Pseudo-Anosov mapping classes are sign-stable

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joint work with Tsukasa Ishibashi

(II) Ishibashi (2019)

Cluster algebraic analogue of Nielsen-Thurston classification

$\rightsquigarrow$ cluster pseudo-Anosov.

↑ weaker than usual pseudo-Anosov.

i.e. cluster pA $\not\Rightarrow$ pA

<u>Today</u>

Giving complete translation of pA property into cluster alg.

via sign stability.

Main Theorem [Ishibashi - K, in prep]

Let Σ be a punctured surface without boundary comp.

For $\phi \in \underline{MC(\Sigma)}$, TFAE:

§1

(1) ϕ is pseudo-Anosov

§2

(2) $\phi \curvearrowright \mathbb{S}\mathcal{X}_\Sigma(\mathbb{R}^{trop})$: NS dynamics

§3

(3) Any representation paths of ϕ are sign-stab on $\Omega_\Sigma^{\mathbb{Q}} \subset \mathcal{X}_\Sigma(\mathbb{R}^{trop})$

§4

Contents

§1. Exchange graph

Cayley graph

$$\mathbb{E}_I := \mathbb{T}_I \times \mathcal{C}(\mathfrak{S}_I ; \{\text{transpositions}\})$$

$$\mathbf{s} : \mathbb{T}_I \ni t \longmapsto (N^{(t)}, B^{(t)})$$: seed pattern

$$\rightsquigarrow \mathcal{X}_{\mathbf{s}} = \bigcup_{t \in \mathbb{T}_I} \mathcal{X}_{(t)}$$

$$\rightsquigarrow \tilde{\mathbf{s}} : \mathbb{E}_I \ni (t, \sigma) \longmapsto \Big(\underset{\bigoplus_{i \in I} e_i^{(t,\sigma)}}{\underset{\|}{N^{(t,\sigma)}}} , \underset{\left(b^{(t)}_{\sigma^{-1}(i), \sigma^{-1}(j)}\right)_{i,j \in I}}{\underset{\|}{B^{(t,\sigma)}}} \Big)$$

: labeled seed pattern

$$\rightsquigarrow \mathcal{X}_{\mathbf{s}} = \bigcup_{(t,\sigma) \in \mathbb{E}_I} \mathcal{X}_{(t,\sigma)} \qquad \Big(\bigcup_{\sigma \in \mathfrak{S}_I} \mathcal{X}_{(t,\sigma)} \simeq \mathcal{X}_{(t)} \Big)$$

$$(t,\sigma)\sim_{\mathrm{triv}}(t',\sigma') :\Longleftrightarrow \begin{cases}\bullet\ B^{(t,\sigma)}=B^{(t',\sigma')}\\ \bullet\ \forall\gamma:(t,\sigma)\to(t',\sigma') : \text{path in } \mathbb{E}_I :\end{cases}$$

$$\begin{tikzcd} \mathcal{X}_{(t,\sigma)} \arrow[r, "\mu_\gamma"'] \arrow[rr, bend right, "\mathrm{id}"'] & \mathcal{X}_{(t',\sigma')} \arrow[r, "\simeq"] & \mathcal{X}_{(t,\sigma)} \end{tikzcd}$$

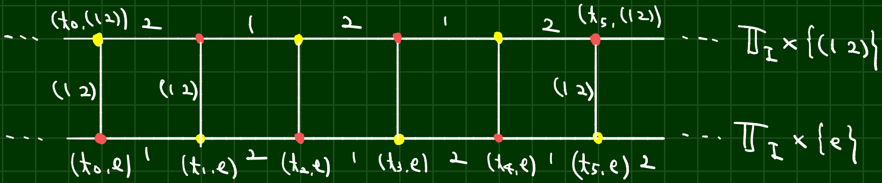
$\rightsquigarrow \mathrm{Exch}_s := \mathbb{E}_I/\sim_{\mathrm{triv}}$: labeled exchange graph.

$$\rightsquigarrow s_{\mathrm{Ex}} : \mathrm{Exch}_s \to \quad \underset{\substack{\downarrow \\ (t,\sigma)}}{v} \longmapsto \Big(N^{(v)},\ \underset{\substack{\| \\ B^{(t,\sigma)}}}{B^{(v)}}\Big)$$ "seed pattern"

$$\rightsquigarrow \mathcal{X}_s = \bigcup_{v\in \mathrm{Exch}_s} \mathcal{X}_{(v)}$$

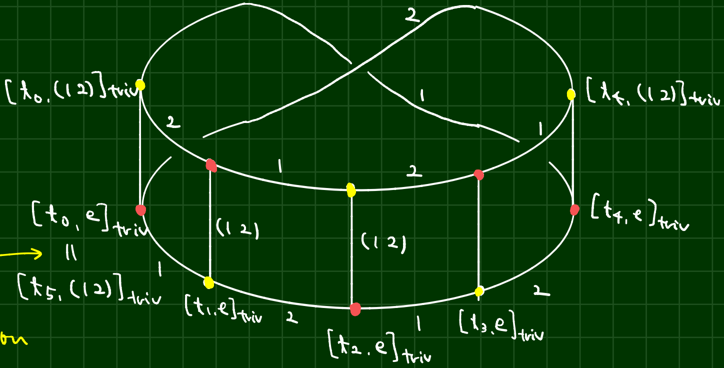
Example: A_2 quiver

$\mathbb{E}_I$:



$\bullet$: $1 \longleftarrow 2$, $\bullet$: $1 \longrightarrow 2$.

