Intervals of s-torsion pairs in extriangulated categories with negative first extensions

Joint work with Adachi and Enomoto (Tsukamoto, 2021.4.23) Mods

§1. Intro.

Recall $\mathcal{A}$: ab. cat. (exact) $\quad \mathcal{D}$: triang. cat. with Σ

$(\mathcal{X},\mathcal{Y})$: torsion pair in $\mathcal{A}$ (t-structure on $\mathcal{D}$)

:$\Leftrightarrow$ · $\forall M \in \mathcal{A}$ ($\mathcal{D}$), $\exists\, 0 \to X \to M \to Y \to 0$: ex. ($X \in \mathcal{X}$, $Y \in \mathcal{Y}$)

$X \to M \to Y \to \Sigma X$: triang

· $\mathcal{A}(\mathcal{X},\mathcal{Y})=0$ ($\mathcal{D}$)

· $\Sigma\mathcal{X} \subseteq \mathcal{X} \Leftrightarrow \mathrm{Ext}^1(\mathcal{X},\mathcal{Y})=0$

① Introduce "s-torsion pairs" in an extriang. cat with a negative first extension

as a com. gen. of $\begin{cases}\text{tors. pair} \\ t\text{-str.}\end{cases}$

ThmA [Happel-Reiten-Smalø'96, Woolf'10]

$(\mathcal{U},\mathcal{V}) \in t\text{-str}\,\mathcal{D}$ $\quad \mathcal{H} := \mathcal{U} \cap \Sigma\mathcal{V}$: the heart

$\Rightarrow \exists$ bij:

$\{(\mathcal{U}',\mathcal{V}') \in t\text{-str}\,\mathcal{D} \mid \Sigma\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{U}\} \rightleftarrows \mathrm{tors}\,\mathcal{H}$

ThmB [Asai-Pfeifer'19, Tattar'21]

$(\mathcal{T}_1,\mathcal{F}_1),(\mathcal{T}_2,\mathcal{F}_2) \in \mathrm{tors}\,\mathcal{A}$ s.t. $\mathcal{T}_1 \subseteq \mathcal{T}_2$

$\mathcal{H} := \mathcal{T}_2 \cap \mathcal{F}_1$

$\Rightarrow \exists$ bij:

$\{(\mathcal{T},\mathcal{F}) \in \mathrm{tors}\,\mathcal{A} \mid \mathcal{T}_1 \subseteq \mathcal{T} \subseteq \mathcal{T}_2\} \rightleftarrows \mathrm{tors}\,\mathcal{H}$

② Give a bij of s-torsion pairs.

ThmA $\quad$ ThmB

§2. Extriangulated cats

- $\mathscr{C} = (\underset{\text{add. cat.}}{\mathscr{C}}, \underset{\text{add. bifunc.}}{\mathbb{E}}, \underset{\text{"realization"}}{\mathfrak{s}})$

$\mathbb{E}: \mathscr{C}^{op} \times \mathscr{C} \to Ab$

: extriangulated cat if it satisfies certain axioms (See, [Nakaoka-Palu'19])

For $\forall \delta \in \mathbb{E}(C, A)$

$\mathfrak{s}(\delta) = [A \to B \to C]$: equiv. class of seqs of morphisms

$A \xrightarrow{f} B \xrightarrow{g} C$ and $A \xrightarrow{f'} B' \xrightarrow{g'} C$ are equiv.

$$:\Longleftrightarrow \begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, "\exists b", "\cong"'] & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & B' \arrow[r, "g'"'] & C \end{tikzcd}$$

- $A \xrightarrow{f} B \xrightarrow{g} C$: $\mathfrak{s}$-conflation

$:\Longleftrightarrow \exists \delta \in \mathbb{E}(C, A)$ s.t. $\mathfrak{s}(\delta) = [A \to B \to C]$

In this case, we write $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$

Ex. (1) $\mathcal{D}$: triang. cat.

- $\mathbb{E}(C, A) := \mathcal{D}(C, \Sigma A)$
- $\mathfrak{s}(\delta) := [A \to B \to C] \xrightarrow{\delta} \Sigma A$

$\delta \in \mathcal{D}(C, \Sigma A)$

(2) $\mathcal{E}$: ex. (ab). cat.

