Sheaf quantization and cluster coordinate

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Our Goal (not yet reached) — Some geometric interpretation & updates

on Iwaki-Nakanishi's Construction.

&

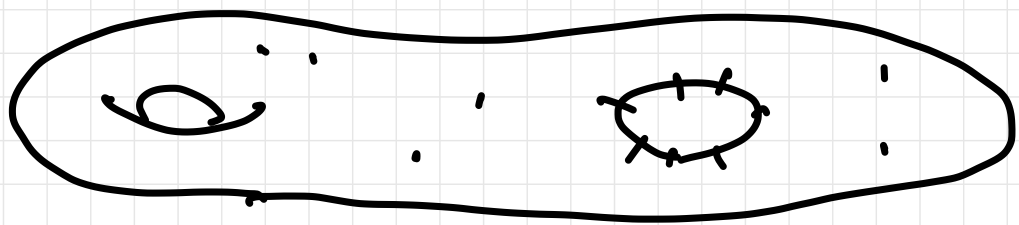
Exact WKB analysis & cluster algebra.

Todays aim

To outline our program.

(mostly consisting of observations).

Geometry: Two variants of Teichmüller space for bounded marked surfaces.

$S =$ 

We'll work / $\mathbb{C}$.

① Moduli of framed PSL_2-local systems.

Around each puncture/marked point, 1-dim subspace $\subset \mathbb{C}^2$.

② Moduli of decorated twisted SL_2-local systems.

will be explained later.

Around each puncture/marked point, a vector. $\subset \mathbb{C}^2$.

$\exists$ "forgetful" map =: ensemble map

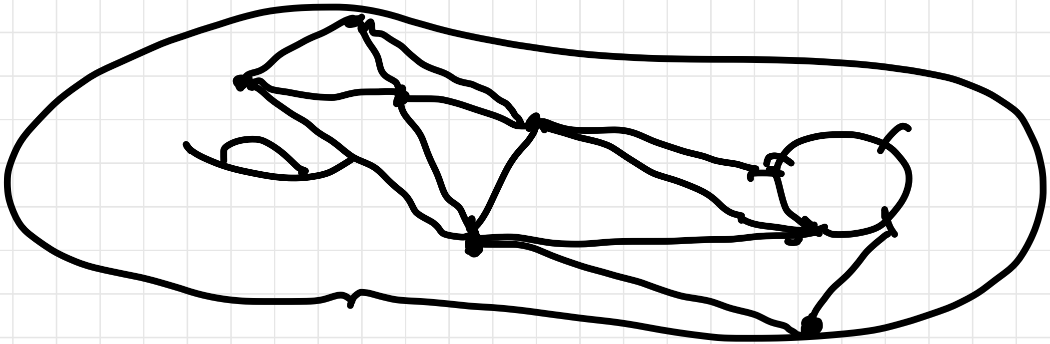
$$\mathcal{M}^{\mathrm{dec,tw}}_{SL_2}(S) \longrightarrow \mathcal{M}^{\mathrm{framed}}_{PSL_2}(S)$$

$$\left(\begin{array}{ccc} \text{vector } u & \longmapsto & \mathbb{C}\cdot u \\ \underset{SL_2\text{ loc system}}{L} & \longmapsto & \underset{PSL_2\text{-loc system}}{[L]} \end{array} \right)$$

<u>Combinatorial description of this map.</u>

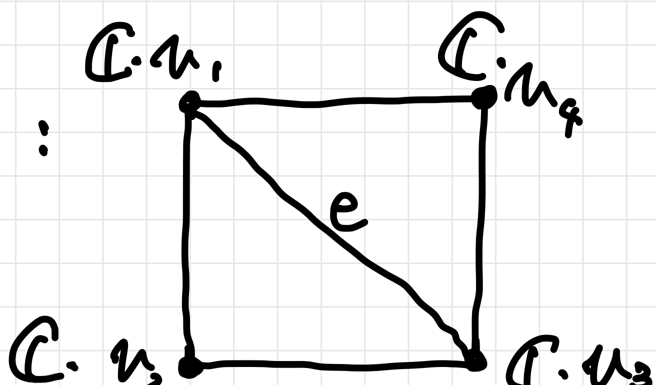
(Fock, Fock-Goncharov).