$\mathbb{E}\mathrm{xch}_s$:



pentagon relation

$\Sigma = (\Sigma, P)$: a punctured surface (without boundary comp.)

the set of punctures

s.t. $\chi(\Sigma) < 0$, $\ldots$

Δ_0 : triangulation of Σ : fix

$\rightsquigarrow \ s_\Sigma : \mathbb{T}_\Sigma \to \underset{\text{fix}}{t_0} \longmapsto (N^{\Delta_0}, B^{\Delta_0}) \rightsquigarrow \mathbb{E}\mathrm{xch}_\Sigma$

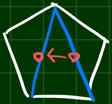
$\mathbb{T}\mathrm{ri}^{\bowtie}(\Sigma)$: the graph of

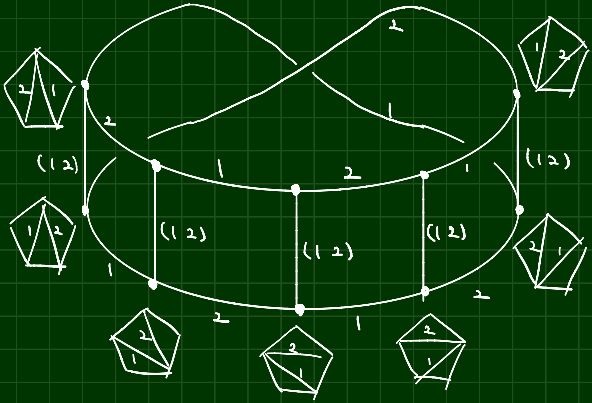
- vertices : labeled tagged triangulations
- edges : labeled flips or actions of transpositions for labelings.

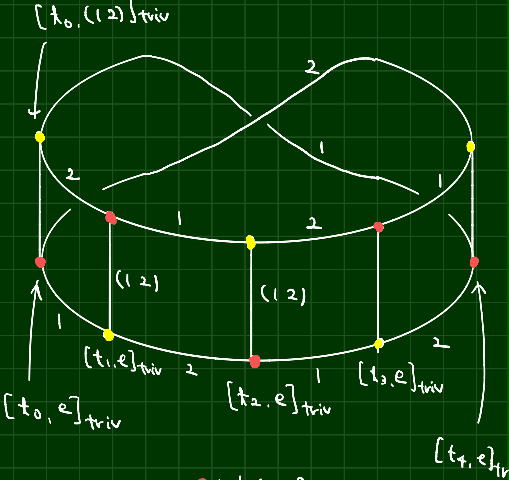
<u>Thm</u> (Fomin–Shapiro–Thurston, Fomin–Thurston)

$$\mathbb{T}\mathrm{ri}^{\bowtie}(\Sigma) \simeq \mathbb{E}\mathrm{xch}_\Sigma$$

Example

Σ = pentagon :  : A_2 quiver





● : $1 \leftarrow 2$

● : $1 \rightarrow 2$

Cluster modular group

$\mathbf{s}$: seed pattern $\rightsquigarrow \mathbb{E}\mathrm{xch}_{\mathbf{s}}$

$\Gamma_{\mathbf{s}} := \{\text{edge paths in } \mathbb{E}\mathrm{xch}_{\mathbf{s}}\} / \text{parallel transl.} \ldots$

· composition = concatenation of representation paths

Thm

∘ If Σ is neither a once-punctured torus nor a sphere w/ 4 punct's.

$$\Gamma_\Sigma \cong \begin{cases} MC(\Sigma) & \text{if } p=1 \\ MC(\Sigma) \ltimes (\mathbb{Z}/2)^p & \text{if } p>1 \end{cases}$$

tag change

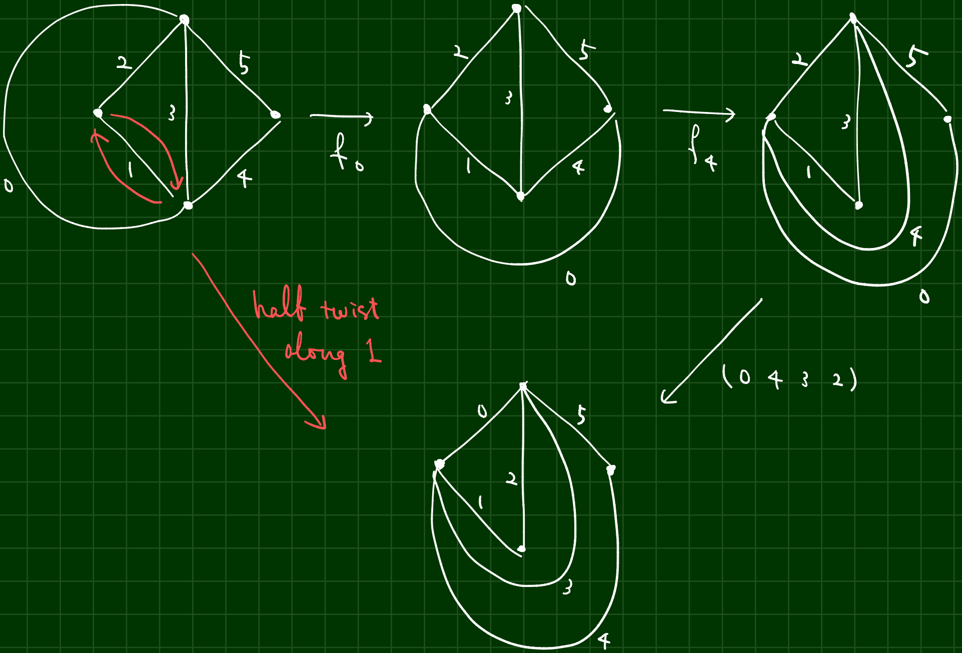
(∘ Σ is a once-punctured torus, $\Gamma_\Sigma \cong MC(\Sigma)/(\mathbb{Z}/2)$

