2024-03-13 Advances in Cluster Algebra 2024
@ Nagoya Univ

Skein and Cluster algebras with coefficients for unpunctured surfaces

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joint work with
Tsukasa Ishibashi and Shunsuke Kano
(arXiv : 2312.02861 + ongoing projects)

oriented, connected, triangulable

Theorem. (Muller '16) Σ : a marked surface without punctures
If # marked points ≥ 2 $\Rightarrow$ $\mathscr{S}^q_\Sigma[\partial^{-1}] = \mathscr{A}^q_\Sigma$ in $\mathrm{Frac}\,\mathscr{S}^q_\Sigma$

the Kauffman bracket skein algebra localized by boundary arcs

the quantum cluster algebra associated with Δ of Σ

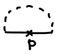
$\boxed{1}$ Review : $\mathscr{S}^q_\Sigma[\partial^{-1}] = \mathscr{A}^q_\Sigma$ (coefficient-free case)

- the Kauffman bracket skein algebra

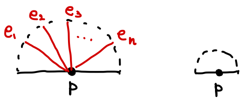
$\mathbb{Z}_q := \mathbb{Z}[q^{\pm 1/2}]$
Σ : a marked surface
$M \subset \partial\Sigma$: a set of marked points

Def. tangle diagram on Σ :$\Leftrightarrow$ local diagram at $p \in \Sigma$ is the one of the following

$(p \in \Sigma \setminus \partial\Sigma)$



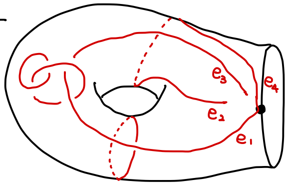
$(p \in \partial\Sigma \setminus M)$



$(p \in M)$

with an order of half edges
$e_1, e_2, \cdots, e_n$

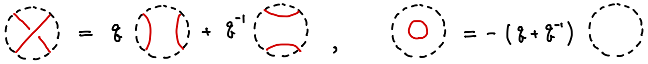
e.g



$e_1 = e_4$ ↑ high
e_3
e_2 ↓ low

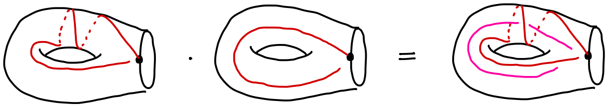
Def. $\mathcal{S}^q_\Sigma = \mathbb{Z}_q\{\text{tangle diagrams on } \Sigma\} \Big/$ • isotopy relative to $\partial\Sigma$
• skein relation

- the Kauffman bracket skein rel.



- clasped skein rel. (Jones-Wenzl)

- multiplication

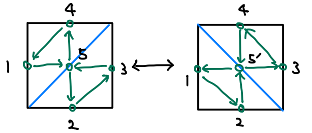


- multi-curves (disjoint union of simple curves) make a basis of $\mathcal{S}^q_\Sigma$.

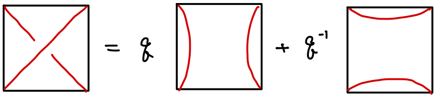
▨ One can construct q-seeds $(s^\Delta_\Sigma)_{\Delta\in Tri(\Sigma)}$ in $\mathrm{Frac}\,\mathcal{S}^q_\Sigma$

- $A^\Delta := \{\gamma \in \Delta\}$: a cluster in $\mathrm{Frac}\,\mathcal{S}^q_\Sigma$
- compatibility and q-exchange matrices obtained from Δ is consistent with skein relations

mutation = flip



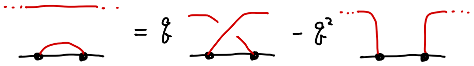
q-exchange rel.



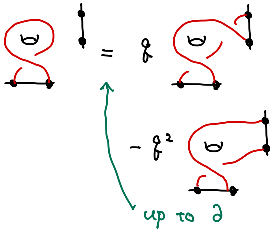
$A_5 A_{5'} = q A_1 A_3 + q^{-1} A_2 A_4$

Fact $\mathcal{S}^q_\Sigma$ is an Ore domain $\rightsquigarrow$ $\mathcal{A}^q_\Delta \subset \mathcal{S}^q_\Sigma[\partial^{-1}] \subset \mathrm{Frac}\,\mathcal{S}^q_\Sigma$

Lemma (sticking trick)



e.g.



$\rightsquigarrow$ $\forall\, \alpha \in \mathcal{S}^q_\Sigma$, $\exists$ monomial x in ∂

s.t. $x\alpha$ = polynomial of ideal arcs (simple) $\in \mathcal{A}^q_\Delta$

i.e. $\mathcal{S}^q_\Sigma[\partial^{-1}] \subset \mathcal{A}^q_\Delta$ ▨

Theorem (IKY 2023) $\# M \geq 2$. $\mathcal{W}$: taut.