Fix an ideal triangulation,



For each internal edge, we have a rational function on the moduli:

PSL_2 :

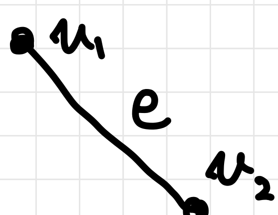


Given a framed local system $\mathcal{L}$,

$$\mathcal{X}_e(\mathcal{L}) := \frac{\det(v_1, v_2)\det(v_3, v_4)}{\det(v_1, v_4)\det(v_2, v_3)}$$

$$\mathcal{M}^{\text{framed}}_{PSL_2}(S) \dashrightarrow (\mathbb{C}^\times)^{\#\{\text{internal edges}\}}$$

Similarly, given a twisted local system

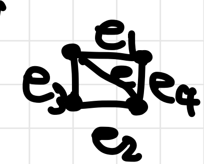


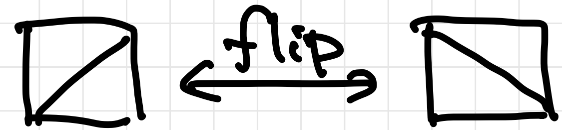
$$\mathcal{A}_e\left(\begin{matrix}\text{dec}\\ \text{loc}\end{matrix}\right) := \det(v_1, v_2)$$

$$\mathcal{M}^{\mathrm{dec,tw}}_{SL_2}(S) \dashrightarrow (\mathbb{C}^\times)^{\#\{\mathrm{edges}\}}$$

Then we get a commutative diagram.

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{dec,tw}}_{SL_2}(S) \arrow[r, dashed] \arrow[d, "\text{ensemble}"] \arrow[dr, phantom, "\circlearrowright"] & (\mathbb{C}^\times)^{\#\{\mathrm{edges}\}} \arrow[d, "X_e = \frac{A_{e_1} A_{e_2}}{A_{e_3} A_{e_4}}"] \\
\mathcal{M}^{\mathrm{framed}}_{PSL_2}(S) \arrow[r, dashed] & (\mathbb{C}^\times)^{\#\{\text{internal edges}\}}
\end{tikzcd}$$



Feature: • $\mathcal{X}, \mathcal{A}$-coordinates behave well under 

.

• The resulting glued-up maps are birational:

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{framed}}_{SL_2}(S) \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{X}(S) \arrow[d] \\
\mathcal{M}^{\mathrm{tw,dec}}_{PSL_2}(S) \arrow[r, dashed] & \mathcal{A}(S)
\end{tikzcd}$$

Conclusion: We could (birationally) completely descibe the ensemble map in a combinatorial way.

Fock-Goncharov: Abstracting this combinatorial construction

$\rightsquigarrow$ cluster ensembles.

A quick review of cluster ensembles

Linear alg data

N: rank n free abelian group equipped with $\{\cdot,\cdot\}$ skew-sym form

M: Dual of N

I : Index st. $\# I = n$

Seed $:= \{e_i\}_{i \in I}$: a basis of N

Seed & $k \in I$ give a new seed

$$\rightsquigarrow \mu_k(e_i) = \begin{cases} -e_k & (i = k) \\ e_i + \max\{0, \{e_i, e_k\}\} e_k & (i \neq k) \end{cases}$$

<u>Algebro-geometric construction</u>

$$\begin{array}{ccccc} (\mathbb{C}^\times)^n & \xleftarrow[\mu_k(\{e_i\})]{\sim} & T_N & \xrightarrow[\{e_i\}]{\sim} & (\mathbb{C}^\times)^n \\ & & \shortparallel & & \\ & & \mathbb{C}^\times \otimes N & & \end{array}$$

not interesting

twist

$$(\mathbb{C}^\times)^n \xleftarrow{\ \mu_k\ } (\mathbb{C}^\times)^n$$

$\{f_i\}$: dual basis of $\{e_i\}$

$$z^{f_i} =: A_i \in \mathbb{C}[M]$$

$$z^{\mu_k(f_i)} =: A_i'$$

$$\mu_k^* A_i' := \begin{cases} A_i & \text{if } i \neq k \\ A_k^{-1}\left(\prod\limits_{\{e_k,e_j\}>0} A_j^{\{e_k,e_j\}} + \prod A_j^{-\{e_k,e_j\}}\right) & \end{cases}$$