hyperelliptic involution

∘ Σ is a sph. w/ 4 punct's, $\Gamma_\Sigma > MC(\Sigma)$

index ≥ 2.)

Example Σ = sph. w/ 4 punct's



§2. pA mapping classes.

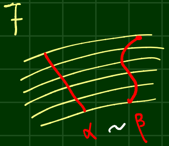
Measured foliations.

- $\mathcal{F} = (\mathcal{F}, \nu)$: a measured foliation on Σ

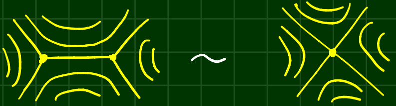
$:\Longleftrightarrow$ $\begin{cases} \cdot\ \mathcal{F} : \text{singular foliation on } \Sigma \\ \quad \text{s.t. } \{\text{singular points of } \mathcal{F}\} \supset P \end{cases}$

$\cdot\ \nu$: transverse measure of $\mathcal{F}$.

i.e. $\nu : \{\text{curves transv. to } \mathcal{F}\}/\sim \ \longrightarrow \mathbb{R}_{>0}$



$\mathcal{MF}(\Sigma) := \{\text{measured foliations on } \Sigma\} \big/ \text{homotopy, Whitehead move}$



Definition

$f : \Sigma \longrightarrow \Sigma$: homeo is pseudo-Anosov (pA)

$:\Longleftrightarrow \ \exists\ \mathcal{F}_f^+, \mathcal{F}_f^-$: measured foliations, $\exists\ \lambda_f > 1$

foli w/ transverse measure.

s.t. $\cdot\ \mathcal{F}_f^+ \pitchfork \mathcal{F}_f^-$

$\cdot\ f(\mathcal{F}_f^{\pm}) = \lambda_f^{\pm 1} \cdot \mathcal{F}_f^{\pm}$.

$\phi \in MC(\Sigma) = \{ f : \Sigma \to \Sigma : \text{ori. pres. homeo} \} / \text{homotopy}$

is pA if $\exists\ f : \Sigma \circlearrowleft$: pA homeo s.t. $\phi = [f]$.

Thm (Nielsen-Thurston classification)

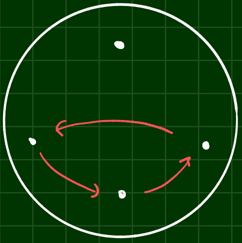
$\forall \phi \in MC(\Sigma)$ is classified into :

$\begin{cases} \cdot \text{ periodic } (:\Leftrightarrow \text{ finite order}) \\ \cdot \text{ reducible } (:\Leftrightarrow \text{ fix some curve system}) \\ \cdot \text{ pA.} \end{cases}$

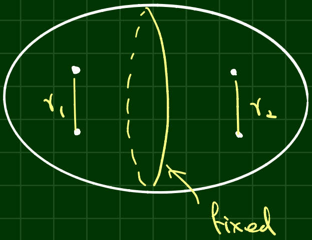
i.e. ϕ is not periodic nor reducible, then pA.

Example. Σ : a sphere w/ 4 punctures.

: half twist along γ.

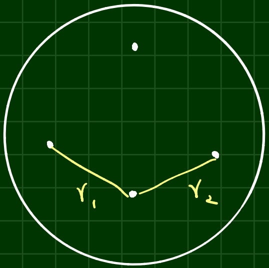


periodic



$$\phi = h_{\gamma_2} h_{\gamma_1}$$

reducible.



$$\phi = h_{\gamma_2}^{-1} h_{\gamma_1}$$

pA.

