Schubert calculus from polyhedral parametrizations of Demazure crystals

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Schubert calculus for G/B

- G: a connected, simply-connected semisimple algebraic group over $\mathbb{C}$,
- $B \subseteq G$: a Borel subgroup,
- B^-: an opposite Borel subgroup.

Let

$$G/B = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} B^- wB/B$$

be the orbit decompositions of G/B, where W is the Weyl group.

Definition

For $w \in W$, the **Schubert variety** X_w and the **opposite Schubert variety** X^w are defined by

- $X_w := \overline{BwB/B} \subseteq G/B$,
- $X^w := \overline{B^- wB/B} \subseteq G/B$.

These are irreducible normal projective varieties.

Schubert calculus for G/B

Let $w_0 \in W$ denote the longest element. Then we have $[X^w] = [X_{w_0 w}]$ in $H^*(G/B;\mathbb{Z})$.

Properties

- $H^*(G/B;\mathbb{Z}) = \sum_{w\in W} \mathbb{Z}[X_w] = \sum_{w\in W} \mathbb{Z}[X^w]$,
- $[X^u]\cdot[X^v] = \sum_{w\in W} c_{u,v}^w [X^w]$ for some $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$.

Aim (of Schubert calculus)

to compute $c_{u,v}^w$ explicitly.

Approach

to realize $[X^w]$ as a concrete combinatorial object.

If $G = SL_{n+1}(\mathbb{C})$, then $[X^w]$ for $w \in W \simeq S_{n+1}$ is represented as a specific polynomial $\mathfrak{S}_w$, called a **Schubert polynomial**.

Schubert polynomials

For $1 \leq i \leq n$ and $f \in \mathbb{Z}[x_1, \ldots, x_{n+1}]$, the **divided difference operator** $\partial_i f$ is defined as follows:

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}},$$

where $s_i := (i\ i+1) \in S_{n+1}$ acts on f as a permutation of variables.

Definition (Lascoux–Schützenberger 1982)

For $w \in S_{n+1}$, the **Schubert polynomial** $\mathfrak{S}_w \in \mathbb{Z}[x_1, \ldots, x_{n+1}]$ is defined by

$$\mathfrak{S}_w := \partial_{i_1} \cdots \partial_{i_r}(x_1^n x_2^{n-1} \cdots x_n),$$

where $(i_1, \ldots, i_r)$ is a reduced word for $w^{-1}w_0 \in S_{n+1}$.

Under the Borel presentation $H^*(GL_{n+1}(\mathbb{C})/B; \mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_{n+1}]/J$, we have $[X^w] = \mathfrak{S}_w + J$ for all $w \in S_{n+1}$.

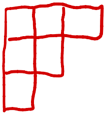
Reduced pipe dreams

- $Y_n := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n,\ 1 \leq j \leq n-i+1\}$ whose elements are arranged as
$$((n,1),(n-1,1),(n-1,2),(n-2,1),\ldots,(1,n-1),(1,n)),$$
- $\boldsymbol{i}_A = (i_1,\ldots,i_N) := (1,2,1,3,2,1,\ldots,n,n-1,\ldots,1)$.

Let $\mathcal{PD}_n$ denote the set of subsets of Y_n, and associate to each $D \in \mathcal{PD}_n$ a sequence $\boldsymbol{k}_D = (k_1,\ldots,k_{|D|})$ with $1 \leq k_1 < \cdots < k_{|D|} \leq N$ by arranging $1 \leq k \leq N$ such that the k-th element of Y_n is included in D. Then we set $w(D) := w_0 s_{i_{k_1}} \cdots s_{i_{k_{|D|}}} w_0$.

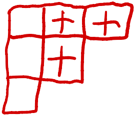
e.g.

$D = \{(1,2),(1,3),(2,2)\} \in \mathcal{PD}_3$

$Y_3 =$ 

4 5 6
2 3
1

||



$\rightsquigarrow k_D = (3,5,6)$

$w(D) = w_0 s_1 s_2 s_1 w_0$

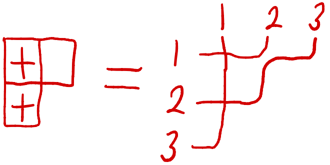
Reduced pipe dreams

Definition (Knutson–Miller 2005)

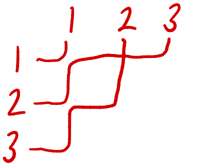
- An element of $\mathcal{PD}_n$ is called a **pipe dream**.
- A pipe dream $D \in \mathcal{PD}_n$ is **reduced** if $(i_{k_1}, \ldots, i_{k_{|D|}})$ is a reduced word, where $\boldsymbol{k}_D = (k_1, \ldots, k_{|D|})$.
- $RP(w) \coloneqq \{$reduced pipe dreams D with $w(D) = w^{-1}\}$.

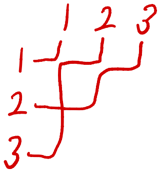
e.g. $\mathcal{PD}_2$ $(W \simeq S_3)$

$RP(s_1 s_2)$ $\quad s_1 s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$



$RP(s_2)$





Reduced pipe dreams

Theorem (Billey–Jockusch–Stanley 1993 and Fomin–Stanley 1994)

For $w \in S_{n+1}$, the following equality holds:

$$\mathfrak{S}_w = \sum_{D \in RP(w)} \boldsymbol{x}^D.$$

e.g. $n = 3$

x_1	x_1	x_1
x_2	x_2	
x_3		

e.g. $n = 2$

x_1	x_1
x_2	

$$\mathfrak{S}_{s_1 s_2} = x = x_1 x_2$$

$$\mathfrak{S}_{s_2} = x + x = x_1 + x_2$$

Gelfand–Tsetlin polytopes

- P_+: the set of dominant integral weights,
- P_{++}: the set of regular dominant integral weights.

For $\lambda \in P_+$, the **Gelfand–Tsetlin polytope** $GT(\lambda)$ is defined to be the set of $(a_1^{(1)}, a_1^{(2)}, a_2^{(1)}, a_1^{(3)}, a_2^{(2)}, a_3^{(1)}, \ldots, a_1^{(n)}, \ldots, a_n^{(1)}) \in \mathbb{R}^N$ satisfying the following inequalities:

$$\begin{array}{ccccccccc}
a_1^{(0)} & & a_2^{(0)} & & \cdots & & a_n^{(0)} & & a_{n+1}^{(0)} \\
& a_1^{(1)} & & a_2^{(1)} & & \cdots & & a_n^{(1)} & \\
& & a_1^{(2)} & & \cdots & & a_{n-1}^{(2)} & & \\
& & & \ddots & & \cdots & & & \\
& & & & a_1^{(n-1)} & & a_2^{(n-1)} & & \\
& & & & & a_1^{(n)}. & & &
\end{array}$$

$a_k^{(0)} = \sum_{1 \le \ell \le k} \langle \lambda, h_\ell \rangle$

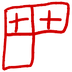
- $GT(\lambda)$ is an integral convex polytope for all $\lambda \in P_+$,
- $GT(\lambda)$ is N-dimensional for all $\lambda \in P_{++}$.