Similarly,

$$\begin{tikzcd} & T_M \arrow[dl, "\{\mu_k f_i\}"', "\sim"] \arrow[dr, "\{f_i\}", "\sim"'] & \\ (\mathbb{C}^\times)^n & & (\mathbb{C}^\times)^n \arrow[ll, dashed, "\mu_k"] \end{tikzcd}$$

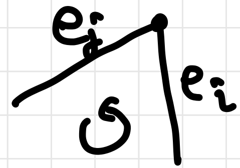
$$X_i := z^{e_i}, \quad X_i' := z^{\mu_k(e_i)}$$

$$\mu_k^* X_i' := \begin{cases} X_i \left(1 + X_k^{-\operatorname{sgn}(\{e_i, e_k\})}\right)^{-\{e_i, e_k\}} & (i \neq k) \\ X_k^{-1} & (i = k) \end{cases}$$

$\rightsquigarrow$ Gluing up them : $\mathcal{A} \xrightarrow{p} \mathcal{X}$

$$p^* X_i = \prod_j A_j^{\{e_i, e_j\}}$$

For surfaces, I = edges.



$\{e_i, e_j\} = +1.$

$\rightsquigarrow$ recovers the Teichmüller case.

<u>One more important actor</u> : $\mathcal{A}_{\mathrm{prin}}$

(a deformation of $\mathcal{A}$-variety)

Linear algebra data

$N \oplus M$, skew-symform

$$\{(n_1,m_1),(n_2,m_2)\} := \{n_1,n_2\} + \langle n_1, m_2\rangle - \langle n_2, m_1\rangle$$

$\rightsquigarrow$ Define $\mathcal{A}$-variety using this data but mutations only for N.

Coordinate: $A_1^p, A_2^p, \ldots, A_n^p, \underbrace{x_1, \ldots, x_n}_{\text{not mutated}}$

$\rightsquigarrow$

$$\begin{tikzcd}
\mathcal{A}_{\mathrm{prin}} \arrow[d] \arrow[r, hookleftarrow] & \mathcal{A} \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\
T_M \arrow[r, hookleftarrow] & \{1\}
\end{tikzcd}$$

$\mathcal{A}_{\mathrm{prin}}$ is important in the cluster theory,
(e.g. Gross-Hacking-Keel-Kontsevich proof of the conjectures.)
but geometric meaning is not clear.

__Our picture__

$\boxed{\text{Cluster}}$

$$
\begin{tikzcd}
T_M & \mathcal{A}_{\mathrm{prin}} \arrow[l] \arrow[r] & \mathcal{X} \\
1 \arrow[u, hook] & \mathcal{A} \arrow[l] \arrow[u, hook] \arrow[ur] &
\end{tikzcd}
$$

$\boxed{\text{Teichmüller}}$ work in progress.

$$
\begin{tikzcd}
T_M & \text{moduli of decorated SQs} \arrow[l] \arrow[r] & \mathcal{M}^{\mathrm{framed}}_{PSL_2}(S) \\
1 \arrow[u, hook] & \mathcal{M}^{\mathrm{dec,tw}}_{SL_2}(S) \arrow[l] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Before going to SQs...

<u>Framed local systems as constructible sheaves.</u>

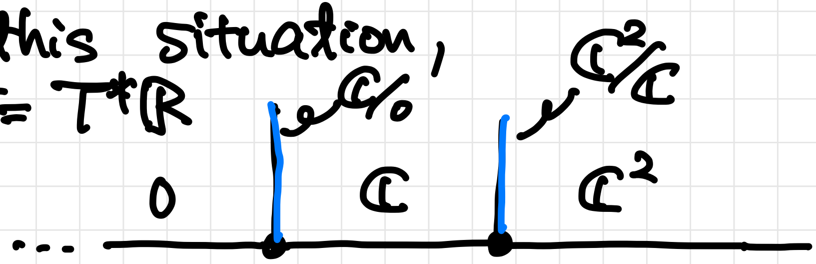
a Constructible sheaf: "a precewise local system"

<u>e.g.</u> . $\mathbb{C} \leftarrow \mathbb{C}^2 \rightarrow \mathbb{C}^3 \leftarrow \mathbb{C}^4 \rightarrow \mathbb{C}^2$