- $\mathbb{E}(C, A) := \{\text{conflations } 0 \to A \to B \to C \to 0\} / \simeq$
- $\mathfrak{s} = \mathrm{Id}$

Notation
- $\mathscr{C} := (\mathscr{C}, \mathbb{E}, \mathfrak{s})$: extriang. cat.
- subcat = full & cl. under $\simeq$
- $\mathcal{X}, \mathcal{Y} \leq \mathscr{C}$

$\mathcal{X} * \mathcal{Y} := \{ M \in \mathscr{C} \mid \exists\, \underset{\mathcal{X}}{\underset{\cap}{X}} \to M \to \underset{\mathcal{Y}}{\underset{\cap}{Y}} \dashrightarrow : \mathfrak{s}\text{-conf} \}$

Note $*$: associative

Def.2.1

A negative first extension on $\mathcal{C}$ consists of the following data.:

(NE1) $\mathbb{E}^{-1}: \mathcal{C}^{op} \times \mathcal{C} \to Ab$: add. bifunc.

(NE2) $\forall \delta \in \mathbb{E}(C,A)$, $\exists$ nat. trans:

$\delta_\sharp : \mathbb{E}^{-1}(-,C) \to \mathcal{C}(-,A)$

$\delta^\sharp : \mathbb{E}^{-1}(A,-) \to \mathcal{C}(C,-)$

s.t. $A \to B \to C \overset{\delta}{\dashrightarrow}$: $\mathfrak{s}$-confl. & $W \in \mathcal{C}$

$\mathbb{E}^{-1}(W,A) \to \mathbb{E}^{-1}(W,B) \to \mathbb{E}^{-1}(W,C)$

$\xrightarrow{(\delta_\sharp)_W} \mathcal{C}(W,A) \to \mathcal{C}(W,B)$ &

$\mathbb{E}^{-1}(C,W) \to \mathbb{E}^{-1}(B,W) \to \mathbb{E}^{-1}(A,W)$

$\xrightarrow{(\delta^\sharp)_W} \mathcal{C}(C,W) \to \mathcal{C}(B,W)$ are ex.

Cf.

$\mathbb{E}: \mathcal{C}^{op} \times \mathcal{C} \to Ab$

$\rightsquigarrow$ $\forall \delta \in \mathbb{E}(C,A)$, $\exists$ nat. trans.

$\delta_\sharp : \mathcal{C}(-,C) \to \mathbb{E}(-,A)$

$\delta^\sharp : \mathcal{C}(A,-) \to \mathbb{E}(C,-)$ by Yoneda lem.

For $A \to B \to C \overset{\delta}{\dashrightarrow}$: $\mathfrak{s}$-confl.,

$\mathcal{C}(W,A) \to \mathcal{C}(W,B) \to \mathcal{C}(W,C) \xrightarrow{(\delta_\sharp)_W} \mathbb{E}(W,A)$

$\to \mathbb{E}(W,B) \to \mathbb{E}(W,C)$ &

$\mathcal{C}(C,W) \to \mathcal{C}(B,W) \to \mathcal{C}(A,W) \xrightarrow{(\delta^\sharp)_W} \mathbb{E}(C,W)$

$\to \mathbb{E}(B,W) \to \mathbb{E}(A,W)$ are ex.

Rem2.2 $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$: extriang. cat. with $\mathbb{E}^{-1}$, $\mathcal{C}' \subseteq \mathcal{C}$: subcat.

$\mathcal{C}'$: extension-closed (i.e., $\mathcal{C}' * \mathcal{C}' \subseteq \mathcal{C}'$) $\Rightarrow$ $(\mathcal{C}', \mathbb{E}|_{\mathcal{C}'}, \mathfrak{s}|_{\mathcal{C}'})$: extriang. cat. with $\mathbb{E}^{-1}|_{\mathcal{C}'}$

Ex2.3 (1) $\mathcal{D}$: triang cat.

- $\mathbb{E}^{-1}(C,A) := \mathcal{D}(C,\Sigma^{-1}A)$
- $\forall A\to B\to C \overset{\delta}{\dashrightarrow}$: $\mathfrak{s}$-confl.

