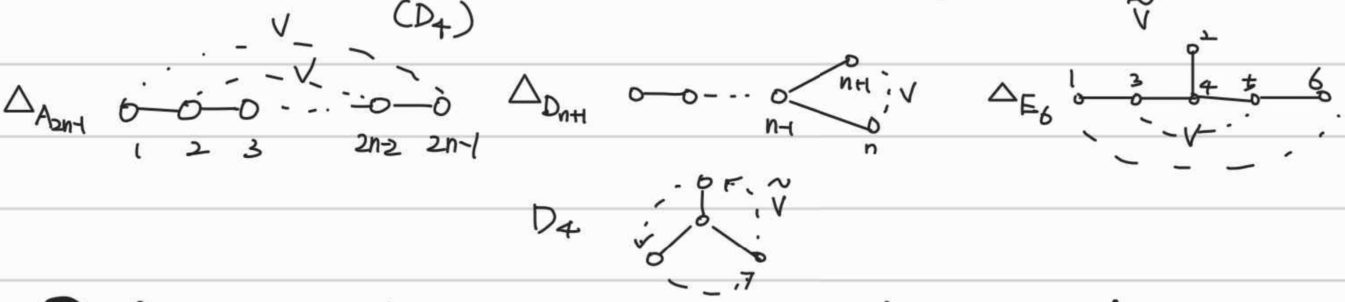
Quantum tori assoc. w. sequences & their application III. (beyond reduced sequence).

Today, $\mathfrak{g}$ is of finite type; i.e $A_n, B_n, C_n, D_{n+1} (n\geq 3), E_{6,7,8}, F_4, G_2$

A_{2n-1}, D_{n+1}, E_6 has non-trivial Dynkin diagram auto $\vee$ s.t. $c_{i,\vee(i)}=0$ $\forall i\in I$.
(D_4) $\tilde{\vee}$



Rmk • id is also Dynkin dia. auto. **Throughout this talk** $\sigma = \vee, \tilde{\vee}$ or id.

• BCFG & $E_{7,8}$ do not have non-trivial Dynkin dia. auto.

• We will use ▲ for Dyn dia of BCFG

• Coxeter elts & its generalization via σ.

$\mathfrak{g}$: f.d simple Lie alg with $I=\{1...n\}$

Def [Coxeter elts of $\mathfrak{g}$] $\tau\in W$ Coxeter elt if $\tau = s_{i_1}\cdots s_{i_n}$ s.t $\{i_1,\dots i_n\}=I$.

Let σ be a Dyn. dia auto on $\Delta_{\mathfrak{g}}$. I^σ = set of σ-orbits of I
$= \{\bar{i}_1 \dots \bar{i}_r\}$

Def [σ-Coxeter elt] $\tau\sigma \in W\rtimes\langle\sigma\rangle$ σ-Coxeter elt if $\tau = s_{i_1}\cdots s_{i_r}$ s.t
$\{\bar{i}_1, \bar{i}_2, \dots \bar{i}_r\} = I^\sigma$.

Rmk If $\sigma=$id, {σ-Coxeter elts} = {Coxeter elts}. (and $r=n$)

Ex $\mathfrak{g}=A_3$ • $s_1s_2s_3$, $s_3s_2s_1$, $s_1s_3s_2$, $s_2s_1s_3$ are Coxeter elts.

• $s_1s_2\vee$, $s_3s_2\vee$, $s_2s_1\vee$, $s_2s_3\vee$ are $\vee$-Coxeter elts.

• Beyond reduced sequences I.

Since $\mathfrak{g}$ is of finite type, $\mathbf{i}=(i_1,i_2,\dots)\in I^{\mathbb{Z}_{\geq 1}}$ can't be reduced.