Kogan faces and dual Kogan faces

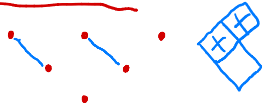
Definition (Kogan 2000)

A face of $GT(\lambda)$ given by equations of the type $a_k^{(l)} = a_k^{(l+1)}$ (resp., $a_k^{(l)} = a_{k+1}^{(l-1)}$) is called a **Kogan face** (resp., a **dual Kogan face**).

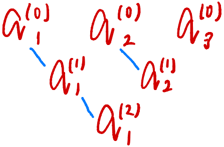
We associate to each $D \in \mathcal{PD}_n$ a Kogan face $F_D^\vee(GT(\lambda))$ and a dual Kogan face $F_D(GT(\lambda))$.

e.g. $n = 2$, $D =$  $\rightsquigarrow$

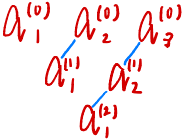
$F_D^\vee(GT(\lambda))$



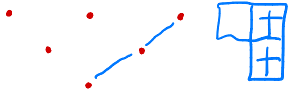
Kogan face



dual Kogan face



$F_D(GT(\lambda))$



Polytope rings

- $\mathscr{P}$: the semigroup of convex polytopes in $\mathbb{R}^N$,
- $G(\mathscr{P})$: the Grothendieck group of $\mathscr{P}$.

An element of $G(\mathscr{P})$ is called a **virtual polytope**. For $c \in \mathbb{R}_{\geq 0}$ and $Q \in \mathscr{P}$, define $cQ \in \mathscr{P}$ by $cQ := \{cx \mid x \in Q\}$; this induces an $\mathbb{R}$-linear space structure on $G(\mathscr{P})$. Let $\Lambda \subseteq G(\mathscr{P})$ be a $\mathbb{Z}$-lattice of finite rank such that the N-dimensional volume Vol_N induces a homogeneous polynomial vol_Λ of degree N on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, called the **volume polynomial**. Then we define a $\mathbb{Z}$-algebra R_Λ, called a **polytope ring**, by

$$R_\Lambda := \mathrm{Sym}(\Lambda)/J,$$

where J is the homogeneous ideal of $\mathrm{Sym}(\Lambda)$ given by

$$J := \{D \in \mathrm{Sym}(\Lambda) \mid D \cdot \mathrm{vol}_\Lambda = 0\},$$

where $\mathrm{Sym}(\Lambda)$ is regarded as a ring of differential operators on $\mathbb{R}[\Lambda \otimes_{\mathbb{Z}} \mathbb{R}]$.

Polytope rings

Let $Q \in \mathscr{P}$ be an N-dimensional smooth integral convex polytope, and $\Lambda \subseteq G(\mathscr{P})$ the $\mathbb{Z}$-lattice generated by all integral convex polytope analogous to Q.

Theorem (Khovanskii–Pukhlikov 1993)

The polytope ring R_Λ is isomorphic to $H^*(X(Q); \mathbb{Z})$.

Since $GT(\lambda + \mu) = GT(\lambda) + GT(\mu)$ for all $\lambda, \mu \in P_+$, the map

$$P_+ \to \mathscr{P}, \quad \lambda \mapsto GT(\lambda),$$

induces an injective group homomorphism $\gamma \colon P \hookrightarrow G(\mathscr{P})$, where P denotes the weight lattice. Then let $R_{GT} = \mathrm{Sym}(\Lambda_{GT})/J_{GT}$ denote the polytope ring associated with $\Lambda_{GT} \coloneqq \gamma(P)$.

Theorem (Borel presentation; see Kaveh 2011)

The polytope ring R_{GT} is isomorphic to $H^*(SL_{n+1}(\mathbb{C})/B; \mathbb{Z})$.

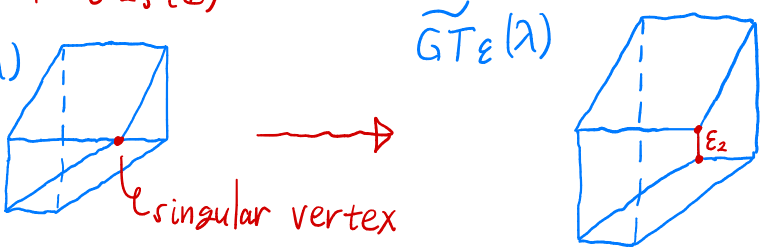
Polytope rings

For $\lambda \in P_{++}$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}^n$ such that $0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n$, we define a smooth integral convex polytope $\widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda)$ to be the set of $(a_1^{(1)}, a_1^{(2)}, a_2^{(1)}, a_1^{(3)}, a_2^{(2)}, a_3^{(1)}, \ldots, a_1^{(n)}, \ldots, a_n^{(1)}) \in \mathbb{R}^N$ satisfying the following inequalities:

$$a_j^{(i)} + \epsilon_{i+1} \geq a_j^{(i+1)} \geq a_{j+1}^{(i)} \quad \text{for } 0 \leq i \leq n-1 \text{ and } 1 \leq j \leq n-i.$$

e.g. $G = SL_3(\mathbb{C})$

$GT(\lambda)$



$\rightsquigarrow X(\widetilde{GT}_\varepsilon(\lambda)) \longrightarrow X(GT(\lambda))$: small resolution

Polytope rings

Let $R_{\widetilde{GT}} = \mathrm{Sym}(\Lambda_{\widetilde{GT}})/J_{\widetilde{GT}}$ be the polytope ring for the $\mathbb{Z}$-lattice $\Lambda_{\widetilde{GT}}$ generated by all integral convex polytopes analogous to $\widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda)$. Since $\widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda) + GT(\mu) = \widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda + \mu)$, we have $\Lambda_{GT} \subseteq \Lambda_{\widetilde{GT}}$.

Proposition (Kiritchenko–Smirnov–Timorin 2012)

There exist

- a $\mathbb{Z}$-module $M_{\widetilde{GT},GT}$,
- a surjective $\mathbb{Z}$-module homomorphism $\pi\colon R_{\widetilde{GT}} \twoheadrightarrow M_{\widetilde{GT},GT}$,
- an injective $\mathbb{Z}$-module homomorphism $\iota\colon R_{GT} \hookrightarrow M_{\widetilde{GT},GT}$

such that

- if $\pi(f') = \iota(f)$ and $\pi(g') = \iota(g)$ for some $f', g' \in R_{\widetilde{GT}}$ and $f, g \in R_{GT}$, then it holds that $\pi(f'g') = \iota(fg)$,
- $\iota(D \bmod J_{GT}) = \pi(D' \bmod J_{\widetilde{GT}})$ for all $D \in \mathrm{Sym}(\Lambda_{GT})$, where $D' \in \mathrm{Sym}(\Lambda_{\widetilde{GT}})$ is the image of D.

Polytope rings

For $D \in \mathcal{PD}_n$, we set

- $[F_D^\vee(GT)] \coloneqq \pi([F_D^\vee(\widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda))])$,
- $[F_D(GT)] \coloneqq \pi([F_D(\widetilde{GT}_{\boldsymbol{\epsilon}}(\lambda))])$,

which are independent of the choices of $\boldsymbol{\epsilon}$ and λ.

Theorem (Kiritchenko–Smirnov–Timorin 2012)

For $w \in W$, the cohomology class $[X^w] = [X_{w_0 w}]$ in $H^*(SL_{n+1}(\mathbb{C})/B; \mathbb{Z}) \simeq R_{GT} \hookrightarrow M_{\widetilde{GT}, GT}$ is described as follows:

$$[X^w] = \sum_{D \in RP(w^{-1})} [F_D^\vee(GT)] = \sum_{D \in RP(w_0 w w_0)} [F_D(GT)].$$

Aim of this talk

Problem

Generalize this story to other Lie types.