§3 Dynamics on the space of measured foliations

- $\exists\ \mathbb{R}_{>0} \curvearrowright \mathcal{MF}(\Sigma) \rightsquigarrow \mathcal{PMF}(\Sigma) := (\mathcal{MF}(\Sigma) \setminus \{0\}) / \mathbb{R}_{>0}$.

$$\mathcal{MF}(\Sigma) \ni \mathcal{F}, \qquad \mathcal{PMF}(\Sigma) \ni [\mathcal{F}]$$

$$\overline{\mathcal{T}(\Sigma)} := \mathcal{T}(\Sigma) \sqcup \underbrace{\mathcal{PMF}(\Sigma)}_{\partial \mathcal{T}(\Sigma)} \ : \text{Thurston compactification}$$

$$\overline{\mathcal{T}(\Sigma)} \curvearrowleft MC(\Sigma)$$

<u>Thm</u> [Nielsen–Thurston]

ϕ is periodic $\iff$ $\exists$ fixed pt $\in \mathcal{T}(\Sigma)$

reducible $\iff$ $\exists$ "rational" fixed pt $\in \partial\mathcal{T}(\Sigma)$

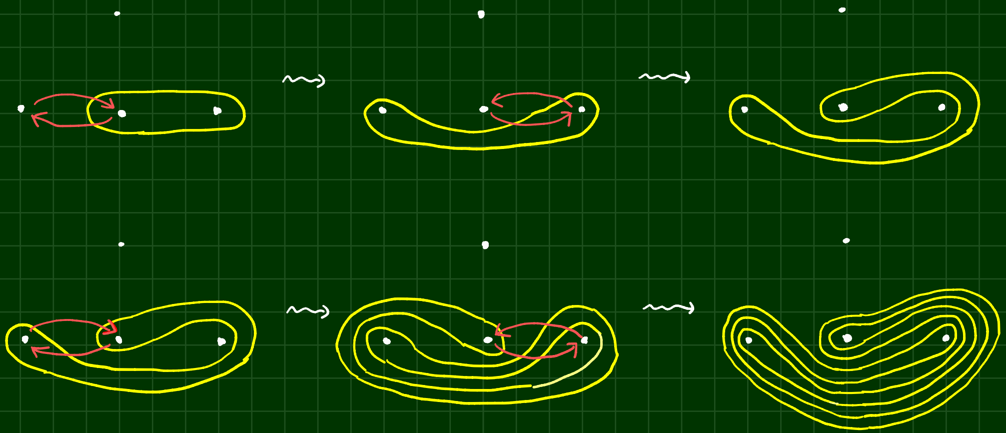
pA $\iff$ $\nexists$ fixed pt $\in \mathcal{T}(\Sigma)$, $\exists_2$ fixed pts $\in \partial \mathcal{T}(\Sigma)$

More precisely, ϕ : pA $\iff$ $\phi \curvearrowright \partial\mathcal{T}(\Sigma)$: NS dynamics i.e. $\exists\, x_\phi^\pm \in \partial\mathcal{T}(\Sigma)$. $\forall\, x \in \partial\mathcal{T}(\Sigma)\setminus\{x_\phi^\mp\}$:

$[\mathcal{F}_\phi^\pm]$

$$\lim_{n\to\infty} \phi^{\pm n}(x) = x_\phi^\pm .$$

e.g. Σ = a sphere w/ 4-punctures

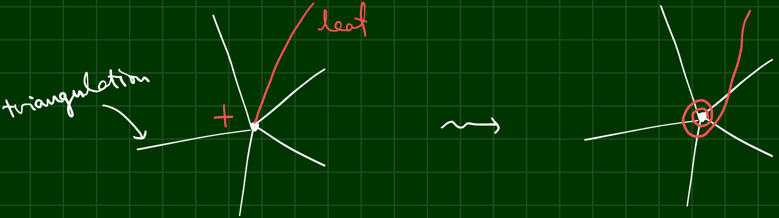


$\rightsquigarrow \mathcal{F}_\phi^+$

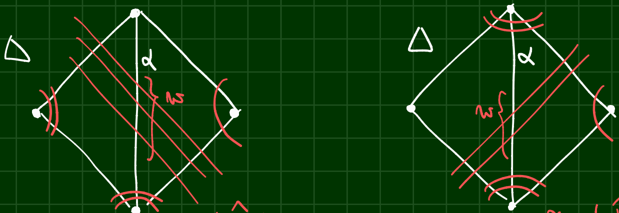
$\mathcal{X}$ - laminations

$\mathcal{L}^x(\Sigma, \mathbb{Q})$

$$:= \left\{ \hat{L} = \left(\bigsqcup_j w_j l_j , \sigma \right) \;\middle|\; \begin{array}{l} l_j : \text{pairwise disjoint} \\ \qquad \text{closed curve or ideal curve.} \\ w_j \in \mathbb{Q}_{>0}, \quad \sigma \in \{+,-,0\}^P \end{array} \right\} \Big/ \sim$$



shear coordinate :



$x_\alpha(\hat{L}) = +w$ $\qquad$ $x_\alpha(\hat{L}) = -w$

Prop (Fock-Goncharov)

$$\begin{tikzcd}
\mathcal{L}^x(\Sigma,\mathbb{Q}) \arrow[r, "\sim", "(x_\alpha)_{\alpha\in\Delta}"'] \arrow[d, hook] & \mathbb{Q}^\Delta \arrow[d, hook, "\text{completion}"] \\
\mathcal{L}^x(\Sigma,\mathbb{R}) \arrow[r, "\sim"] & \mathbb{R}^\Delta
\end{tikzcd}$$

$(x_\alpha)_{\alpha\in\Delta} : \mathcal{L}^x(\Sigma,\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}^\Delta$: bij

