_{sl_3,\Sigma}$
- $\mathscr{S}^q_{sl_3,\Sigma}[\partial^{-1}] \subset \mathscr{A}^q_{sl_3,\Sigma}$

∂-localized

quantum cluster alg

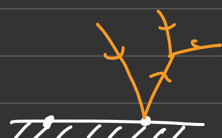
$(\hat{L}, (\delta_E)_{E\in\mathbb{B}}) \in \mathcal{L}^p_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z})$ : dominant

$:\Longleftrightarrow$ $\delta_E \in P_+^\vee$ (dominant coweight), $\forall E \in \mathbb{B}$

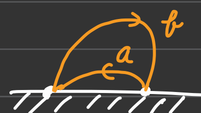
<u>Theorem</u> (I.–Kano '22)

1) $\mathbb{I}^q_{\mathscr{X}} : \mathcal{L}^p_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z})_+ \xrightarrow{\ \sim\ } \mathrm{BWeb}_{\mathfrak{sl}_3, \Sigma} \subset \mathscr{S}^q_{\mathfrak{sl}_3, \Sigma}$

dominant

 $\longmapsto$ 

E E

$\delta_E = a\varpi_1^\vee + b\varpi_2^\vee \longmapsto$ 


E

which is extended to $\mathbb{I}^q_{\mathscr{X}} : \mathcal{L}^p_{\mathfrak{sl}_3}(\Sigma, \mathbb{Z}) \hookrightarrow \mathscr{S}^q_{\mathfrak{sl}_3, \Sigma}[\partial^{-1}]$

again giving a $\mathbb{Z}_q$-basis.

2) $y_i := -x_i^{\Delta}(\hat{L}, \delta) \geq 0, \quad \forall i \in I$

$$\Rightarrow \mathbb{I}_{\mathcal{X}}^{q}(\hat{L}, \delta) = \Big[ \prod_{i \in I(\Delta)} (A_i^{\Delta})^{y_i} \Big] \quad : \text{cluster monomial}$$

3) If $\Sigma$ = 3-gon or 4-gon,

Image $\mathbb{I}_{\mathcal{X}}^{q}$ consists of cluster monomials

& gives a $\mathbb{Z}_q$-basis of $\mathscr{S}_{\mathfrak{sl}_3, \Sigma}^{q}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sl}_3, \Sigma}^{q} = \mathscr{U}_{\mathfrak{sl}_3, \Sigma}^{q}$

[I.-Yuasa '21]

4) In the classical limit $q^{1/2} = 1$,

Image $\mathbb{I}_{\mathcal{X}}^{1}$ gives a $\mathbb{Z}$-basis of

$$\mathscr{S}_{\mathfrak{sl}_3, \Sigma}^{1}[\partial^{-1}] = \mathscr{A}_{\mathfrak{sl}_3, \Sigma} = \mathscr{U}_{\mathfrak{sl}_3, \Sigma} = \mathcal{O}(\mathcal{A}_{SL_3, \Sigma}^{\times}).$$

[I.-Oya-Shen '22]

<u>Conjecture</u> Image $\mathbb{I}_{\mathcal{X}}^{q}$ is "parametrized by tropical points" in the sense of F. Qin.

# Future works:

- Weyl group action $W(\mathfrak{sl}_3)^{M_\circ} \curvearrowright \mathcal{L}^x_{\mathfrak{sl}_3}(\Sigma, \mathbb{Q})$

(in preparation)

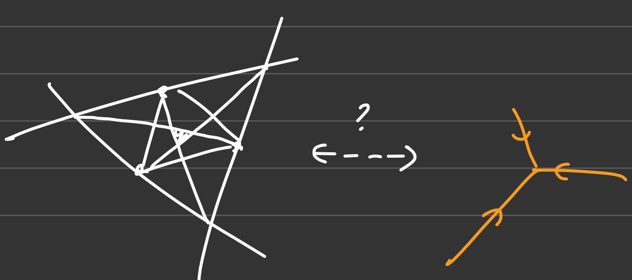
cf. $W(\mathfrak{sl}_3)^{M_\circ} \curvearrowright \mathcal{X}_{PGL_3,\Sigma}$ [GS'18]

$\rightsquigarrow$ "higher tags"? cf. [Fraser–Pylyavskyy '21]

- Geometric model for $\mathcal{L}^x_{\mathfrak{sl}_3}(\Sigma, \mathbb{R}) \cong \mathcal{X}^{uf}_{\mathfrak{sl}_3,\Sigma}(\mathbb{R}^T)$ ?

cf. $\mathcal{X}_{PGL_3,\Sigma}(\mathbb{R}_{>0})$ = moduli space of convex $\mathbb{RP}^2$-str's on $\Sigma$ [FG'07]

It should describe deformations / degenerations of convex $\mathbb{RP}^2$-str's.




Thank you!