- **Commutation moves** : For seqs $\mathbf{i}, \mathbf{j} \in I^{\mathbb{Z}_{\geq 1}}$ $\mathbf{j}$ can be obtained from $\mathbf{i}$ by a <u>commutation move</u> if $\exists\, k \geq 1$ s.t

$$i_k = j_{k+1}, \quad i_{k+1} = j_k, \quad c_{i_k, i_{k+1}} = 0 \quad \text{and} \quad i_s = j_s \quad \forall s \notin \{k, k+1\}.$$

- **Infinite seqs from τ** : Let σ-Cox elt $\tau\sigma = s_{j_1} \cdots s_{j_r} \sigma$ given. Define $\mathbf{i} = (i_1, i_2, \dots) \in I^{\mathbb{Z}_{\geq 1}}$ s.t

$$i_k = j_k \quad \text{for } 1 \leq k \leq r \quad \text{and} \quad i_t = \sigma(i_{t-r}) \quad \text{for } t > r.$$

We set $[\Delta^\sigma] = \{\mathbf{j} \mid \mathbf{j}$ can be obtained from $\mathbf{i}$ via comm. moves$\}$.

- **Quantum affine algebras**

$\Bbbk =$ alg. closed field containing $\bigcup_{m \geq 1} \mathbb{C}((q^{1/m}))$.

$U_q'(\hat{\mathfrak{g}})$: untwisted quantum affine alg corr to $\mathfrak{g}$. (Ex. $\mathfrak{g} = B_n \longrightarrow \hat{\mathfrak{g}} = B_n^{(1)}$).
(w/o degree operator)

Note. • $U_q'(\hat{\mathfrak{g}})$ has a Hopf alg. str. and admits non-trivial f.d rep'ns.
• Index set of $\hat{\mathfrak{g}} = I_{\mathfrak{g}} \sqcup \{0\}$.

For each $(i, x) \in I \times \Bbbk^\times$, $\exists!$ simple $U_q'(\hat{\mathfrak{g}})$-module $V(\varpi_i)_x$ called a **fundamental representation**.

- **Categories.**

$\mathcal{C}_{\hat{\mathfrak{g}}}$: Category of f.d integrable $U_q'(\mathfrak{g})$-modules.

thus this cat is enough.

Known $\exists$ <u>skeleton monoidal subcategory</u> $\mathcal{C}_{\hat{\mathfrak{g}}}^0$ of $\mathcal{C}_{\hat{\mathfrak{g}}}$ in the following sense : For every prime simple module $M \in \mathcal{C}_{\hat{\mathfrak{g}}}$, $\exists\, x \in \Bbbk^\times$ s.t $M_x \in \mathcal{C}_{\hat{\mathfrak{g}}}^0$.

- $\hat{\mathfrak{g}} \longleftrightarrow [\Delta, \sigma]$ <u>Interesting correspondence.</u>

$A_n^{(1)} \longleftrightarrow [\Delta_{A_n}, \mathrm{id}]$ $\quad B_n^{(1)} \longleftrightarrow [\Delta_{A_{2n-1}}, \vee]$ $\quad C_n^{(1)} \longleftrightarrow [\Delta_{D_{n+1}}, \vee]$

$D_n^{(1)} \leftrightarrow \underline{[\Delta_{D_n}, id]}$ $E_{6,7,8}^{(1)} \leftrightarrow \underline{[\Delta_{E_{6,7,8}}, id]}$ $F_4^{(1)} \leftrightarrow \underline{[E_6, \vee]}$, $G_2^{(1)} \leftrightarrow \underline{[D_4, \tilde{\vee}]}$

Rmk Every Δ appearing in the corr$\Leftrightarrow$ is of symmetric type even though $\hat{\mathfrak{g}}$ is not.
Also $(I_\Delta)^\sigma$ can be identified with $I_{\mathfrak{g}}$ in a canonical way.
(Ex $\hat{\mathfrak{g}} = B_n^{(1)} \to \Delta = A_{2n-1}$ $\sigma = \vee$ on $\Delta_{A_{2n-1}}$, $I_\Delta = \{1 \dots 2n-1\}$ $(I_\Delta)^\vee = \{1 \dots n\}$).