Billey–Haimann (1995) and Fomin–Kirillov (1996) introduced several kinds of Schubert polynomials for other classical types.

Aim

to discuss generalizations of reduced Kogan faces and reduced dual Kogan faces through Kashiwara's crystal bases.
More precisely, we give the following relations:

$$\text{reduced Kogan faces} \longleftrightarrow \text{Demazure crystals},$$

$$\text{reduced dual Kogan faces} \longleftrightarrow \text{opposite Demazure crystals}.$$

Borel–Weil theory

- G: a connected, simply-connected semisimple algebraic group over $\mathbb{C}$,
- $B \subseteq G$: a Borel subgroup,
- $I = \{1, \ldots, n\}$: an index set for the vertices of the Dynkin diagram,
- P_+: the set of dominant integral weights.

Theorem (Borel–Weil theory)

There exists a natural bijective map

$$P_+ \xrightarrow{\sim} \{\text{globally generated line bundles on } G/B\}, \quad \lambda \mapsto \mathcal{L}_\lambda,$$

such that $H^0(G/B, \mathcal{L}_\lambda)^*$ is the irreducible highest weight G-module with highest weight λ.

Crystal bases

- $\{G_\lambda^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\lambda)\}$: the upper global basis of $H^0(G/B, \mathcal{L}_\lambda)$,
- $\{G^{\mathrm{up}}(b) \mid b \in \mathcal{B}(\infty)\}$: the upper global basis of $\mathbb{C}[U^-]$, where U^- denotes the unipotent radical of B^-.

The index sets $\mathcal{B}(\lambda)$ and $\mathcal{B}(\infty)$ equipped with the Kashiwara operators $\{\tilde{e}_i \mid i \in I\} \cup \{\tilde{f}_i \mid i \in I\}$ are called **crystal bases**. Define I-colored directed graph structures on $\mathcal{B}(\lambda)$ and $\mathcal{B}(\infty)$ by

$$b \xrightarrow{i} b' \text{ if and only if } b' = \tilde{f}_i b.$$

If $G = SL_3(\mathbb{C})$ and $\lambda = \varpi_1 + \varpi_2$, then $\mathcal{B}(\lambda)$ is given by

$$\begin{tikzcd}
 & \circ \arrow[r, "2"] & \circ \arrow[r, "2"] & \circ \arrow[dr, "1"] & \\
b_\lambda\ \circ \arrow[ur, "1"] \arrow[dr, "2"'] & & & & \circ \\
 & \circ \arrow[r, "1"] & \circ \arrow[r, "1"] & \circ \arrow[ur, "2"'] &
\end{tikzcd}$$

String parametrizations

- $R(w)$: the set of reduced words for $w \in W$,
- $w_0 \in W$: the longest element.

For $b \in \mathcal{B}(\lambda)$ (resp., $b \in \mathcal{B}(\infty)$) and $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$, define $\Phi_{\boldsymbol{i}}(b) = (a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^N$, called the **string parametrization**, by

$$\begin{aligned} a_1 &:= \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_1}^a b \neq 0\}, \\ a_2 &:= \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_2}^a \tilde{e}_{i_1}^{a_1} b \neq 0\}, \\ &\vdots \\ a_N &:= \max\{a \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_{i_N}^a \tilde{e}_{i_{N-1}}^{a_{N-1}} \cdots \tilde{e}_{i_1}^{a_1} b \neq 0\}. \end{aligned}$$

String parametrizations

If $G = SL_3(\mathbb{C})$, $\lambda = \varpi_1 + \varpi_2$, and $\boldsymbol{i} = (1, 2, 1)$, then the string parametrizations $\Phi_{\boldsymbol{i}}(b)$, $b \in \mathcal{B}(\lambda)$, are given by

$$
\begin{tikzcd}
 & (1,0,0) \arrow[r, "2"] & (0,1,1) \arrow[r, "2"] & (0,2,1) \arrow[dr, "1"] & \\
(0,0,0) \arrow[ur, "1"] \arrow[dr, "2"] & & & & (1,2,1) \\
 & (0,1,0) \arrow[r, "1"] & (1,1,0) \arrow[r, "1"] & (2,1,0) \arrow[ur, "2"] &
\end{tikzcd}
$$

String polytopes

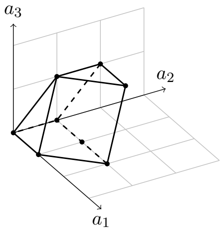
We set

$$\mathcal{S}_{\boldsymbol{i}}(\lambda) := \bigcup_{k \in \mathbb{Z}_{>0}} \{(k, \Phi_{\boldsymbol{i}}(b)) \mid b \in \mathcal{B}(k\lambda)\} \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}^N,$$

$\mathcal{C}_{\boldsymbol{i}}(\lambda)$: the smallest real closed cone containing $\mathcal{S}_{\boldsymbol{i}}(\lambda)$,

$$\Delta_{\boldsymbol{i}}(\lambda) := \{\boldsymbol{a} \in \mathbb{R}^N \mid (1, \boldsymbol{a}) \in \mathcal{C}_{\boldsymbol{i}}(\lambda)\}.$$

The set $\Delta_{\boldsymbol{i}}(\lambda)$ is called **Berenstein–Littelmann–Zelevinsky's string polytope**. If $G = SL_3(\mathbb{C})$, $\lambda = \varpi_1 + \varpi_2$, and $\boldsymbol{i} = (1, 2, 1)$, then the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$ is given by



String polytopes

For $\lambda \in P_{++}$, the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$ is N-dimensional. Let $\mathcal{C}_{\boldsymbol{i}} \subseteq \mathbb{R}^N$ denote the smallest real closed cone containing $\Delta_{\boldsymbol{i}}(\lambda)$, which is independent of the choice of $\lambda \in P_{++}$ and called the **string cone**. Then we have

$$\mathcal{C}_{\boldsymbol{i}} \cap \mathbb{Z}^N = \Phi_{\boldsymbol{i}}(\mathcal{B}(\infty)),$$

and the string polytope $\Delta_{\boldsymbol{i}}(\lambda)$ is given by cutting $\mathcal{C}_{\boldsymbol{i}}$ by the following affine inequalities:

$$a_k \leq \langle \lambda, h_{i_k} \rangle - \sum_{k<j\leq N} \langle \alpha_{i_j}, h_{i_k} \rangle a_j \quad \text{for } 1 \leq k \leq N.$$

For $1 \leq k \leq N$, let $F_k(\Delta_{\boldsymbol{i}}(\lambda))$ denote the facet of $\Delta_{\boldsymbol{i}}(\lambda)$ given by

$$a_k = \langle \lambda, h_{i_k} \rangle - \sum_{k<j\leq N} \langle \alpha_{i_j}, h_{i_k} \rangle a_j.$$

Semi-toric degenerations of Richardson varieties

- $X_w^v := X_w \cap X^v$: the **Richardson variety** for $v, w \in W$ with $v \leq w$,
- $\mathcal{B}_w(\lambda) \subseteq \mathcal{B}(\lambda)$: the **Demazure crystal** for $X_w \subseteq G/B$,
- $\mathcal{B}^v(\lambda) \subseteq \mathcal{B}(\lambda)$: the **opposite Demazure crystal** for $X^v \subseteq G/B$.