We are interested in a particular class of Constructible sheaves

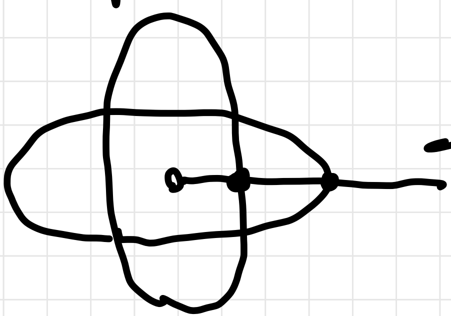
On $\mathbb{R}_{>0}$ $\quad 0 = 0 \hookrightarrow \mathbb{C} = \mathbb{C} \hookrightarrow \mathbb{C}^2$ $\quad$ Expressing a flag.

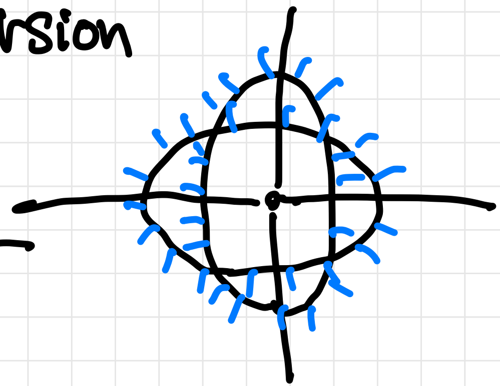
For this situation, $\mathbb{R}^2 = T^*\mathbb{R}$

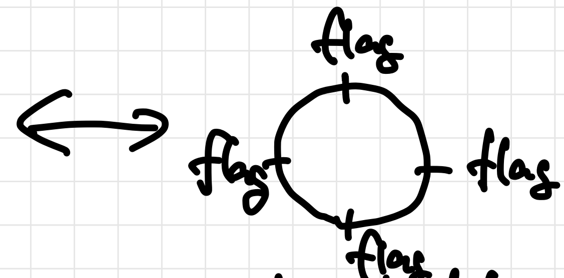


Microsupport & Microstalk in microlocal sheaf theory

"Radar" version



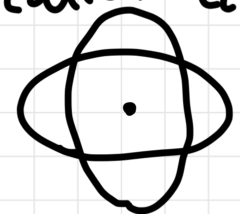




marked boundary

framed local system

= Constructible sheaf with stratification like



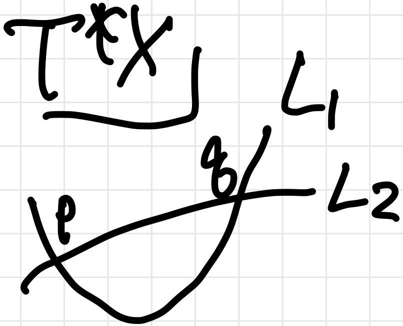
around marked points.

A philosophy from microlocal sheaf theory

Constructible sheave should be studied using their microsupport Lagrangian & "sheaves" on them.

$\omega = d\lambda$
$\lambda|_L = \text{exact}$

A realization



Nadler–Zaslow equivalence

sheaf $\longmapsto$ microsupport
exact

$$\text{Constructible sheaves} \simeq \mathrm{Fuk}(T^*X)$$

$\cup$
Lag brane $\{\text{Lag} + \text{local system}\}$

$$\mathrm{Hom}(L_1, L_2) \simeq \mathbb{C}\cdot p \oplus \mathbb{C}_q$$

i.e.,

$$\mathcal{M}^{\text{framed}}(S) = \text{A moduli of Lagrangian branes.}$$

The Lagrangians appeared here are very degenerate.

$\exists$ A way to treat smooth Lagrangians sheaf theoretically.