- locally reduce seq

For each $\hat{\mathfrak{g}} \leftrightarrow [\Delta, \sigma]$, we can take $\mathbf{i} \in [\Delta^\sigma]$ which is <u>localled reduced</u>:
For each $k \in \mathbb{Z}_{\geq 1}$ $(i_k, i_{k+1} \dots i_{k+\ell-1})$ is a red. seq of $w_0 \in W_\Delta$ $(\ell = \ell(w_0))$.
$\Rightarrow$ In this case, $i_{k+\ell} = i_k^*$ where $w_0(\alpha_i) = -\alpha_{i^*}$.

prop [.. Hernandez-Leclerc, Fujita-O] For such $\mathbf{i}$, we have an injective map

$$\phi_{\mathbf{i}} : \mathbb{Z} \hookrightarrow I_\Delta \times \mathbb{Z} \qquad k \longmapsto i_k, p_k$$

s.t

- $\phi_{\mathbf{i}}(k) = i_{k+\ell}^*, p_{k+\ell} + |\sigma| h^\vee$ where $h^\vee$: dual Coxeter # of $\mathfrak{g}$.
- $\{ V(\varpi_{\bar{i}})_{(q_s)^p} \mid (i,p) \in \mathrm{Im}(\phi_{\mathbf{i}}) \}$ generates $\mathscr{C}^0_{\hat{\mathfrak{g}}}$.

We set $\mathscr{C}_{\mathfrak{g}}^- :=$ Category gen'd by $\{ V(\varpi_{\bar{i}})_{q^p} \mid (i,p) \in \phi_{\mathbf{i}}(\mathbb{Z}_{\geq 1}) \}$.

- **Block decomposition of $\mathscr{C}^0_{\hat{\mathfrak{g}}}$**

Recall $\beta_k^{\mathbf{i}}$ for $k \geq 1$. For loc. red. $\mathbf{i} \in [\Delta^\sigma]$, we set

$$\beta_k^{\mathbf{i}} = w_0(\beta_{k+\ell}^{\mathbf{i}}) \quad \text{for } k < 1.$$

We set $V_{\mathbf{i}}(\beta_k) = V(\varpi_{\bar{i}_k})_{q^{p_k}}$ and assign $wt_{\mathbf{i}}(V_{\mathbf{i}}(\beta_k)) = \beta_k^{\mathbf{i}}$.

we fix $\mathbf{i} \in [\Delta^\sigma]$ loc. red. one for a while.

Thm [Chari-Moura, Kashiwara-Kim-O-Park, ...]

$$\mathscr{C}^0_{\hat{\mathfrak{g}}} = \bigoplus_{\beta \in Q_\Delta} \mathscr{C}_{\mathfrak{g}}[\beta]$$ (block decomposition via Q_Δ not $Q_{\mathfrak{g}}$!)
$\Rightarrow$ any simple module is homo w.r.t Q_Δ.

• q-character and (q,t)-character.

Thm [Frenkel–Reshetikhin]

- $\exists$ an inj. alg homo $\chi_q : K(\mathscr{C}^\circ_{\mathfrak{g}}) \longrightarrow \mathbb{Z}[X^{\pm 1}_{i,p} \mid (i,p) \in \mathrm{Im}(\phi_{\mathbf{i}})]$. ← q-character map. $\chi_q(M)$ for a module M is called the q-character of M.
- $K(\mathscr{C}^\circ_{\mathfrak{g}}) \simeq \mathbb{Z}[[V(\beta_k)] \mid k \in \mathbb{Z}]$

⤳ $K(\mathscr{C}^\circ_{\mathfrak{g}})$ is comm. However $M \otimes N \not\simeq N \otimes M$ for $M, N \in \mathscr{C}^\circ_{\mathfrak{g}}$ in general!

• Quantum Grothendieck ring.

t : indeterminate

Thm [Nakajima, Varagnolo–Vasserot, Hernandez $+ \alpha$ ...]

$\exists$ t-quantization $K_t(\mathscr{C}^\circ_{\mathfrak{g}})$ of $\chi_q(K(\mathscr{C}^\circ_{\mathfrak{g}}))$ satisfying the following properties:

① $K_t(\mathscr{C}^\circ_{\mathfrak{g}})$ is embedded into the quantum torus $\mathcal{X}_t$ gen'd by $\{\widetilde{X}^{\pm 1}_{i,p} \mid (i,p) \in \phi_{\mathbf{i}}(\mathbb{Z})\}$

$$\widetilde{X}_{i,p}\widetilde{X}_{i,p}^{-1} = \widetilde{X}_{i,p}^{-1}\widetilde{X}_{i,p} = 1 \quad \text{and} \quad \widetilde{X}_{i,p}\widetilde{X}_{j,s} = t^{-N^{\mathbf{i}}(k,l)}\widetilde{X}_{j,s}\widetilde{X}_{i,p} \;\text{——}\; ✩$$

where $\phi_{\mathbf{i}}^{-1}(i,p) = k$ & $\phi_{\mathbf{i}}^{-1}(j,s) = l$.