Theorem (Caldero 2002)

For $\boldsymbol{i} \in R(w_0)$ and $\lambda \in P_{++}$, there exists a flat degeneration of G/B to the normal toric variety $X(\Delta_{\boldsymbol{i}}(\lambda))$ corresponding to $\Delta_{\boldsymbol{i}}(\lambda)$.

Theorem (Morier-Genoud 2008)

Let $\boldsymbol{i} \in R(w_0)$, $\lambda \in P_{++}$, and $v, w \in W$ such that $v \leq w$.

(1) There naturally exists a union $\Delta_{\boldsymbol{i}}(\lambda, X_w^v)$ of faces of $\Delta_{\boldsymbol{i}}(\lambda)$ such that $\Delta_{\boldsymbol{i}}(\lambda, X_w^v) \cap \mathbb{Z}^N = \Phi_{\boldsymbol{i}}(\mathcal{B}_w(\lambda) \cap \mathcal{B}^v(\lambda))$.

(2) The Richardson variety X_w^v degenerates into the union of irreducible normal toric varieties corresponding to the faces of $\Delta_{\boldsymbol{i}}(\lambda, X_w^v)$.

Relation with Lusztig polytopes

For $\boldsymbol{i} \in R(w_0)$, let $b_{\boldsymbol{i}} \colon \mathbb{Z}_{\geq 0}^N \to \mathcal{B}(\infty)$ denote the **Lusztig parametrization**. Since $b_{\boldsymbol{i}}$ is bijective, we obtain the inverse map $\Upsilon_{\boldsymbol{i}} \coloneqq b_{\boldsymbol{i}}^{-1} \colon \mathcal{B}(\infty) \to \mathbb{Z}_{\geq 0}^N$, which induces a parametrization $\Upsilon_{\boldsymbol{i}} \colon \mathcal{B}(\lambda) \to \mathbb{Z}_{\geq 0}^N$. Replacing $\Phi_{\boldsymbol{i}}$ by $\Upsilon_{\boldsymbol{i}}$ in the definition of string polytopes, we obtain the **Lusztig polytope** $\widehat{\Delta}_{\boldsymbol{i}}(\lambda)$. Define $\Omega_{\boldsymbol{i},\lambda} \colon \mathbb{R}^N \to \mathbb{R}^N$, $(t_1, \ldots, t_N) \mapsto (t'_1, \ldots, t'_N)$, for $\lambda \in P_+$ by

$$t'_k \coloneqq \langle \lambda, h_{i_k} \rangle - t_k - \sum_{k<j\leq N} \langle \alpha_{i_j}, h_{i_k} \rangle t_j \quad \text{for } 1 \leq k \leq N.$$

Theorem (Morier-Genoud 2008)

For $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$ and $\lambda \in P_+$, the equality

$$\Omega_{\boldsymbol{i},\lambda}(\Delta_{\boldsymbol{i}}(\lambda)) = \widehat{\Delta}_{\boldsymbol{i}^*}(\lambda)$$

holds, where $\boldsymbol{i}^* = (i_1^*, \ldots, i_N^*)$ is given by $w_0(\alpha_i) = -\alpha_{i^*}$ for $i \in I$.

Relation with Lusztig polytopes

For $\lambda \in P_{++}$, the smallest real closed cone containing $\widehat{\Delta}_{\boldsymbol{i}}(\lambda)$ is $\mathbb{R}_{\geq 0}^N$, and the Lusztig polytope $\widehat{\Delta}_{\boldsymbol{i}}(\lambda)$ is given by cutting $\mathbb{R}_{\geq 0}^N$ by some affine inequalities depending on λ. Under the unimodular transformation

$$\Omega_{\boldsymbol{i},\lambda} \colon \Delta_{\boldsymbol{i}}(\lambda) \to \widehat{\Delta}_{\boldsymbol{i}^*}(\lambda),$$

the facet $F_k(\Delta_{\boldsymbol{i}}(\lambda))$ of $\Delta_{\boldsymbol{i}}(\lambda)$ corresponds to the facet $F_k(\widehat{\Delta}_{\boldsymbol{i}^*}(\lambda))$ of $\widehat{\Delta}_{\boldsymbol{i}^*}(\lambda)$ given by $a_k = 0$. If we set

$$\widehat{\Delta}_{\boldsymbol{i}^*}(\lambda, X_{w_0 v}^{w_0 w}) := \Omega_{\boldsymbol{i},\lambda}(\Delta_{\boldsymbol{i}}(\lambda, X_w^v)),$$

then we have

$$\widehat{\Delta}_{\boldsymbol{i}^*}(\lambda, X_{w_0 v}^{w_0 w}) \cap \mathbb{Z}^N = \Upsilon_{\boldsymbol{i}^*}(\mathcal{B}_{w_0 v}(\lambda) \cap \mathcal{B}^{w_0 w}(\lambda)).$$

Relation with Lusztig polytopes

For $\boldsymbol{k} = (k_1, \ldots, k_\ell)$ with $1 \leq k_1 < \cdots < k_\ell \leq N$, we set

$$F_{\boldsymbol{k}}(\Delta_{\boldsymbol{i}}(\lambda)) \coloneqq F_{k_1}(\Delta_{\boldsymbol{i}}(\lambda)) \cap \cdots \cap F_{k_\ell}(\Delta_{\boldsymbol{i}}(\lambda)),$$
$$F_{\boldsymbol{k}}(\widehat{\Delta}_{\boldsymbol{i}}(\lambda)) \coloneqq F_{k_1}(\widehat{\Delta}_{\boldsymbol{i}}(\lambda)) \cap \cdots \cap F_{k_\ell}(\widehat{\Delta}_{\boldsymbol{i}}(\lambda)).$$

Similarly, we define a face $F_{\boldsymbol{k}}(\mathbb{R}^N_{\geq 0})$ of $\mathbb{R}^N_{\geq 0}$. For $w \in W$, let $\mathcal{B}_w(\infty) \subseteq \mathcal{B}(\infty)$ denote the **Demazure crystal**. Then it holds that

- $\Upsilon_{\boldsymbol{i}}(\mathcal{B}_w(\infty))$ is a union of sets of the form $F_{\boldsymbol{k}}(\mathbb{R}^N_{\geq 0}) \cap \mathbb{Z}^N$,
- $\widehat{\Delta}_{\boldsymbol{i}}(\lambda, X_w)$ is a union of faces of $\widehat{\Delta}_{\boldsymbol{i}}(\lambda)$ of the form $F_{\boldsymbol{k}}(\widehat{\Delta}_{\boldsymbol{i}}(\lambda))$,
- $\Delta_{\boldsymbol{i}}(\lambda, X^w)$ is a union of faces of $\Delta_{\boldsymbol{i}}(\lambda)$ of the form $F_{\boldsymbol{k}}(\Delta_{\boldsymbol{i}}(\lambda))$.