$\mathcal{L}^x(\Sigma,\mathbb{R}) \ni$ $\mathcal{X}$-lamination

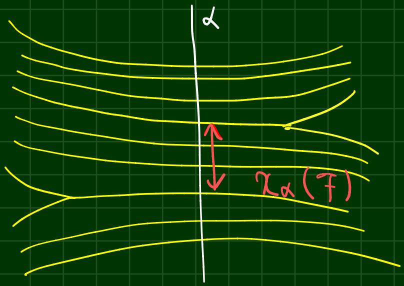
$\mathcal{L}^x(\Sigma,\mathbb{R}) \cong \mathcal{X}_\Sigma(\mathbb{R}^{trop})$

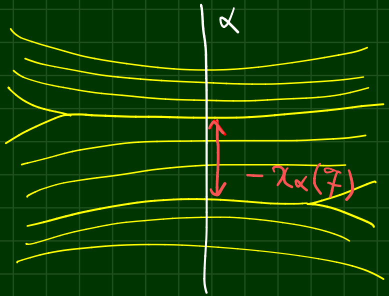
$$\begin{tikzcd}
 & \mathcal{L}^x(\Sigma;\mathbb{R}) \arrow[dl, "(x_\alpha)_\alpha"'] \arrow[dr, "(x_{\alpha'})_{\alpha'}"] & \\
\mathbb{R}^\Delta \arrow[rr, "\text{trop. cluster } \mathcal{X}\text{-transf.}"'] \arrow[rr, phantom, bend left=20, "\circlearrowleft"] & & \mathbb{R}^{f_k\Delta}
\end{tikzcd}$$

$\mathrm{Exch}_\Sigma \cong \mathrm{Tri}^{\bowtie}(\Sigma)$,

$$x_{\alpha'}(\hat{L}) = \begin{cases} -x_k(\hat{L}) & : \alpha = k \\ x_\alpha(\hat{L}) + [\operatorname{sgn}(x_k(\hat{L}))\, b^\Delta_{\alpha k}]_+\, x_k(\hat{L}) & : \alpha \neq k. \end{cases}$$

- shear coordinate for measured foliations





$\rightsquigarrow \mathcal{MF}(\Sigma) \hookrightarrow \mathcal{X}_\Sigma(\mathbb{R}^{trop})$.

codim $= \# P$

- For $\phi \in MC(\Sigma)$: pA. $\mathcal{F}_\phi^\pm$ are arational

  i.e. $\nexists$ saddle connection

  a leaf connecting singular pts

Prop

$\mathcal{F} \in \mathcal{MF}(\Sigma)$: arational

$\Longleftrightarrow$ $\mathcal{F} \in \mathcal{X}_\Sigma(\mathbb{R}^{trop})$: $\mathcal{X}$-filling.

$\left(:\Longleftrightarrow \forall \Delta : \text{triangulation of } \Sigma,\ \forall \alpha \in \Delta ; \quad x_\alpha(\mathcal{F}) \neq 0. \right)$

$\rightsquigarrow$ strict sign of a path γ

§4. Main theorem

Thm (Ishibashi - K.)

Σ : punctured surface w/o. ∂

$\phi \in \Gamma_\Sigma$: pA $\left(:\Leftrightarrow \begin{array}{ccc} \Gamma_\Sigma & \to & MC(\Sigma) \\ \rotatebox{90}{$\in$}\ \phi & \mapsto & \underline{\phi} : \text{pA} \end{array} \right)$

TFAE :

(1) ϕ : pA

(2) $\phi \curvearrowright \mathbb{S}\mathcal{X}_\Sigma(\mathbb{R}^{trop})$: NS dynamics.

(3) For any representation path γ of ϕ,

γ is sign-stable on $\Omega_\Sigma^{\mathbb{Q}}$

$$\Omega_\Sigma^{\mathbb{Q}} = \mathbb{R}_{>0} \cdot \underbrace{\mathcal{X}_\Sigma(\mathbb{Q}^{trop})}_{\text{rational } \mathcal{X}\text{-laminations}}.$$

Sketch of proof

(1) $\Longleftrightarrow$ (2)

$$\begin{tikzcd}
\mathcal{MF}(\Sigma) \arrow[r, hook] \arrow[d, "\phi^u"] & \mathcal{X}_\Sigma(\mathbb{R}^{\mathrm{trop}}) \arrow[r, two heads] \arrow[d, "\phi^x"] & \mathbb{R}^P \arrow[d, "\sigma"] \\
\mathcal{MF}(\Sigma) \arrow[r, hook] & \mathcal{X}_\Sigma(\mathbb{R}^{\mathrm{trop}}) \arrow[r, two heads] & \mathbb{R}^P
\end{tikzcd}$$

σ : permutation of punctured

fin. order

$$\rightsquigarrow \; E_\gamma(\hat{L}) \sim \left(\begin{array}{c|c} E^u & * \\ \hline 0 & \sigma \end{array}\right).$$

pres. mat at $\hat{L} \in \mathcal{X}_\Sigma(\mathbb{R}^{\mathrm{trop}})$

(1) $\Longleftrightarrow$ (3) $\quad \gamma$: sign-stab on $\Omega_\Sigma^{\mathbb{Q}}$

$$\Longleftrightarrow \; \forall \hat{L} \in \mathcal{X}_\Sigma(\mathbb{Q}^{\mathrm{trop}}), \; \exists \, \boldsymbol{\varepsilon}_\gamma^{\mathrm{stab}} \in \{+,-\}^{h(\gamma)}$$

rationality of $\mathcal{F}_\phi^x$

$$\text{s.t. } \; \boldsymbol{\varepsilon}_\gamma(\phi^n \hat{L}) = \boldsymbol{\varepsilon}_\gamma^{\mathrm{stab}} \; \text{ for } n \gg 0.$$