② $K_t(\mathscr{C}^\circ_{\mathfrak{g}})|_{t=1} = \chi_q(K(\mathscr{C}^\circ_{\mathfrak{g}}))$. We call $K_t(\mathscr{C}^\circ_{\mathfrak{g}})$ the <u>quantum Grothendieck ring</u>!

<u>**Rmk**</u> • We saw ✩ two days ago.

• By taking "<u>reduced part</u>", we can obtain the same quantum torus containing $A_{\mathbb{A}}(\mathfrak{n}(w))$ of type Δ. (ADE!!)

• Categorification.

For $a \le b$ • take a successive subseq $\mathbf{i}[a,b] = (i_a, i_{a+1}, \dots, i_b)$ of $\mathbf{i}$.

• $\mathscr{C}^{\mathbf{i}[a,b]}_{\mathfrak{g}}$: the category gen'd by $\{V_{\mathbf{i}}(\beta_k) \mid a \le k \le b\}$.

Let $b-a+1 \le \ell(w_0)$ of $w_0 \in W_\Delta$. ⤳ $\mathbf{i}[a,b] \in R(w)$ for some $w \in W_\Delta$.

Thm [HL, KKKO, ..]

For $a \le b$ with $b-a+1 \le \ell(w_0)$, we have the following:

• $K_t(\mathscr{C}^{\mathbf{i}[a,b]}_{\mathfrak{g}}) \simeq A_{\mathbb{A}}(\mathfrak{n}_\Delta(w))$ $(\simeq K(\mathscr{C}_w))$

- $\exists$ an exact monoidal f'tor from $K(R^{\wedge}\text{gmod})$ to $\mathscr{C}_{\hat{\mathfrak{g}}}^{\mathbf{i}[1,\ell]}$ sending simples to simples. In particular, it sends

$$S_k^{\mathbf{i}} \longrightarrow V_{\mathbf{i}}(\beta_k)$$

$$M^{\mathbf{i}}[t,s] \longrightarrow W_{\mathbf{i}}[t,s] \quad \text{in } \mathscr{C}_{\hat{\mathfrak{g}}}^{\mathbf{i}[1,\ell]} \quad \text{for an } \mathbf{i}\text{-box } [t,s] \subseteq [0,\ell].$$

(Kirillov-Reshetikhin module)

<u>R-matrix, Λ-invariant.</u> (Recall them for q. Hecke alg).

For simple $M,N \in \mathscr{C}_{\hat{\mathfrak{g}}}$, $\exists$ a non-zero $U_q'(\hat{\mathfrak{g}})$-module homo

$$\mathbf{r}_{M,N} : M \otimes N \longrightarrow N \otimes M.$$

We call $\mathbf{r}_{M,N}$ the R-matrix.

<u>Thm</u> (Kashiwara-Kim-O-Park). For each simple $M,N \in \mathscr{C}_{\hat{\mathfrak{g}}}$, we can associated a $\mathbb{Z}$-invariant $\Lambda(M,N)$ determined by $\mathbf{r}_{M,N}$.

↓ (before $1 \le k \le \ell \le r$ for quiver Hecke alg)

<u>prop</u> (KKOP) For $k \le \ell$,

$$\Lambda(V_{\mathbf{i}}(\beta_\ell), V_{\mathbf{i}}(\beta_k)) = N^{\mathbf{i}}(\ell,k).$$

For $\mathbf{i} \in [\Delta^6]$ loc. red, let $\mathbf{i}_\infty = (\dots i_0, i_1, i_2 \dots)$ s.t $i_k = i_{k+\ell}^*$ for $k < 1$.

Then we can define $\mathbf{i}$-box $[k,\ell]$ for $k \le \ell$ s.t $i_k = i_\ell$.