First main result

For $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$ and $w \in W$ with $\ell := \ell(w)$, we set

$$R(\boldsymbol{i}, w) \coloneqq \{(k_1, \ldots, k_\ell) \mid 1 \leq k_1 < \cdots < k_\ell \leq N,\ (i_{k_1}, \ldots, i_{k_\ell}) \in R(w)\}.$$

Example

Let $G = SL_4(\mathbb{C})$, $\boldsymbol{i} := (2, 1, 3, 2, 3, 1) \in R(w_0)$, and $w := s_1 s_3 = s_3 s_1$. Then we have $R(\boldsymbol{i}, w) = \{(2, 3), (2, 5), (3, 6), (5, 6)\}$.

Theorem (F.)

For $\boldsymbol{i} \in R(w_0)$, $w \in W$, and $\lambda \in P_+$, the following equalities hold:

$$\Delta_{\boldsymbol{i}}(\lambda, X^w) = \bigcup_{\boldsymbol{k} \in R(\boldsymbol{i}, w)} F_{\boldsymbol{k}}(\Delta_{\boldsymbol{i}}(\lambda)),$$

$$\widehat{\Delta}_{\boldsymbol{i}}(\lambda, X_w) = \bigcup_{\boldsymbol{k} \in R(\boldsymbol{i}^*, w_0 w)} F_{\boldsymbol{k}}(\widehat{\Delta}_{\boldsymbol{i}}(\lambda)).$$

Relation with quantum nilpotent subalgebras

For $w \in W$, Kimura (2012) proved that the quantum nilpotent subalgebra $U_q^-(w)$ is compatible with the upper global basis and we have

$$\mathscr{B}(U_q^-(w)) \subseteq \mathcal{B}_{w^{-1}}(\infty).$$

For $\boldsymbol{i} = (i_1, \ldots, i_N) \in R(w_0)$, it holds that

$$\Upsilon_{\boldsymbol{i}}(\mathcal{B}_{w^{-1}}(\infty)) = \bigcup_{\boldsymbol{k} \in R(\boldsymbol{i}^*, w_0 w^{-1})} F_{\boldsymbol{k}}(\mathbb{R}_{\geq 0}^N) \cap \mathbb{Z}^N.$$

If $(i_1, \ldots, i_\ell) \in R(w)$, then we have

$$(\ell + 1, \ldots, N) \in R(\boldsymbol{i}^*, w_0 w^{-1}),$$

and the face $F_{(\ell+1,\ldots,N)}(\mathbb{R}_{\geq 0}^N)$ corresponds to the set $\mathscr{B}(U_q^-(w))$. More precisely, we have

$$\Upsilon_{\boldsymbol{i}}(b) = (c_1, \ldots, c_\ell, 0, \ldots, 0)$$

for $b \in \mathscr{B}(U_q^-(w))$, where $(c_1, \ldots, c_\ell)$ is the PBW-parametrization of b associated with $(i_1, \ldots, i_\ell) \in R(w)$.

Relation with dual Kogan faces

Let $G = SL_{n+1}(\mathbb{C})$, and

$$\boldsymbol{i}_A := (1, 2, 1, 3, 2, 1, \ldots, n, n-1, \ldots, 1) \in R(w_0).$$

Theorem (Littelmann 1998)

For $\lambda \in P_+$, the string polytope $\Delta_{\boldsymbol{i}_A}(\lambda)$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $GT(\lambda)$.

Under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$, the faces $F_{\boldsymbol{k}}(\Delta_{\boldsymbol{i}_A}(\lambda))$, $\boldsymbol{k} \in R(\boldsymbol{i}_A, w)$, correspond to the reduced dual Kogan faces $F_D(GT(\lambda))$, $D \in RP(w_0 w w_0)$.

Corollary

For $w \in W$ and $\lambda \in P_+$, the union $\bigcup_{D \in RP(w_0 w w_0)} F_D(GT(\lambda))$ of reduced dual Kogan faces for $RP(w_0 w w_0)$ gives a polyhedral parametrization of $\mathcal{B}^w(\lambda)$ under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$.

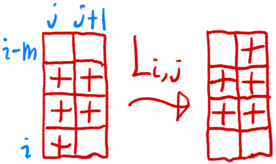
Ladder moves

Definition (Bergeron–Billey 1993)

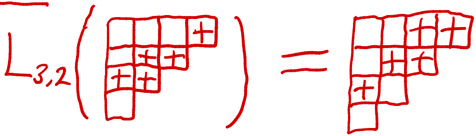
For $(i, j) \in Y_n$, the **ladder move** $L_{i,j}$ is defined as follows. Take $D \in \mathcal{PD}_n$ satisfying the following conditions:

- $(i, j) \in D$, $(i, j+1) \notin D$,
- there exists $1 \leq m < i$ such that $(i-m, j), (i-m, j+1) \notin D$ and $(i-k, j), (i-k, j+1) \in D$ for all $1 \leq k < m$.

Then define $L_{i,j}(D) \in \mathcal{PD}_n$ by $L_{i,j}(D) := D \cup \{(i-m, j+1)\} \setminus \{(i, j)\}$.



e.g.



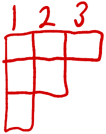
Ladder moves

Arrange the elements of Y_n as

$$((p_1, i_1), \ldots, (p_N, i_N)) \coloneqq ((n, 1), (n-1, 2), (n-1, 1), \ldots, (1, 2), (1, 1)).$$

We associate to each $D \in \mathcal{PD}_n$ a sequence $\boldsymbol{k}'_D = (k_1, \ldots, k_{N-|D|})$ with $1 \leq k_1 < \cdots < k_{N-|D|} \leq N$ by arranging $1 \leq k \leq N$ such that $(p_k, i_k) \in Y_n \setminus D$. For $w \in W$, define $D(w) \in \mathcal{PD}_n$ by the condition that $\boldsymbol{k}'_{D(w)}$ is the minimum element in $R(\boldsymbol{i}_A, w)$ with respect to the lexicographic order.

e.g.

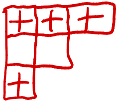
$Y_3 =$ 

$$\begin{array}{ccc} 6 & 5 & 4 \\ 3 & 2 & \\ 1 & & \end{array}$$

$w = s_2 s_1 \in \mathfrak{S}_4$

$\to \boldsymbol{i}_A = (1, \dot{2}, \dot{1}, 3, 2, 1)$

$\to \boldsymbol{k}'_{D(w)} = (2, 3)$

$\to D(w) =$ 

Ladder moves

Proposition

For $w \in W \simeq \mathfrak{S}_{n+1}$, the following equality holds:

$$D(w) = \{(i, j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m(i)\},$$

where $m(i)$ is the cardinality of $\{i < j \leq n+1 \mid w^{-1}w_0(j) < w^{-1}w_0(i)\}$.

Let $\mathscr{L}(D(w))$ denote the set of elements of $\mathcal{PD}_n$ obtained from $D(w)$ by applying sequences of ladder moves.

Theorem (Bergeron–Billey 1993)

For $w \in W$, the following equality holds:

$$\mathscr{L}(D(w)) = RP(w^{-1}w_0).$$

Transposed mitosis operators

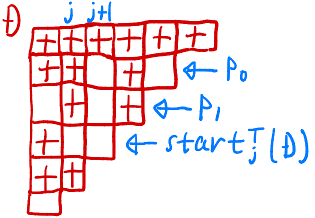
For $1 \leq j \leq n$ and $D \in \mathcal{PD}_n$, we set

- $\mathrm{start}_j^\top(D) := \min\{1 \leq i \leq n-j+1 \mid (i,j) \notin D\} \cup \{n-j+2\}$,
- $\mathcal{J}_j^\top(D) := \{1 \leq i < \mathrm{start}_j^\top(D) \mid (i,j+1) \notin D\}$.