Cor $\phi \in MC(\Sigma)$: pA

$$\Longrightarrow \quad \mathcal{E}^{a}_{\phi} = \mathcal{E}^{x}_{\phi} = \mathcal{E}^{top}_{\phi} = \log \lambda_{\phi}.$$

proof.

$$E^{\mathcal{E}}_{\gamma} \sim \begin{pmatrix} E^{u} & * \\ 0 & \sigma \end{pmatrix}, \qquad \check{E}^{\mathcal{E}}_{\gamma} := ((E^{\mathcal{E}}_{\gamma})^{T})^{-1} \sim \begin{pmatrix} E^{u} & 0 \\ 0 & \sigma \end{pmatrix}$$

pres. mat of ϕ on $\mathcal{X}_{\Sigma}(\mathbb{R}^{trop})$ pres. mat of ϕ on $\mathcal{A}_{\Sigma}(\mathbb{R}^{trop})$

$$\therefore \quad P(E^{\mathcal{E}}_{\gamma}) = P(E^{u}) \cdot P(\sigma) = P(\check{E}^{\mathcal{E}}_{\gamma})$$

$\rightsquigarrow$ palindromicity of $P(E^{\mathcal{E}}_{\gamma})$.

$\mathcal{X}_{\Sigma}(\mathbb{Z}^{trop})$

sign-stab on Ω^{can}_{Δ} $\Longleftrightarrow$ sign-stab on $\mathbb{R}_{\geq 0} \cdot \{ l_{\Delta} \} \subset \Omega^{\mathbb{Q}}_{\Sigma}$

Rmk

(1) $\varepsilon_\gamma(l_\Delta) =$ tropical sign of γ.

$\therefore$ tropical sign of representative of pA

is converges to $\underset{\sim}{\varepsilon_\gamma^{stab}}$

$\uparrow$ $\exists$ geometric meaning

(train track splitting)

(2) When Σ has boundary components

$\phi \in MC(\Sigma)$: pA $:\Longleftrightarrow$ $\phi|_{\bar{\Sigma}}$ is pA

$\bar{\Sigma} = \Sigma \setminus \overline{\text{collar nbd of } \partial\Sigma}$

$\Longrightarrow$ ϕ is weakly sign-stable on $\mathcal{X}_\Sigma(\mathbb{R}^{trop}) \setminus \Delta_\phi$

(In this case aradional $\cancel{\Longleftrightarrow}$ $\mathcal{X}$-filling)

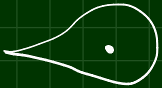
§5 Train track

A train track "=" A combinatorial model of measured foliations

$\tau \subset \Sigma$: "smoothly" embedded (trivalent) graph

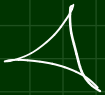
is a (complete) train track

$:\Longleftrightarrow$ Each component of $\Sigma \setminus \tau$ is



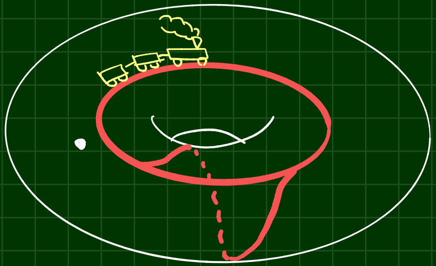
once punctured
monogon

or

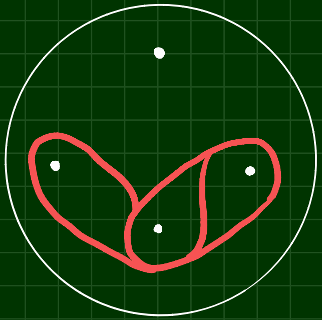


unpunctured
triangle

Examples of train tracks



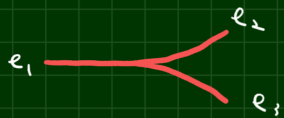
once punctured torus



sphere w/ 4-punctures

For a train track τ, its measure is a map

$\nu : \{\text{edges of } \tau\} \longrightarrow \mathbb{R}_{\geq 0}$ s.t.



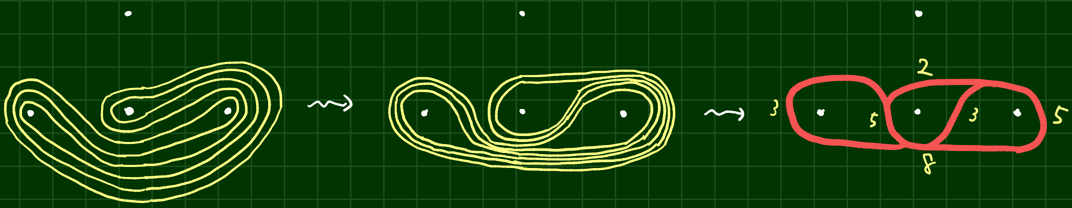
$$\nu(e_1) = \nu(e_2) + \nu(e_3)$$

(measured foliation)

$$\tau > \mathcal{F} \; :\Longleftrightarrow \; \exists\, r : \Sigma \setminus \{\text{singular pts of } \mathcal{F}\} \searrow \tau .$$

$\mathcal{F} = (\mathcal{F}, \nu)$

s.t. ... "smoothness"



For $\underset{\substack{\Downarrow\\ p}}{e} \subset \tau$: edge, $\nu(r^{-1}(p))$ gives a measure of τ.