Ansatz

Sheaf quantization $\ni SQ$ ←- - - - -→ (?) $Fuk^{\text{nonexact}}(T^*X) \ni (L, \text{local system})$

Using this analogy:

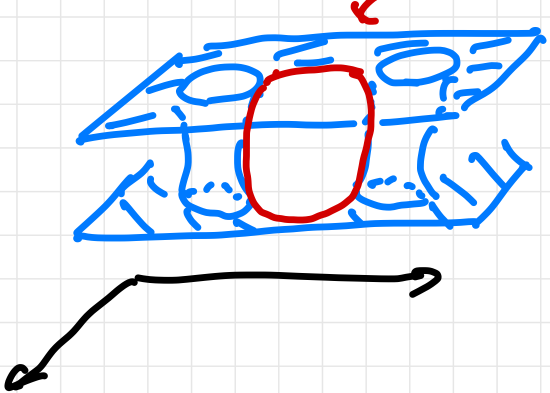
T_M ← $e^{\int_\gamma \lambda}$ ← period — Moduli of ^decorated Lagrangian branes. $\xrightarrow{\text{Conic degeneration}} \mathcal{M}^{\text{framed}}_{SL_2}$

$\uparrow$ $\uparrow$ Conic limit boundary

1 ← $\mathcal{M}^{\text{dec.tw}}_{SL_2}$

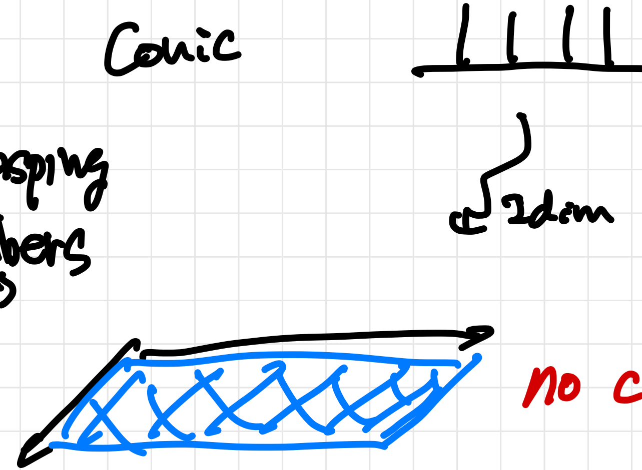
1 $\mathbb{C}^\times_\hbar$ $\hbar \to 0$ $\hbar \to \infty$

non-conic period



Collapsing fibers

Conic



no cycles.

Relation to Iwaki–Nakanishi

Schrödinge equation $\underline{((\hbar\partial)^2 - Q)}\psi = 0$ (+ decoration)

[k] $\rightsquigarrow$ An SQ of the spec curve.

$$\{\xi^2 - Q = 0\} \subset T^*X$$

diff eqn $\rightsquigarrow$ sol $\rightsquigarrow$ local system

Riemann-Hilbert: Constructible sheaf

$\hbar$-diff eqn $\rightsquigarrow$ "solution sheaf"

K + Iwaki-Nakanishi
Iwaki, Allegretti ⇝ The corresponding local system
is Iwaki-Nakanishi's Voros symbol
= Fock-Goncharov coordinate
of solution framed local system
of $((\hbar\partial)^2 - Q)\psi = 0$

Our future goal

Recognize Iwaki-Nakanishi as a "monodromy map".

(Allegretti + Bridgeland : Quadratic diffs $\xrightarrow[\text{depending on } \hbar]{}$ moduli of framed loc system $\to$ cluster variety.

$$\begin{tikzcd}
 & & \mathbb{C}_\hbar^\times \cup \{\infty\} \arrow[dll, bend right] \arrow[dll, bend right=40] \arrow[dl] \arrow[d, "AB"] \\
\frac{1}{\hbar} \overline{\ \ } \, \mathcal{T}_M \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & SQs \arrow[l] \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathcal{M}^{\text{framed}}(S) \\
\left\{ e^{-\left(\int_{\gamma_i} \frac{\lambda}{\hbar}\right)} \right\} & L \arrow[l, mapsto] &
\end{tikzcd}$$

Scaling of $\hbar$ $\Longleftrightarrow$ Scaling of symplectic form

$\Longleftrightarrow$ Scaling of cotangent fibers

$\lambda = \sum \xi_i dx_i$

$\xrightarrow[\text{limit}]{}$ Conic Lagrangians.

If time permits, I'll describe SQ a bit more precisely.

$$X \rightsquigarrow X \times \mathbb{R}_t$$

base manifold

$$SQ \in \left\{ \begin{array}{c} \text{Sheaves} \\ \text{on } X \times \mathbb{R}_t \end{array} \right\}$$

<u>Conditions</u>.