For $i \in \mathcal{J}_j^\top(D)$, we define a pipe dream $D_i(j) \in \mathcal{PD}_n$ as follows: write $\{1 \leq p \leq i \mid (p,j+1) \notin D\} = \{p_0 < p_1 < \cdots < p_r = i\}$, and set

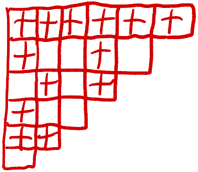
$$D_i(j) := L_{p_r,j} \cdots L_{p_1,j}(D \setminus \{(p_0,j)\}).$$

e.g. $n=6$, $j=2$

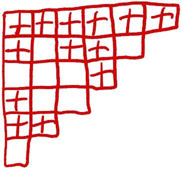


$\mathcal{J}_j^\top(D) = \{2, 3\}$

$D_2(2)$



$D_3(2)$



Transposed mitosis operators

Definition (Knutson–Miller 2005)

For $1 \leq j \leq n$ and $D \in \mathcal{PD}_n$, the **transposed mitosis operator** $\mathrm{mitosis}_j^\top$ sends D to

$$\mathrm{mitosis}_j^\top(D) \coloneqq \{D_i(j) \mid i \in \mathcal{J}_j^\top(D)\}.$$

If $\mathcal{J}_j^\top(D) = \emptyset$, then we regard $\mathrm{mitosis}_j^\top(D)$ as the empty set. For a subset $\mathcal{A} \subseteq \mathcal{PD}_n$, we set $\mathrm{mitosis}_j^\top(\mathcal{A}) \coloneqq \bigcup_{D \in \mathcal{A}} \mathrm{mitosis}_j^\top(D)$.

Theorem (Knutson–Miller 2005 and Miller 2003)

For $w \in W$ and $(j_1, \ldots, j_\ell) \in R(w)$, it holds that

$$\mathscr{L}(D(w)) = \mathrm{mitosis}_{j_\ell}^\top \cdots \mathrm{mitosis}_{j_1}^\top(Y_n).$$

Faces of string cones

Littelmann (1998) proved that the string cone $\mathcal{C}_{\boldsymbol{i}_A}$ coincides with the set of $(a_1^{(1)}, a_1^{(2)}, a_2^{(1)}, a_1^{(3)}, a_2^{(2)}, a_3^{(1)}, \dots, a_1^{(n)}, \dots, a_n^{(1)}) \in \mathbb{R}^N$ satisfying the following inequalities:

$$a_1^{(1)} \geq 0, \qquad a_1^{(2)} \geq a_2^{(1)} \geq 0, \qquad \dots, \qquad a_1^{(n)} \geq \cdots \geq a_n^{(1)} \geq 0.$$

We associate to each $D \in \mathcal{PD}_n$ a face $F_D^{\vee}(\mathcal{C}_{\boldsymbol{i}_A})$ of $\mathcal{C}_{\boldsymbol{i}_A}$ and a face $F_D^{\vee}(\Delta_{\boldsymbol{i}_A}(\lambda))$ of $\Delta_{\boldsymbol{i}_A}(\lambda)$ as the following example.

e.g.

$n=3 \longrightarrow$

$a_3^{(1)}=0$	$a_2^{(2)}=a_3^{(1)}$	$a_1^{(3)}=a_2^{(2)}$
$a_2^{(1)}=0$	$a_1^{(2)}=a_2^{(1)}$	
$a_1^{(1)}=0$		

Second main result

Theorem (F.)

For $w \in W$ and $\lambda \in P_+$, the following equalities hold:

$$\Phi_{\boldsymbol{i}_A}(\mathcal{B}_w(\infty)) = \bigcup_{D \in \mathscr{L}(D(w))} F_D^\vee(\mathcal{C}_{\boldsymbol{i}_A}) \cap \mathbb{Z}^N,$$

$$\Delta_{\boldsymbol{i}_A}(\lambda, X_w) = \bigcup_{D \in \mathscr{L}(D(w))} F_D^\vee(\Delta_{\boldsymbol{i}_A}(\lambda)).$$

Corollary

For $w \in W$ and $\lambda \in P_+$, the union $\bigcup_{D \in \mathscr{L}(D(w))} F_D^\vee(GT(\lambda))$ of reduced Kogan faces for $\mathscr{L}(D(w))$ gives a polyhedral parametrization of $\mathcal{B}_w(\lambda)$ under the unimodular affine transformation $\Delta_{\boldsymbol{i}_A}(\lambda) \simeq GT(\lambda)$.

Second main theorem

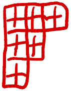
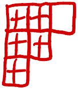
key : Kashiwara embedding Ψ_{i_A}

e.g. $G = SL_4(\mathbb{C})$

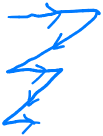
$\mathcal{B}_i : \cdots \xrightarrow{i} (-1)_i \xrightarrow{i} (0)_i \xrightarrow{i} (1)_i \xrightarrow{i} \cdots$

$\rightsquigarrow \Psi_{i_A} : \mathcal{B}(\infty) \hookrightarrow \{b_\infty\} \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_3 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \simeq \mathbb{Z}^6$

Compute $\Psi_{i_A}(\mathcal{B}_{s_3 s_2}(\infty)) = \Psi_{i_A}(\mathcal{B}_{s_2 s_3}(\infty))$

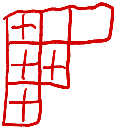
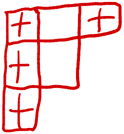
$\Psi_{i_A}(\mathcal{B}_e(\infty)) =$  $\xrightarrow{\tilde{f}_3}$ $\Psi_{i_A}(\mathcal{B}_{s_3}(\infty)) =$ 

B_1	B_2	B_3
B_1	B_2	
B_1		



$\downarrow \tilde{f}_2$

$L_{2,2}$

$\Psi_{i_A}(\mathcal{B}_{s_2 s_3}(\infty)) =$  $\cup$ 

Skew pipe dreams

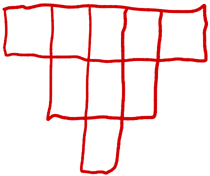
Let $G = Sp_{2n}(\mathbb{C})$, and set

- $SY_n := \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n,\ i \leq j \leq 2n - i\}$,
- $\mathcal{SPD}_n$: the set of subsets of SY_n.

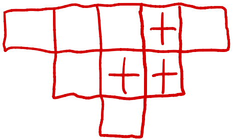
An element of $\mathcal{SPD}_n$ is called a **skew pipe dream** (Kiritchenko 2016).

e.g.