If $C_{\mathfrak{j}} := \ell^{-1}(\mathfrak{j})$ $(\mathfrak{j} \in J)$ is a multi-curve. then a multi-lamination $\mathbb{L}(\mathcal{W}) = (L_{\mathfrak{j},\varepsilon})_{\mathfrak{j} \in J, \varepsilon \in \{\pm\}}$ with $L_{\mathfrak{j},\varepsilon} = \bigcup_{\xi \in C_{\mathfrak{j}}} \gamma_{\xi,\varepsilon}$ on Σ realizes $\mathcal{S}^{q}_{\Sigma,\mathcal{W}}[\partial^{-1}]$.

i.e. $\mathcal{S}^{q}_{\Sigma,\mathcal{W}}[\partial^{-1}] = \mathscr{A}^{q}_{\Sigma,\mathcal{W}} = \mathscr{A}^{q}_{\Sigma,\mathbb{L}(\mathcal{W})}$.

Theorem (IKY 2023) $\# M \geq 2$, ${}^{\forall}\mathbb{L}$: a multi-lamination on Σ, there exists a wall system $\mathcal{W}(\mathbb{L})$ s.t. $\mathcal{S}^{q}_{\Sigma,\mathcal{W}(\mathbb{L})}|_{Z_-=1}[\partial^{-1}] = \mathscr{A}^{q}_{\Sigma,\mathbb{L}}$

Remark The coefficients of $\mathscr{A}^{q}_{\Sigma,\mathbb{L}}$ is "normalized"

$\rightsquigarrow$ If $C_{\mathfrak{j}}$ is a multi-curve $({}^{\forall}\mathfrak{j} \in J)$. then $\mathcal{S}^{q}_{\Sigma,\mathcal{W}}$ has normalized coefficients.

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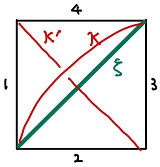
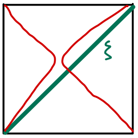
e.g. (exchange relation) ξ : a wall $\longrightarrow$ coefficients $\{Z_{\xi,\pm}\}$

$\ell(\xi) = \mathfrak{j}$ $(J = \{\mathfrak{j}\})$

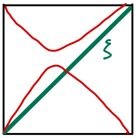
$KK' =$



$= q$

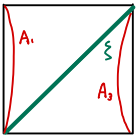


$+ q^{-1}$

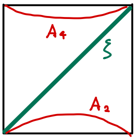


minimal position with respect to ξ

$= q Z_{\mathfrak{j},+}$



$+ q^{-1} Z_{\mathfrak{j},-}$



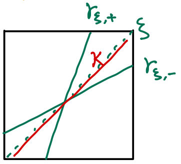
$$(P^{\Delta}_{\mathcal{W}})^{+}_{\alpha} := \prod_{\mathfrak{j} \in J} \prod_{\xi \in C_{\mathfrak{j}}} Z_{\mathfrak{j},+}^{\overset{-1}{\overset{\shortparallel}{[x^{\Delta}_{\alpha}(\gamma_{\xi,+})]_{+}}}} Z_{\mathfrak{j},-}^{\overset{+1}{[x^{\Delta}_{\alpha}(\gamma_{\xi,-})]_{+}}} = Z_{\mathfrak{j},-}$$

$$(P^{\Delta}_{\mathcal{W}})^{-}_{\alpha} := \prod_{\mathfrak{j} \in J} \prod_{\xi \in C_{\mathfrak{j}}} Z_{\mathfrak{j},+}^{\overset{1}{[-x^{\Delta}_{\alpha}(\gamma_{\xi,+})]_{+}}} Z_{\mathfrak{j},-}^{\overset{-1}{[-x^{\Delta}_{\alpha}(\gamma_{\xi,-})]_{+}}} \,\text{"}\, Z_{\mathfrak{j},+}$$

$$= q \underbrace{Z_{\mathfrak{j},+}}_{(P^{\Delta}_{\mathcal{W}})^{-}_{K}} A_1 A_3 + q^{-1} \underbrace{Z_{\mathfrak{j},-}}_{(P^{\Delta}_{\mathcal{W}})^{+}_{K}} A_2 A_4 \quad \xrightarrow[q=1,\ Z_{\xi,-}=1]{} \quad A_{\gamma} A_{\gamma'} = Z_{\mathfrak{j},+} A_1 A_3 + A_2 A_4$$

$$(P^{\Delta}_{\mathcal{W}})^{-}_{K} \oplus (P^{\Delta}_{\mathcal{W}})^{+}_{K} = 1$$

Corresponding multi-lamination :