$(\delta_\sharp^{-1})_W : \mathbb{E}^{-1}(W,C)=\mathcal{D}(W,\Sigma^{-1}C) \xrightarrow{\mathcal{D}(W,\Sigma^{-1}\delta)} \mathcal{D}(W,A)$

$(\delta^\sharp_{-1})_W : \mathbb{E}^{-1}(A,W)=\mathcal{D}(A,\Sigma^{-1}W)\simeq \mathcal{D}(\Sigma A,W) \xrightarrow{\mathcal{D}(\delta,W)} \mathcal{D}(C,W)$

(2) $\mathcal{E}$: ex. cat

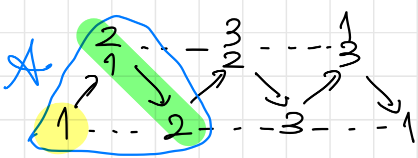
- $\mathbb{E}^{-1}(C,A)=0$
- $(\delta_\sharp^{-1})_W = 0 = (\delta^\sharp_{-1})_W$

Rem2.4 Neg. first exts are not unique.

eg. $\Lambda := K\left(\begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "x"] & \\ \cdot \arrow[ur, "x"] & & \cdot \arrow[ll, "x"] \end{tikzcd}\right)/x^3$, $\mathcal{D} := \underline{\mathrm{mod}}\Lambda$: stable cat

$\Rightarrow \mathcal{A}$: extriang. cat. with $\left\{ \begin{array}{l} \mathbb{E}^{-1}=\mathcal{D}(-,\Sigma^{-1})|_{\mathcal{A}} \\ \neq 0 \\ \mathbb{E}'^{-1}=0 \end{array} \right.$

Rem2.2



$\mathbb{E}^{-1}(2,1)=\mathcal{D}(2,\underbrace{\Sigma^{-1}1}_{=\,\frac{3}{2}})\neq 0$

$1 \xrightarrow{f} \frac{2}{1} \xrightarrow{g} 2 \dashrightarrow$: $\mathfrak{s}$-confl., $W\in\mathcal{A}$

$0 \to \mathcal{A}(W,1) \xrightarrow{\mathcal{A}(W,f)} \mathcal{A}(W,\frac{2}{1})$

§3. Main result

$\mathcal{C}$: extriang. cat. with $\mathbb{E}^{-1}$.

Def.3.1 $\mathcal{T},\mathcal{F}\subseteq\mathcal{C}$: subcats

$(\mathcal{T},\mathcal{F})$: s-torsion pair

:⇔ (STP1) $\mathcal{C}=\mathcal{T}*\mathcal{F}$

i.e. $\forall C\in\mathcal{C}$, $\exists!\ \underset{\mathcal{T}}{T}\to C\to \underset{\mathcal{F}}{F} \dashrightarrow$: $\mathfrak{s}$-confl. (up to isom of $\mathfrak{s}$-confls.)

(STP2) $\mathcal{C}(\mathcal{T},\mathcal{F})=0$

(STP3) $\mathbb{E}^{-1}(\mathcal{T},\mathcal{F})=0$

Ex3.2 $(\mathrm{add}\{\frac{2}{1},2\}, \mathrm{add}\{1\})$ in $\mathcal{A}$ satisfies (STP1) & (STP2), but does not satisfy (STP3) ☹ $\mathbb{E}^{-1}(2,1)\neq 0$

Rem3.3 • stors$\mathcal{C}$: the set of s-torsion pairs in $\mathcal{C}$

$(\mathcal{T}_1,\mathcal{F}_1)\le(\mathcal{T}_2,\mathcal{F}_2) :\Leftrightarrow \mathcal{T}_1\subseteq\mathcal{T}_2 \Leftrightarrow \mathcal{F}_1\supseteq\mathcal{F}_2$

⇝ (stors$\mathcal{C}$, $\le$) : poset

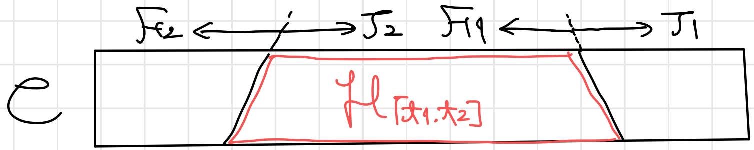
- $(\mathcal{T},\mathcal{F})\in$ stors$\mathcal{C}$ $\Rightarrow$ $\mathcal{F}=\mathcal{T}^\perp$, $\mathcal{T}={}^\perp\mathcal{F}$

$\mathcal{T}^\perp = \{X\in\mathcal{C} \mid \mathcal{C}(\mathcal{T},X)=0\}$

$t_1 := (\mathcal{T}_1, \mathcal{F}_1) \le t_2 := (\mathcal{T}_2, \mathcal{F}_2) \in \mathrm{stors}\,\mathcal{C}$

Def. 3.4

- $\mathrm{stors}[t_1, t_2] := \{ t \in \mathrm{stors}\,\mathcal{C} \mid t_1 \le t \le t_2 \}$: interval in $\mathrm{stors}\,\mathcal{C}$
- $\mathcal{H}_{[t_1,t_2]} := \mathcal{T}_2 \cap \mathcal{F}_1$: the heart of $\mathrm{stors}[t_1, t_2]$