$\mathcal{D} = \{(1,4), (2,3), (2,4)\} \in \mathcal{SPD}_3$

$SY_3 =$ 

shifted Young diagram



Symplectic Gelfand–Tsetlin polytopes

For $\lambda \in P_+$ and $1 \leq k \leq n$, we set $b_k^{(1)} := \sum_{1 \leq \ell \leq n-k+1} \langle \lambda, h_\ell \rangle$, and write $b_{n+1}^{(1)} = b_n^{(2)} = \cdots = b_2^{(n)} := 0$. Then the **symplectic Gelfand–Tsetlin polytope** $SGT(\lambda)$ is defined to be the set of

$$(a_1^{(1)}, \underbrace{b_1^{(2)}, a_2^{(1)}, a_1^{(2)}}_{3}, \underbrace{b_1^{(3)}, b_2^{(2)}, a_3^{(1)}, a_2^{(2)}, a_1^{(3)}}_{5}, \ldots, \underbrace{b_1^{(n)}, \ldots, b_{n-1}^{(2)}, a_n^{(1)}, \ldots, a_1^{(n)}}_{2n-1})$$

satisfying the following inequalities:

$$\begin{array}{ccccccccc}
b_1^{(1)} & & b_2^{(1)} & & \cdots & b_n^{(1)} & & b_{n+1}^{(1)} \\
& a_1^{(1)} & & a_2^{(1)} & & & a_n^{(1)} & \\
& & b_1^{(2)} & & \cdots & b_{n-1}^{(2)} & & b_n^{(2)} \\
& & & a_1^{(2)} & & & a_{n-1}^{(2)} & \\
& & & & \ddots & & & \vdots \\
& & & & & b_1^{(n)} & & b_2^{(n)} \\
& & & & & & a_1^{(n)}. &
\end{array}$$

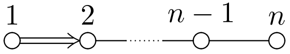
Symplectic Gelfand–Tsetlin polytopes

- $SGT(\lambda)$ is an integral convex polytope for $\lambda \in P_+$, and
- $SGT(\lambda)$ is N-dimensional if $\lambda \in P_{++}$.

We define $\boldsymbol{i}_C = (i_1, \ldots, i_N) \in R(w_0)$ by

$$\boldsymbol{i}_C := (1, \underbrace{2, 1, 2}_{3}, \underbrace{3, 2, 1, 2, 3}_{5}, \ldots, \underbrace{n, n-1, \ldots, 1, \ldots, n-1, n}_{2n-1}),$$

where

C_n 

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Theorem (Littelmann 1998)

For $\lambda \in P_+$, the string polytope $\Delta_{\boldsymbol{i}_C}(\lambda)$ is unimodularly equivalent to the symplectic Gelfand–Tsetlin polytope $SGT(\lambda)$.

String cones

Littelmann (1998) proved that the string cone $\mathcal{C}_{\boldsymbol{i}_C}$ coincides with the set of

$$(a_1^{(1)}, b_1^{(2)}, a_2^{(1)}, a_1^{(2)}, b_1^{(3)}, b_2^{(2)}, a_3^{(1)}, a_2^{(2)}, a_1^{(3)}, \dots, b_1^{(n)}, \dots, b_{n-1}^{(2)}, a_n^{(1)}, \dots, a_1^{(n)})$$

satisfying the following inequalities:

$$a_1^{(1)} \geq 0, \quad b_1^{(2)} \geq a_2^{(1)} \geq a_1^{(2)} \geq 0, \quad \dots,$$
$$b_1^{(n)} \geq \cdots \geq b_{n-1}^{(2)} \geq a_n^{(1)} \geq \cdots \geq a_1^{(n)} \geq 0.$$

We associate to each $D \in \mathcal{SPD}_n$ a face $F_D^\vee(\mathcal{C}_{\boldsymbol{i}_C})$ of $\mathcal{C}_{\boldsymbol{i}_C}$ and a face $F_D^\vee(\Delta_{\boldsymbol{i}_C}(\lambda))$ of $\Delta_{\boldsymbol{i}_C}(\lambda)$ as the following example.

e.g.

$n = 3 \longrightarrow$

$a_1^{(3)} = 0$	$a_2^{(2)} = a_1^{(3)}$	$a_3^{(1)} = a_2^{(2)}$	$b_2^{(2)} = a_3^{(1)}$	$b_1^{(3)} = b_2^{(2)}$
	$a_1^{(2)} = 0$	$a_2^{(1)} = a_1^{(2)}$	$b_1^{(2)} = a_2^{(1)}$	
		$a_1^{(1)} = 0$		

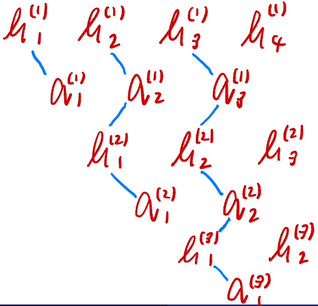
Symplectic Kogan faces

Definition

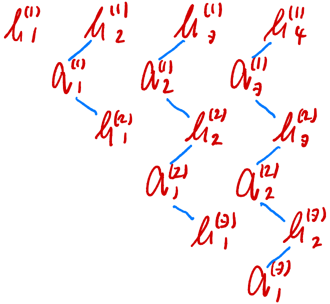
A face of $SGT(\lambda)$ given by equations of the types $a_k^{(l)} = b_{k-1}^{(l+1)}$ and $b_k^{(l)} = a_k^{(l)}$ (resp., $a_k^{(l)} = b_k^{(l+1)}$ and $b_k^{(l)} = a_{k-1}^{(l)}$) is called a **symplectic Kogan face** (resp., a **dual symplectic Kogan face**).

e.g. $n=3$

symp. Kogan face



dual symp. Kogan face



Ladder moves

We arrange the elements of SY_n as

$$\begin{aligned}&((p_1, q_1), \ldots, (p_N, q_N))\\&:= ((n, n), (n-1, n+1), (n-1, n), (n-1, n-1), \ldots, (1, 2), (1, 1)).\end{aligned}$$

Definition

For $(i, j) \in SY_n$, the **ladder move** $L_{i,j}$ is defined as follows: write $(i, j) = (p_k, q_k)$, and take $D \in \mathcal{SPD}_n$ satisfying the following conditions:

- $(i, j) \in D$, $(i, j+1) \notin D$,
- there exists $k < \ell \leq N$ such that $q_\ell \in \{j, 2n - j\}$, $(p_\ell, q_\ell), (p_\ell, q_\ell + 1) \notin D$, and $(p_r, q_r), (p_r, q_r + 1) \in D$ for all $k < r < \ell$ such that $q_r \in \{j, 2n - j\}$.

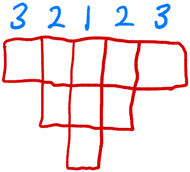
Then we define $L_{i,j}(D) \in \mathcal{SPD}_n$ by

$$L_{i,j}(D) := D \cup \{(p_\ell, q_\ell + 1)\} \setminus \{(i, j)\}.$$

Ladder moves

e.g. $n=3$

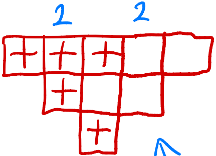
$SY_3 =$



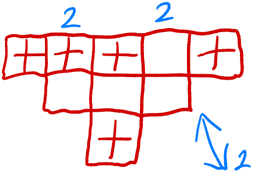
(p_k, q_k)

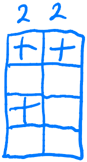
$$\begin{array}{ccccc} 9 & 8 & 7 & 6 & 5 \\ & 4 & 3 & 2 & \\ & & 1 & & \end{array}$$

$D =$

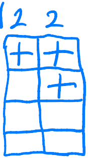


$\xrightarrow{L_{2,2}}$





ladder move in type A



Ladder moves

We associate to each $D \in \mathcal{SPD}_n$ a sequence $\boldsymbol{k}'_D = (k_1, \ldots, k_{N-|D|})$ with $1 \leq k_1 < \cdots < k_{N-|D|} \leq N$ by arranging $1 \leq k \leq N$ such that $(p_k, q_k) \in SY_n \setminus D$. For $w \in W$, define $D(w) \in \mathcal{SPD}_n$ by the condition that $\boldsymbol{k}'_{D(w)}$ is the minimum element in $R(\boldsymbol{i}_C, w)$ with respect to the lexicographic order.