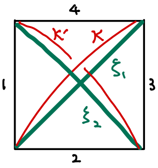
$\mathbb{L}(\mathcal{W}) = \{\gamma_{\xi,+}\} \sqcup \{\gamma_{\xi,-}\}$: multi-lamination

$x^{\Delta}_{K}(\gamma_{\xi,+}) = -1$

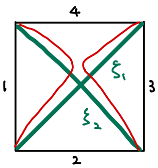
$x^{\Delta}_{K}(\gamma_{\xi,-}) = +1$

e.g. (non-normalized case) $C = \{\xi_1, \xi_2\}$, $J = \{j\}$, $\ell(\xi_1) = \ell(\xi_2) = j$

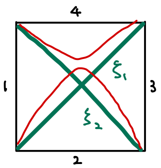
$\kappa\kappa' =$



$= q$



$+ q^{-1}$



$x^{\Delta}_{\kappa}(\xi_{1,+}) = -1$

$x^{\Delta}_{\kappa}(\xi_{1,-}) = +1$

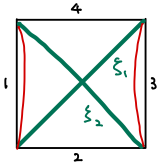
$x^{\Delta}_{\kappa}(\xi_{2,+}) = +1$

$x^{\Delta}_{\kappa}(\xi_{2,-}) = -1$

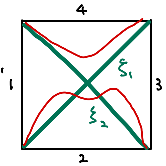
$(p^{\Delta}_{w,\kappa})^{-} = Z_{j,+} Z_{j,-}$

$(p^{\Delta}_{w,\kappa})^{+} = Z_{j,-} Z_{j,+}$

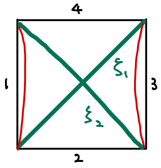
$= q Z_{j,+} Z_{j,-}$



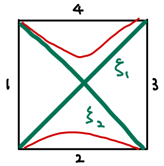
$+ q^{-1}$



$= q Z_{j,+} Z_{j,-}$



$+ q^{-1} Z_{j,+} Z_{j,-}$



$$\rightsquigarrow A_\kappa A_{\kappa'} = q \underbrace{Z_{j,+} Z_{j,-}}_{(p^{\Delta}_{w;\kappa})^{-}} A_1 A_3 + q^{-1} \underbrace{Z_{j,+} Z_{j,-}}_{(p^{\Delta}_{w;\kappa})^{+}} A_2 A_4 \rightsquigarrow A_\kappa A_{\kappa'} = Z_{j,+} A_1 A_3 + Z_{j,+} A_2 A_4$$

$(p^{\Delta}_{w;\kappa})^{-} \oplus (p^{\Delta}_{w;\kappa})^{+} = Z_{j,+} Z_{j,-}$

$Z_{j,+} \oplus Z_{j,+} = Z_{j,+} \neq 1$

non-normalized

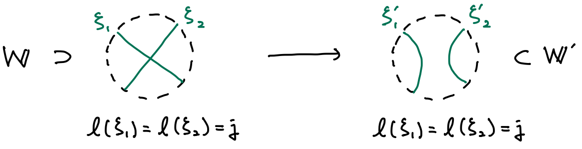
$\boxed{4}$ Quasi-homomorphisms from resolution of crossings

$\parallel$

“$\mathbb{Z}_q\mathbb{P}$-algebra homomorphism $\Psi : \mathcal{A}_S \to \mathcal{A}_{S'}$ which rescales coefficients.”

(See Fraser '16 for details)

Let $W = (C, J, \ell)$ be a wall system, and assume that $C_j = \ell^{-1}(j)$ has a crossing. We define $W' = (C', J, \ell')$ by the following deformation :



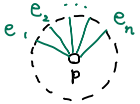
(ξ_1 and ξ_2 , ξ'_1 and ξ'_2 may become same wall)

__Theorem__ (IKY. 2023)

$\Psi : \mathcal{S}^{q}_{\Sigma,W} \longrightarrow \mathcal{S}^{q}_{\Sigma,W'}$, $\Psi([\alpha]_W) = [\alpha]_{W'}$ is a well-defined $\mathbb{Z}_{q,W}$-algebra homomorphism. Moreover, it is a quasi-homomorphism through $\mathcal{S} = \mathcal{A}$ if W and W' are taut, and $\#M \geq 2$.

$\boxed{5}$ generalization of wall systems

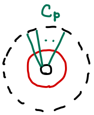
① Punctured surface : Let us consider walls incident to puncture.