Rem 3.5 $\mathcal{H}_{[t_1,t_2]}$: extriang. cat. with $\mathbb{E}'|_{\mathcal{H}_{[t_1,t_2]}}$

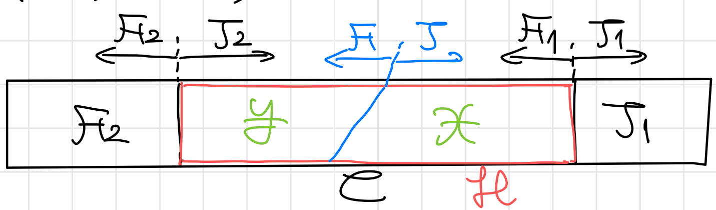
Thm 3.6 [Adachi–Enomoto–T]

$\exists$ bijs (as posets)

$$\mathrm{stors}[t_1,t_2] \underset{\Psi}{\overset{\Phi}{\underset{\sim}{\rightleftarrows}}} \mathrm{stors}\,\mathcal{H}_{[t_1,t_2]} =: \mathcal{H}$$

$$(\mathcal{T}, \mathcal{F}) \longmapsto (\mathcal{T} \cap \mathcal{F}_1, \mathcal{T}_2 \cap \mathcal{F})$$

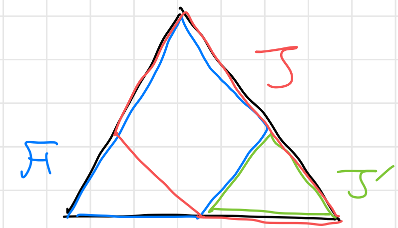
$$(\mathcal{T}_1 * \mathcal{X}, \mathcal{Y} * \mathcal{F}_2) \longleftarrow\!\!| \ (\mathcal{X}, \mathcal{Y})$$



Lem 3.7 $(\mathcal{T}', \mathcal{F}') \le (\mathcal{T}, \mathcal{F}) \in \mathrm{stors}\,\mathcal{C}$

(1) $\mathcal{T} = \mathcal{T}' * \underbrace{(\mathcal{T} \cap \mathcal{F}')}_{\text{heart}}$ (2) $\mathcal{F}' = \underbrace{(\mathcal{T} \cap \mathcal{F}')}_{} * \mathcal{F}$

($\subseteq$) Use (STP3)



Sketch of proof

- Φ : well-defined.

Let $(\mathcal{T}, \mathcal{F}) \in \mathrm{stors}[t_1, t_2]$.

Then $\mathcal{T} \cap \mathcal{F}_1,\ \mathcal{T}_2 \cap \mathcal{F} \subseteq \mathcal{H}$

(STP1) $\mathcal{H} = (\mathcal{T} \cap \mathcal{F}_1) * (\mathcal{T}_2 \cap \mathcal{F})$

($\subseteq$) $\mathcal{H} = \mathcal{T}_2 \cap \mathcal{F}_1 = \underbrace{(\mathcal{T} * (\mathcal{T}_2 \cap \mathcal{F}))}_{\text{Lem3.7}} \cap \underbrace{((\mathcal{T} \cap \mathcal{F}_1) * \mathcal{F})}_{}$

$\mathcal{T} \cap \mathcal{F}_1 \qquad \mathcal{T}_2 \cap \mathcal{F}$

$\ \ \cup\!\!\!\downarrow \qquad\qquad \cup\!\!\!\downarrow$

Let $H \in \mathcal{H}$

By (STP1) for $(\mathcal{T}, \mathcal{F})$, $\exists!\ T \to H \to F \dashrightarrow$: s.conf. //

(STP2) $\mathcal{C}(\mathcal{T} \cap \mathcal{F}_1, \mathcal{T}_2 \cap \mathcal{F}) = 0$

(STP3) $\mathbb{E}'$ $\qquad \overset{\cap|}{\mathcal{T}} \qquad \overset{\cap|}{\mathcal{F}}$ //