$V(\tau) := \{\text{measures of } \tau\} \simeq \{\mathcal{F} \mid \tau > \mathcal{F}\} \subset \mathcal{MF}(\Sigma)$.

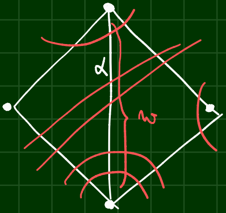
<u>Thm</u> [Thurston]

$\{V(\tau) \mid \tau : \text{recurrent complete train tracks}\}$.

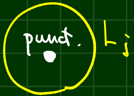
gives a PL atlas of $\mathcal{MF}(\Sigma)$.

$\mathcal{A}$- laminations

$$\mathcal{L}^a(\Sigma, \mathbb{Q}) := \left\{ L = \bigsqcup_j w_j L_j \;\middle|\; \begin{array}{l} L_j : \text{pairwise disjoint closed curves} \\ w_j \in \mathbb{Q}^\times, \quad w_j > 0 \text{ if } L_j \text{ is not peripheral} \end{array} \right\} \Big/ \sim$$



$$a_\alpha(L) := \frac{w}{2}$$



Prop [Fock–Goncharov]

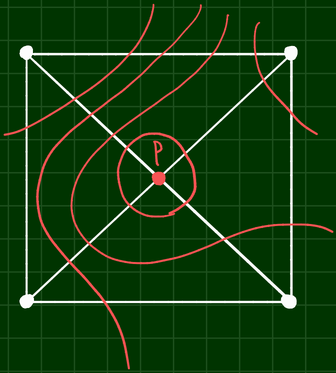
$$\begin{tikzcd} (a_\alpha)_{\alpha \in \Delta} : \mathcal{L}^a(\Sigma, \mathbb{Q}) \arrow[r, "\sim"] \arrow[d, hook] & \mathbb{Q}^\Delta \arrow[d, hook, "\text{completion}"] \\ \mathcal{A}_\Sigma(\mathbb{R}^{trop}) \simeq \mathcal{L}^a(\Sigma, \mathbb{R}) \arrow[r, "\sim"] & \mathbb{R}^\Delta \end{tikzcd}$$

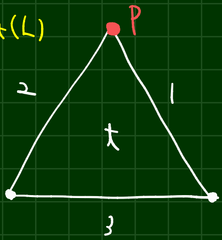
(Tropicalized) Goncharov - Shen potential

$$W_\Delta : \mathcal{A}_\Delta(\mathbb{R}^{\mathrm{trop}}) \longrightarrow \mathbb{R}^P$$

$$\begin{array}{ccc} W_\Delta & & \mathcal{A}_\Delta(\mathbb{R}^{\mathrm{trop}}) \\ \| & & \rotatebox{90}{$\in$} \\ (W_{\Delta,p})_{p\in P} & & L \longmapsto \min\limits_{t}\Big\{ \underbrace{a_1(L)+a_2(L)-a_3(L)}_{= w_t(L)} \Big\}. \end{array}$$

$t \in$ triangles of Δ.





$$W_{\Delta,p}(L)$$

$$= 2\left(\text{peripheral weight around } p \in P\right).$$

$$\rightsquigarrow \mathcal{A}_\Sigma(\mathbb{R}^{\mathrm{trop}}) \supset W^{-1}(0) \simeq \mathcal{MF}(\Sigma).$$

$$\mathrm{Conf}_{\Delta,P} := \left\{ \underline{t} = (t_p)_p \in \prod_{p \in P} \{\text{triangles of } \Delta \text{ around } p\} \;\middle|\; t_{p_1} \neq t_{p_2} \right\}.$$

$$\underline{t} \rightsquigarrow V_\Delta(\underline{t}) := \left\{ L \in W^{-1}(0) \;\middle|\; W_{\Delta,p}(L) = w_{t_p}(L) \right\}.$$

Prop

τ : complete train track, $\exists\, \underline{t} \in \mathrm{Conf}_{\Delta,P}$

$$\begin{array}{ccc} \mathcal{U}\mathcal{F}(\Sigma) & \simeq & W^{-1}(0) \\ \cup & & \cup \\ V(\tau) & \simeq & V_\Delta(\underline{t}) \end{array}$$

cluster algebraic interpretation of train tracks.