Thm [KKOP] For commuting $\mathbf{i}$-boxes $[a_1,b_1], [a_2,b_2] \subseteq [-\infty,\infty]$,

$$W_{\mathbf{i}}[a_1,b_1] \otimes W_{\mathbf{i}}[a_2,b_2] \simeq W_{\mathbf{i}}[a_2,b_2] \otimes W_{\mathbf{i}}[a_1,b_1] \text{ simple.}$$

If $a_2^- < a_1 \le b_1 < b_2^+$, we have

$$\Lambda(W_{\mathbf{i}}[a_1,b_1], W_{\mathbf{i}}[a_2,b_2]) = -(\omega_{[a_2^-,a_1]}\varpi_{i_{a_1}} - \omega_{[a_2^-,b_1]}\varpi_{i_{b_1}}, \omega_{[a_2^-,a_2]}\varpi_{i_{a_1}} - \omega_{[a_2^-,b_2]}\varpi_{i_{b_1}})$$

<u>Thm</u> (KKOP) The above results for $\mathbf{i} \in [\Delta^6]$ loc red can be generalized for any loc. red seq $\mathbf{j}$ (not necessarily in $[\Delta^6]$).

(mutation corresponding braid moves)

<u>Rmk</u> If $\mathbf{j}$ is loc. red but $\notin [\Delta^6]$, $W_{\mathbf{j}}[t,s]$ is not KR-module any more!

$\mathfrak{d}$-invariant For simples $M, N \in \mathscr{C}^{\circ}_{\mathfrak{g}}$, $\mathfrak{d}(M,N) := \frac{1}{2}(\Lambda(M,N) + \Lambda(N,M))$.

Thm (KKKO) M, N: simple $U_q'(\hat{\mathfrak{g}})$-modules s.t one of them is simple.

① $\mathfrak{d}(M,N) \geq 0$

② $\mathfrak{d}(M,N) = 0 \iff M \otimes N \simeq N \otimes M$ simple.

③ $\mathrm{hd}(M \otimes N)$ is simple.

(= same w/ q. Hecke algebra).

Thm (Nakajima, Hernandez (for $\mathbf{i} \in [\Delta^6]$), KKOP for $\mathbf{j}$)

Let $\mathbf{j}$ be a loc. red seq., and $[a,b]$ be an $\mathbf{j}$-box s.t $j_a = j_a = j$. Then we have a short exact seq

$$0 \to \underbrace{\bigotimes_{\substack{k \in I_\Delta \\ C^{\Delta}_{j,k} < 0}} M_{\mathbf{j}}[a^+(k), b^-(k)]}_{①} \to M_{\mathbf{j}}[a^+, b] \otimes M_{\mathbf{j}}[a, b^-] \to \underbrace{M_{\mathbf{j}}[a,b] \otimes M_{\mathbf{j}}[a^+, b^+]}_{②} \to 0.$$

where ①, ② are simples. **This s.e.s is known as T-system!**

- **Cluster & q. cluster alg. str related to $\mathscr{C}^{-}_{\mathfrak{g}}$.**

Thm **[Hernandez - Leclerc]**

Let $\mathbf{i} = (i_1, i_2 \dots) \in [\Delta^6]$ associated with $\hat{\mathfrak{g}}$. Then

$$K(\mathscr{C}^{-}_{\mathfrak{g}}) \simeq \mathcal{A}\left(\mathcal{S}^{\mathbf{i}} \simeq (\widetilde{B}^{\mathbf{i}}, \{z_k\})\right) \quad \text{as cluster algebra}$$

(not quantum)

$$[W_{\mathbf{i}}[k_{min}, k]] \longleftrightarrow z_k \qquad {}^{\forall} k \geq 1.$$

Moreover they show that $W_{\mathbf{i}}[a,b]$ appears as a cluster variable for every $\mathbf{i}$-box $[a,b] \subseteq [1,\infty]$ by using T-system as exchange relations.

Thm [Bittmann, FHOO]

$$K_t(\mathscr{C}^{-}_{\mathfrak{g}}) \simeq \mathcal{A}_q(\mathcal{S}^{\mathbf{i}} = (L^{\mathbf{i}}, \widetilde{B}^{\mathbf{i}}, \{\widetilde{z}_k\})) \quad \text{as } q\text{. cluster algebra.}$$

$$[W_{\mathbf{i}}[k_{min}, k]]_t \longleftrightarrow \widetilde{z}_k \qquad {}^{\forall} k \geq 1$$

• Monoidal categorification of $\mathscr{C}_{\mathfrak{g}}^{\circ}$.