· Φ: well-defined

Let $(\mathcal{X},\mathcal{Y}) \in \mathrm{stors}\,\mathcal{H}$

(STP1) $\mathcal{C} = (\mathcal{T}_1 * \mathcal{X}) * (\mathcal{Y} * \mathcal{F}_2)$

$$(\mathrm{RHS}) = \mathcal{T}_1 * \underbrace{(\mathcal{X} * \mathcal{Y})}_{\overset{\shortparallel}{\mathcal{H}}\ \text{by (STP1)}} * \mathcal{F}_2 = (\mathcal{T}_1 * \mathcal{H}) * \mathcal{F}_2 = \mathcal{T}_2 * \mathcal{F}_2 \underset{\text{Lem3.7(1)}}{=} \mathcal{C}$$

(STP2) $\mathcal{C}(\mathcal{T}_1 * \mathcal{X}, \mathcal{Y} * \mathcal{F}_2) = 0$

(STP3) $\mathbb{E}^{-1}$

· $\underset{\mathbb{E}^{-1}}{\mathcal{C}}(\mathcal{T}_1, \underbrace{\mathcal{Y} * \mathcal{F}_2}_{\overset{\cap}{\mathcal{F}_1}}) = 0$

· $\underset{\mathbb{E}^{-1}}{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = 0$, $\underset{\mathbb{E}^{-1}}{\mathcal{C}}(\mathcal{X}, \underset{\mathcal{T}_2}{\overset{\cap}{\mathcal{F}_2}}) = 0 \Rightarrow \underset{\mathbb{E}^{-1}}{\mathcal{C}}(\mathcal{X}, \mathcal{Y} * \mathcal{F}_2) = 0$ by (NE2)

☺ $M \in \mathcal{Y} * \mathcal{F}_2$

$\Rightarrow \exists\, \underset{\mathcal{Y}}{\overset{\in}{Y}} \to M \to \underset{\mathcal{F}_2}{\overset{\in}{F}} \dashrightarrow$ s-confl

$\therefore \mathcal{C}(\mathcal{X}, Y) \to \mathcal{C}(\mathcal{X}, M) \to \mathcal{C}(\mathcal{X}, F)$: ex by (NE2)

$\mathbb{E}^{-1}$: $0 \Longrightarrow \therefore 0 \Longleftarrow 0$ //

· Thm3.6 gives a gen. of ThmA & ThmB.

Cor3.8 (ThmA)

$\mathcal{D}$: triang. cat $\rightsquigarrow$ extriang cat. with $\mathbb{E}^{-1} = \mathcal{D}(-, \Sigma^{-1}-)$

$t_1 := (\underset{\Sigma\mathcal{U}_2}{\mathcal{U}_1}, \underset{\Sigma\mathcal{V}_2}{\mathcal{V}_1}) \le t_2 := \boxed{(\mathcal{U}_2, \mathcal{V}_2) \in t\text{-str}\,\mathcal{D}}$, $\mathcal{H} := \mathcal{U}_2 \cap \mathcal{V}_1$

s.t. $\Sigma\mathcal{U}_2 \le \mathcal{U}_1 \Rightarrow$ exact cat & $\mathbb{E}^{-1}|_{\mathcal{H}} = 0$

$\Rightarrow \exists$ bij :

$$\{(\mathcal{U}, \mathcal{V}) \in t\text{-str}\,\mathcal{D} \mid \overset{\Sigma\mathcal{U}_2}{\mathcal{U}_1} \le \mathcal{U} \le \mathcal{U}_2\} \rightleftarrows \mathrm{stors}\,\mathcal{H} = \mathrm{tors}\,\mathcal{H}$$

☺ $\mathrm{stors}[t_1, t_2] \underset{\text{Thm3.6}}{\overset{\sim}{\rightleftarrows}} \mathrm{stors}\,\mathcal{H}$ //

Cor3.9 (ThmB)

$\mathcal{E}$: ex. cat. ab $t_1 := (\mathcal{T}_1, \mathcal{F}_1) \le t_2 := (\mathcal{T}_2, \mathcal{F}_2) \in \mathrm{tors}\,\mathcal{E}$

$\rightsquigarrow$ extriang. cat. with $\mathbb{E}^{-1} = 0$ $\mathcal{H} := \mathcal{T}_2 \cap \mathcal{F}_1$

$\Rightarrow \exists$ bij :

$$\{(\mathcal{T}, \mathcal{F}) \in \mathrm{tors}\,\mathcal{E} \mid \mathcal{T}_1 \le \mathcal{T} \le \mathcal{T}_2\} \overset{\sim}{\rightleftarrows} \mathrm{tors}\,\mathcal{H}$$

☺ $\mathrm{stors}[t_1, t_2] \underset{\text{Thm3.6}}{\overset{\sim}{\rightleftarrows}} \mathrm{stors}\,\mathcal{H}$ //