(1) microsupport of SQ

$$\{\tau \geq 0\} \subset T^*(X \times \mathbb{R}_t)$$

$$= T^*X \times \mathbb{R}_t \times \mathbb{R}_\tau$$

($\tau = 0$ is quotiented out.)

In this sense, essentially, microsupport of $SQ \subset \{\tau > 0\}$

(2) $\underset{\mathcal{G}}{L} = \rho(\text{microsupp of } SQ \cap \{\tau > 0\})$

the Lagrangian corresponding to SQ

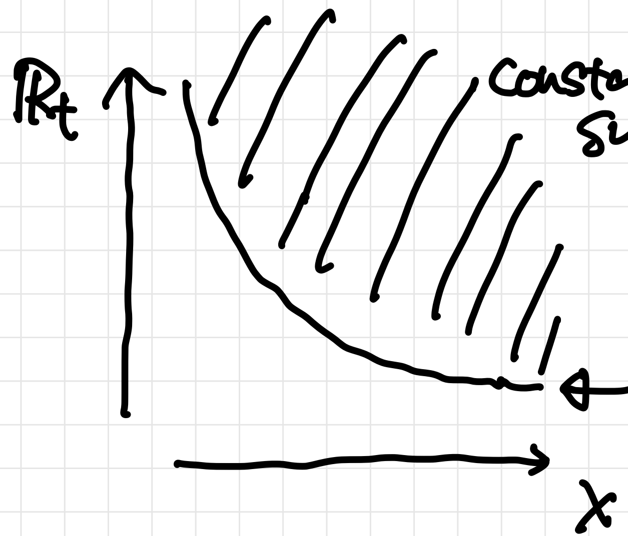
$$\rho : T^*X \times \mathbb{R}_t \times \{\tau > 0\} \to T^*X$$

$$(x, \xi, t, \tau) \longmapsto (x, \xi/\tau).$$

is Lagrangian.

[K]: $((\hbar\partial)^2 - Q)\psi = 0$ + exact WKB analysis $\longleftrightarrow$ $\boxed{SQ \text{ s.t. } \rho(\mathrm{msupp}(SQ)\cap\{\tau>0\}) = L}$

$\hbar$: fix



constant sheaf supp'd on this region

Only locally meaningful

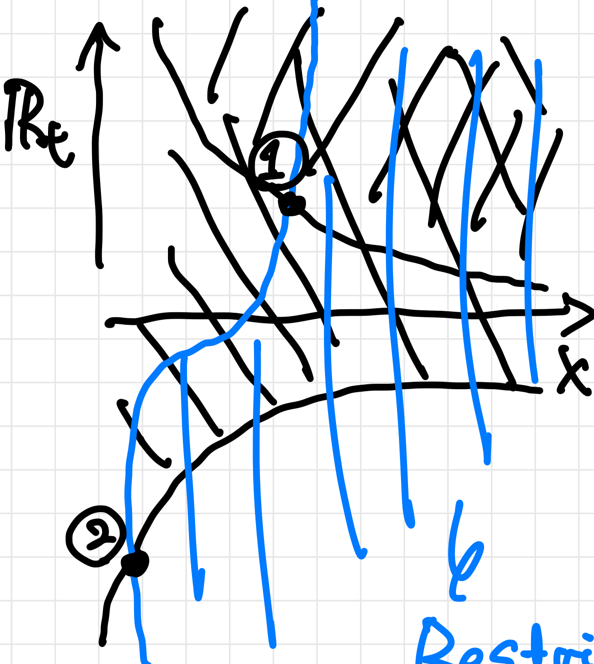
$\pm\mathrm{Re}\int \frac{\sqrt{Q}}{\hbar} = \pm\mathrm{Re}\int \frac{\lambda|_L}{\hbar}$

WKB analysis provides connection formula of
very special class of solutions.