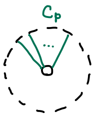
$C_p := \{e_1, e_2, \dots, e_n\}$

- P : the set of punctures
- $R_W := \mathbb{Z}_{q,w}[v_p^{\pm 1} \mid p \in P] \big/ \langle \prod_{e \in C_p} z_{\ell(e),+} - \prod_{e \in C_p} z_{\ell(e),-} \mid p \in P \rangle$

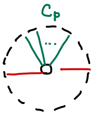
● skein relation at $p \in P$



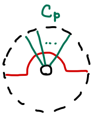
$= Z_p\,(q + q^{-1})$



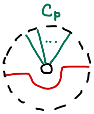
where $Z_p = \prod_{e \in C_p} z_{\ell(e),+} - \prod_{e \in C_p} z_{\ell(e),-}$



$= v_p^{-1} \Big(q^{\frac{1}{2}}$

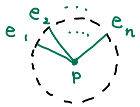


$+\ q^{-\frac{1}{2}} Z_p$



$\Big)$

② Branched walls : we allow endpoints of walls to be interior of Σ. and it may share the same point.

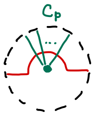


$C_p = \{e_1, \dots, e_n\}$

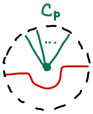
$V(W)$: the set of endpoints of walls in $\Sigma \setminus \partial\Sigma$.

$R_W := \mathbb{Z}[q^{\pm 1/2}, z_{j,\pm}^{\pm 1/2} \mid j \in J]$

● skein relation at $p \in V(W)$

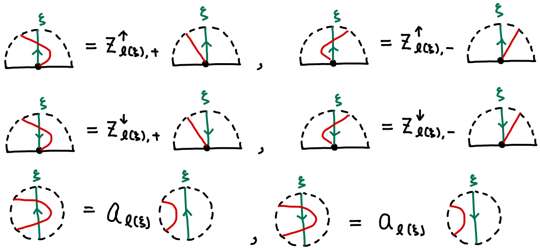


$= \prod_{e \in C_p} z_{\ell(e),+}^{\frac{1}{2}} z_{\ell(e),-}^{-\frac{1}{2}}$

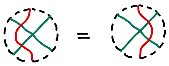


③ Oriented walls : $R := \mathbb{Z}[q^{\pm 1/2}, (Z^{\uparrow}_{j,\pm})^{\pm 1}, (Z^{\downarrow}_{j,\pm})^{\pm 1}] \big/ \langle Z^{\uparrow}_{j,+}Z^{\uparrow}_{j,-} - Z^{\downarrow}_{j,+}Z^{\downarrow}_{j,-} \mid j \in J \rangle$

- skein relation



where $a_j := Z^{\uparrow}_{j,+}Z^{\uparrow}_{j,-} = Z^{\downarrow}_{j,+}Z^{\downarrow}_{j,-}$



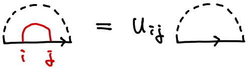
for any orientation of walls

Remark $\exists\ \mathcal{S}^{q}_{\Sigma,W} \hookrightarrow \mathcal{S}^{q}_{\Sigma,\vec{W}}$ where W is an (unoriented) wall system
$\vec{W}$ is double of W :

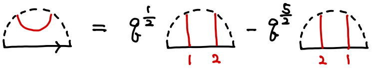
6 Stated skein algebra & Quantum trace (in progress)

Bonahon - Won ('11) constructed a quantum trace map using the stated skein algebra :

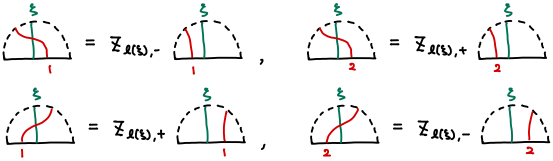
- stated skein relation :



$(u_{ij}) = \begin{pmatrix} 0 & -q^{-\frac{5}{2}} \\ q^{-\frac{1}{2}} & 0 \end{pmatrix}$



Definition (stated skein relation of a walled surface)



- We are going to construct $\mathrm{Tr}^{\Delta}_{q,W} : \mathcal{S}^{q,\text{stated}}_{\Sigma,W} \longrightarrow$ "Chekhov - Fock algebra" ...
- matrix formula, geometric model of $\mathcal{A}^{q}_{\Sigma,W}$, etc.

$\boxed{7}$ Other direction ...

- Lee - Schiffler ('19) showed that a specialization of F-polynomials gave Jones polynomials of 2-bridge links

  $\rightsquigarrow$ Give a deeper understanding of relationship between F- and Jones polynomial via the skein theory.

- Ishibashi - Y. showed $\mathfrak{sl}_3$, $\mathfrak{sp}_4$-version of Muller's theorem, and I believe that I can give a definition of $\mathfrak{sl}_3$, $\mathfrak{sp}_4$-version of skein algebra of walled surface.

  $\rightsquigarrow$ However, we have no motivation ...