Splitting sequence of train tracks

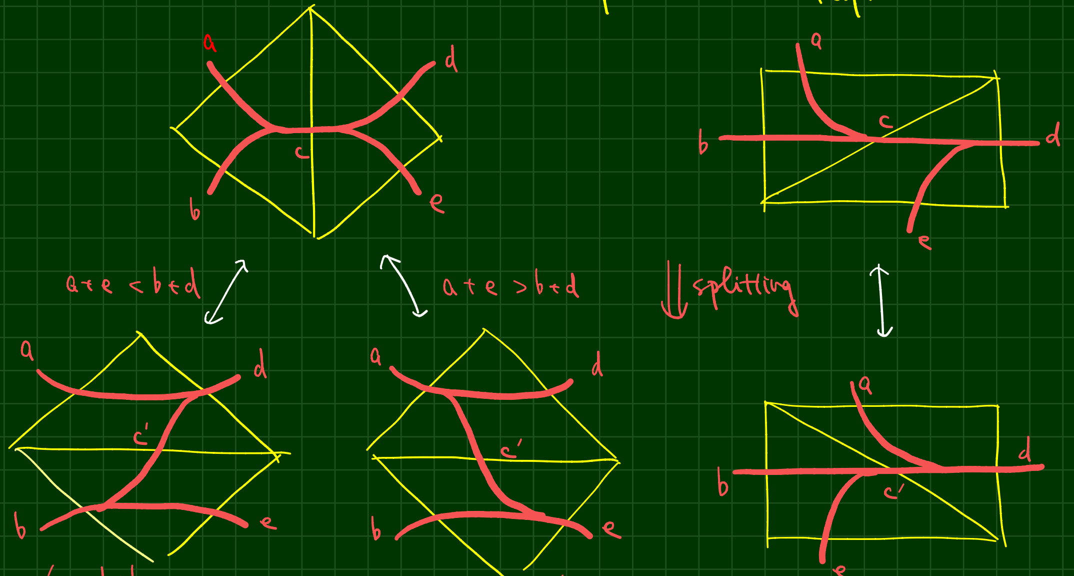
$\phi \in MC(\Sigma)$: pA $\rightsquigarrow$ $\mathcal{F}^+_\phi \in \mathcal{MF}(\Sigma)$.

Take $\tau > \mathcal{F}^+_\phi$ s.t. $\tau > \phi(\tau)$ (existence of suitable retraction)

$$\rightsquigarrow \quad \begin{array}{ccccc} \mathcal{MF}(\Sigma) & = & \mathcal{MF}(\Sigma) & \xrightarrow{\phi_*} & \mathcal{MF}(\Sigma) \\ \cup & & \cup & & \cup \\ V(\phi\tau) & \hookrightarrow & V(\tau) & \longrightarrow & V(\phi\tau) \\ \Psi & & \Psi & & \Psi \\ \mathcal{F} & \longmapsto & \mathcal{F} & \longmapsto & \phi(\mathcal{F}) \end{array}$$

decomposable into elementary moves!

- Elementary moves of a train track correspond to a flip.



$c' = d+b-c$

$c' = a+e-c$

tropical cluster $\mathcal{A}$-transf.

Thm (Penner-Papadopoulos, Agol)

$\forall\, \phi \in MC(\Sigma)$: pA, $\exists\, \tau \subset \Sigma$: train track

s.t. $\tau \rightsquigarrow \phi(\tau)$

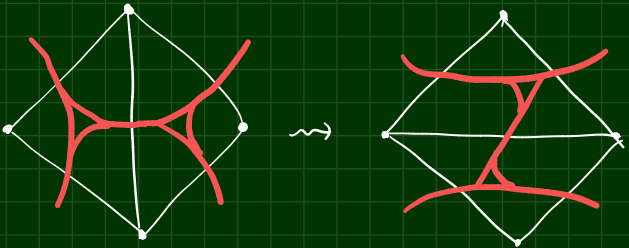
sequence of splittings.

RLS sequence (Penner-Papadopoulos)

maximal splitting seq. (Agol)

Any relationships between stable sign & this sequences?

Problem : in gen.



__Thm__ (K. in prep.)

$\exists\ \Sigma^\circ = (\Sigma, P \cup P^\circ)$.

$\forall\ \phi \in MC(\Sigma)$, $\exists\ \Delta^\circ$: tri. of Σ°. $\exists\ \delta : \Delta^\circ \to \Delta_i^\circ$

$$
\begin{tikzcd}
\mathcal{A}_{\phi(\Delta)}(\mathbb{R}^{trop}) \arrow[rr, hook] \arrow[dd, "\mu_\gamma"'] \arrow[dr, phantom, "\cup"] & & \mathcal{A}_{\phi(\Delta^\circ)}(\mathbb{R}^{trop}) \arrow[r, "\mu_\delta"] & \mathcal{A}_{\phi(\Delta_i^\circ)} \arrow[dd, "\mu_{\gamma^\circ}"] \arrow[dr, phantom, "\cup"] & \\
& V(\phi\tau) \arrow[rrr] \arrow[dd, "E_\gamma^{stab}"] & & & \oslash \arrow[dd, "\text{splitting sequence}"] \\
\mathcal{A}_{\Delta}(\mathbb{R}^{trop}) \arrow[rr, hook] \arrow[dr, phantom, "\cup"] & & \mathcal{A}_{\Delta^\circ}(\mathbb{R}^{trop}) \arrow[r, "\mu_\delta"'] & \mathcal{A}_{\Delta_i^\circ} \arrow[dr, phantom, "\cup"] & \\
& V(\tau) \arrow[rrr] & & & \oslash
\end{tikzcd}
$$

Idea of proof