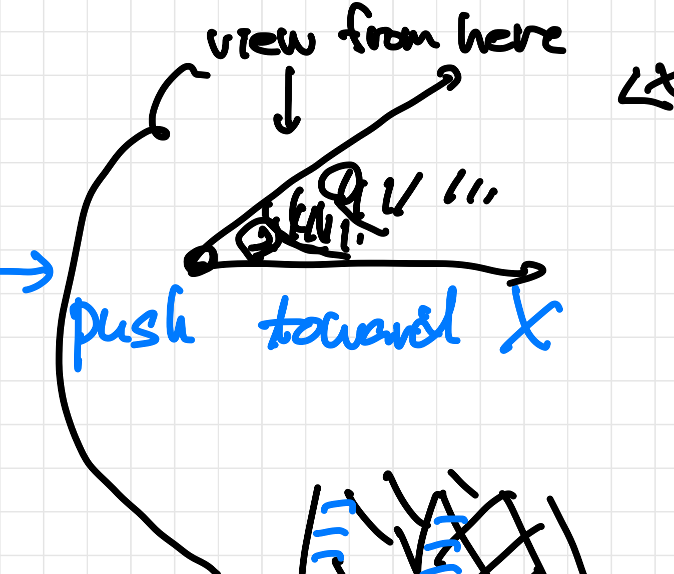
$\hookrightarrow$ This provides the gluing of
the above local sheaves.

(**Rmk.** $\exists$ functorial construction
(in prep. [K])).

$SQ \longmapsto$ framed local system.

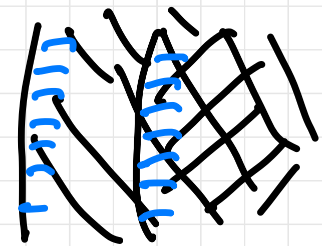


$\longmapsto$



push toward X

Restrict to this region



Started from $\mathbb{C}$,
but end at Novikov ring $\rightsquigarrow$ have to be specialized over $\mathbb{C}$

Rmk More precisely,
need to change the coefficient.

Constructible sheaf on X $\xmapsto[\text{cook up}]{}$ SQ

$\cong$

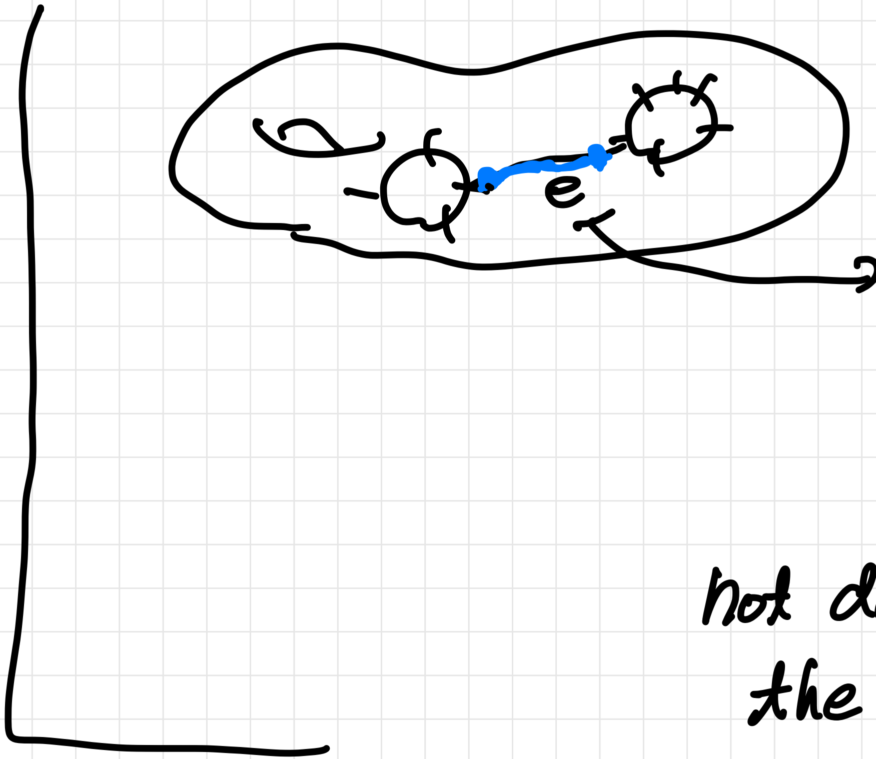
very degenerate

$\mathcal{E}$ on X $\longmapsto$ $\mathcal{E} \boxtimes \mathbb{C}_{\{t \geq 0\}}$

$p(\text{microsupp}(\mathcal{E} \boxtimes \mathbb{C}_{\{t \geq 0\}}) \cap \tau > 0))$

$=$ microsupp of $\mathcal{E}$.

Decoration (a very small new idea)



→ path Voros symbol as an integration over e

not directly related to the local system on L