Proposition

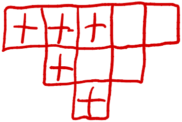
For $w \in W$, there exist $m(1), \ldots, m(n) \in \mathbb{Z}_{\geq 0}$ such that

$$D(w) = \{(i,j) \in SY_n \mid j \leq m(i) + i - 1\}.$$

e.g. $n = 3$, $w = s_2 s_1 s_3 s_2$

$\rightsquigarrow \boldsymbol{i}_C = (1, \dot{2}, \dot{1}, 2, \dot{3}, \dot{2}, 1, 2, 3)$

$\rightsquigarrow k'_{D(w)} = (2, 3, 5, 6)$ $\rightarrow$ $D(w) =$



Upper bounds for transposed mitosis operators

Definition

For $D \in \mathcal{SPD}_n$, we set

$$r_0 \coloneqq \max\{1 \leq r \leq N \mid q_r \in \{n-i+1, n+i-1\},\ (p_r, q_r+1) \notin D\},$$

and assume that

$$\{(p_r, q_r) \mid q_r \in \{n-i+1, n+i-1\},\ r_0 \leq r \leq N\}$$
$$\cup \{(p_r, q_r+1) \mid q_r \in \{n-i+1, n+i-1\},\ r_0+1 \leq r \leq N\} \subseteq D.$$

Then the operator M_i sends D to the set $M_i(D)$ of elements of $\mathcal{SPD}_n$ obtained from $D \setminus \{(p_{r_0}, q_{r_0})\}$ by applying sequences of ladder moves $L_{p,q}$ such that $q \in \{n-i+1, n+i-1\}$.

For a subset $\mathcal{A} \subseteq \mathcal{SPD}_n$, we set $M_i(\mathcal{A}) \coloneqq \bigcup_{D \in \mathcal{A}} M_i(D)$.

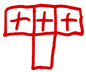
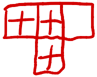
Upper bounds for transposed mitosis operators

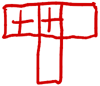
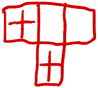
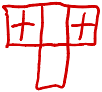
For $w \in W$, define a set $\mathscr{M}(w)$ of skew pipe dreams by

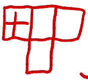
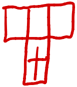
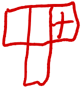
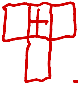
$$\mathscr{M}(w) \coloneqq M_{i_{k_\ell}} \cdots M_{i_{k_1}}(SY_n),$$

where $\boldsymbol{k}'_{D(w)} = (k_1, \ldots, k_\ell)$.

e.g. $n = 2$

$\mathcal{M}(s_1) = \{$  $\}$, $\mathcal{M}(s_2) = \{$  $\}$

$\mathcal{M}(s_1 s_2) = \{$  $\}$, $\mathcal{M}(s_2 s_1) = \{$ ,  $\}$

$\mathcal{M}(s_1 s_2 s_1) = \{$  $\}$, $\mathcal{M}(s_2 s_1 s_2) = \{$ , ,  $\}$

Third main result

Theorem (F.)

For $w \in W$ and $\lambda \in P_+$, the following equalities hold:

$$\Phi_{\boldsymbol{i}_C}(\mathcal{B}_w(\infty)) = \bigcup_{D \in \mathscr{M}(w)} F_D^\vee(\mathcal{C}_{\boldsymbol{i}_C}) \cap \mathbb{Z}^N,$$

$$\Delta_{\boldsymbol{i}_C}(\lambda, X_w) = \bigcup_{D \in \mathscr{M}(w)} F_D^\vee(\Delta_{\boldsymbol{i}_C}(\lambda)).$$

Corollary

For $w \in W$, the cohomology class $[X^w] = [X_{w_0 w}]$ in $H^*(Sp_{2n}(\mathbb{C})/B; \mathbb{Z}) \simeq R_{SGT} \hookrightarrow M_{\widetilde{SGT}, SGT}$ is described as follows:

$$[X^w] = \sum_{\boldsymbol{k} \in R(\boldsymbol{i}_C, w)} [F_{\boldsymbol{k}}(SGT)] = \sum_{D \in \mathscr{M}(w_0 w)} [F_D^\vee(SGT)].$$

Other classical types (B_n and D_n)

Type B_n

There exists a similarity property between string parametrizations of type B_n and of type C_n (Kashiwara 1996). Hence our results for $Sp_{2n}(\mathbb{C})$ (of type C_n) are naturally extended to those for $\mathrm{Spin}_{2n+1}(\mathbb{C})$ (of type B_n) if we use $H^*(\mathrm{Spin}_{2n+1}(\mathbb{C}); \mathbb{Q})$ instead of $H^*(Sp_{2n}(\mathbb{C}); \mathbb{Z})$.

Type D_n

The Gelfand–Tsetlin polytopes $GT_{D_n}(\lambda)$ of type D_n do not have the additivity property, that is, $GT_{D_n}(\lambda + \mu) \neq GT_{D_n}(\lambda) + GT_{D_n}(\mu)$ in general.

Theorem (F. 2019)

The Nakashima–Zelevinsky polytopes $\widetilde{\Delta}_{\boldsymbol{i}}(\lambda)$, $\lambda \in P_+$, of type D_n associated with

$$\boldsymbol{i} = (n, n-1, \ldots, 1, \ldots, n, n-1, \ldots, 1) \in I^{n(n-1)}$$

have the additivity property.

Future directions

Kiritchenko (2016) introduced the notion of **transposed mitosis operators** $\mathrm{mitosis}_i^\top$ for skew pipe dreams.

Conjecture

For $w \in W$, the following equalities hold:

$$\mathscr{L}(D(w)) = \mathscr{M}(w) = \mathrm{mitosis}_{i_{k_\ell}}^\top \cdots \mathrm{mitosis}_{i_{k_1}}^\top (SY_n),$$

where $\boldsymbol{k}'_{D(w)} = (k_1, \ldots, k_\ell)$.

We have checked this conjecture for $Sp_4(\mathbb{C})$ and $Sp_6(\mathbb{C})$.

Future directions

Question

Is there a kind of "Schubert polynomial" of type C_n which is related to the set $\mathscr{M}(w)$?

- Fomin–Kirillov's Schubert polynomial of first kind of type C_n is defined to be the form
$$\sum_{\boldsymbol{k} \in R(\boldsymbol{i}, w)} f_{\boldsymbol{k}}(\boldsymbol{x}),$$
where
$$\boldsymbol{i} = (n, n-1, \ldots, 1, \ldots, n, n-1, \ldots, 1) \in I^{n^2}.$$
- Knutson–Miller (2005) realized the Schubert polynomial $\mathfrak{S}_w$ of type A_n as the multidegree of the opposite matrix Schubert variety $\overline{X}^{w_0 w w_0}$.

Thank you for your attention!