From now on, let $\mathbf{i} \in I^{\mathbb{Z}}$ (not $\mathbb{Z}_{\geq 1}$) be a loc. red seq (not necessarily related to $[\Delta^6]$).

[Thm] (Kashiwara-Kim-O-Park) Let $\{[a_k, b_k]\}_{k\in\mathbb{Z}}$ be a set of $\mathbf{i}$-boxes. We call $\{[a_k, b_k]\}$ an admissible chain of $\mathbf{i}$-boxes if

- $(a_\ell)^- < a_k \leq b_k < (b_\ell)^+$ for $k < \ell$
- $\bigsqcup_k [a_k, b_k] = (-\infty, \infty)$.

Then we have a monoidal seed

$$(\tilde{B}, \{W_{\mathbf{i}}[a_k, b_k]\}_{k\in\mathbb{Z}})$$

s.t $\mathscr{C}_{\mathfrak{g}}^{\circ}$ provides a monoidal categorification of a cluster algebra of skew-symm type $\mathcal{A}(\mathcal{S})$ where $\mathcal{S} = (\tilde{B}, \{[W_{\mathbf{i}}[a_k, b_k]]\})_{k\in\mathbb{Z}}$.

In particular, $W_{\mathbf{i}}[a,b]$ appears

Moreover, they proved that $(\Lambda, \tilde{B})$ is compatible, when

$\Lambda = (\Lambda_{k\ell})$ and $\Lambda_{k\ell} := \Lambda(W_{\mathbf{i}}[a_k, b_k], W_{\mathbf{i}}[a_\ell, b_\ell])$ $k < \ell$.

Rmk • $\tilde{B}$ is diff from $\tilde{B}^{\mathbf{i}}$.

• Computing $\tilde{B}$ is not suggested in the paper.

Consequence. Every cluster monomial $\mathcal{A}(\mathcal{S})$ corresponds to a simple module in $\mathscr{C}_{\mathfrak{g}}^{\circ}$ (This is also proved by Qin).

• The monoidal categorification thm for $\mathscr{C}_{\mathfrak{g}}^{\circ}$ is generalized into the quantum cluster algebra setting by FHOO; i.e., $K_t(\mathscr{C}_{\mathfrak{g}}^{\circ})$ has a q. cluster alg. structure & every q. cluster mono corresponds to a (q,t)-character of a simple module.

Thm [Herandez–Leclerc, Fujita–Hernandez–O–OYA]

Let $\hat{\mathfrak{g}}, \hat{\mathfrak{g}}'$ with the same Δ (EX. $A_{2n-1}^{(1)}$, $B_n^{(1)}$ with Δ of type A_{2n-1}).

Then $K_t(\mathcal{C}^{\circ}_{\hat{\mathfrak{g}}}) \simeq K_t(\mathcal{C}^{\circ}_{\hat{\mathfrak{g}}'})$ and their presentation is given as follows:

It is generated by $\{f_{i,m} \mid (i,m) \in I \times \mathbb{Z}\}$ subject to the following relations:

$$\sum_{r=0}^{1-C_{ij}^{\Delta}} (-1)^r f_{i,m}^{(1-C_{ij}-r)} f_{j,m} f_{i,m}^{(r)} = 0 \quad \text{for } i \neq j.$$

$$f_{i,m} f_{j,p} = q^{(-1)^{p+m+1} C_{ij}^{\Delta}} f_{j,p} f_{i,m} \quad \text{for } p > m+1$$

$$f_{i,p} f_{j,p+1} = q^{C_{ij}^{\Delta}} f_{j,p+1} f_{i,p} + \delta_{ij}(1-q^2)$$

where $(C_{ij}^{\Delta})_{i,j \in I_\Delta}$ denote the finite Cartan matrix of Δ

Roughly. Every str. related to $\mathcal{C}_{\hat{\mathfrak{g}}}$ is (skew) symmetric even in the case when $B_n^{(1)}, C_n^{(1)}, F_4^{(1)}, G_2^{(1)}$.

- **Missing**

We have $\blacktriangle$ of type BCFG, $\mathbf{i} \in [\blacktriangle]$ compatible pair $(\Lambda^{\mathbf{i}}, \widetilde{B}^{\mathbf{i}})$, $N^{\mathbf{i}}$, $T(L^{\mathbf{i}})$ ....
$$T(L^{\mathbf{i}}) \simeq T(\Lambda^{\mathbf{i}})$$

For a while, let $\mathfrak{g}$ be of type BCFG.

- **Quantum virtual Grothendieck ring**

Thm [Kashiwara–O, Jang–Lee–O] $\exists$ alg $\mathbb{Z}[q^{\pm 1/2}]$-subalg $K_q(\mathfrak{g})$ of a quantum torus $T(\Lambda^{\mathbf{i}})$, which is determined by $\mathbf{i} \in [\blacktriangle]$, satisfying the following properties

① $K_q(\mathfrak{g})$ has a q-cluster alg iso to $\mathcal{A}_q(\mathscr{S}^{\mathbf{i}})$.

(non skew-symmetric any more!!)

② $K_q(\mathfrak{g})$ has a presentation as follows:

$$\sum_{r=0}^{1-c_{ij}^{\blacktriangle}} (-1)^r f_{i,m}^{(1-c_{ij}-r)} f_{j,m} f_{i,m}^{(r)} = 0 \quad \text{for } i \neq j.$$

$$f_{i,m} f_{j,p} = q^{(-1)^{p+m+1}(\alpha_i,\alpha_j)} f_{j,p} f_{i,m} \quad \text{for } p > m+1 \qquad \text{— ⊛ } \hat{\mathcal{A}}_q(n)$$

$$f_{i,p} f_{j,p+1} = q^{(\alpha_i,\alpha_j)} f_{j,p+1} f_{i,p} + \delta_{ij}(1 - q^{(\alpha_i,\alpha_i)})$$

where $(c_{ij}^{\blacktriangle})_{i,j \in I_{\blacktriangle}}$ denote the finite Cartan matrix of $\blacktriangle$

- **Rmk** For subseq $\mathbf{i} \in [\Delta^{\sigma}], [\blacktriangle]$, or loc. red seq, we can consider its successive subseq $\mathbf{i}[a,b]$, w/ the sequence $\mathbf{i}[a,b]$, we can generalize the results above to such sub-sequence.

- **Up to now,** we used $\mathbf{i}$ s.t loc. reduced, reduced or comm eq. to loc. red. sequence, and of finite type. However, the quantum torus can be defined for an arbitrary seq $\mathbf{i}$ and $\mathfrak{g}$ beyond finite type.

• Arbitrary seq but finite type.

- Let $\mathbf{i} = (i_1, i_2, \dots, i_r)$ be an arbitrary seq of I of finite type $\mathfrak{g}$. Then $\mathbf{i}$ can be understood as a "red" expression of an element of the **Braid group** $B_{\mathfrak{g}} = \langle r_i \mid i \in I \rangle$

- For notation simplicity, let us write $\mathcal{K}_q(\mathfrak{g}_\Delta)$ for $K_t(\mathcal{C}_{\mathfrak{g}}^{\circ})$ assoc w/ $[\Delta, \sigma]$. Then for $\mathfrak{g}$ of types $A \sim G$, $\mathcal{K}_q(\mathfrak{g})$ has a presentation ⊛, uniformly.

Thm [KKOP for ADE $\mathfrak{g}$, Jang-Lee-O for BCFG $\mathfrak{g}$]

For each $i \in I$ of finite type, $\exists$ an auto T_i on $\mathcal{K}_q(\mathfrak{g})$ given as follows:

$$T_i(f_{j,p}) = \begin{cases} f_{j,p+\delta_{ij}} & \text{if } c_{ij} \geq 0, \\ \dfrac{\sum_{r+s=-c_{ij}} (-1)^r q_i^{c_{ij}/2+r} f_{i,p}^{(s)} f_{j,p} f_{i,p}^{(r)}}{(q_i^{-1} - q_i)^{-c_{ij}}} & \text{if } c_{ij} < 0. \end{cases}$$

Moreover, $\{T_i\}_{i \in I}$ satisfies the Braid relation of $B_{\mathfrak{g}}$.

Rmk • **when $i \neq j$.** T_i = Lustzig (Saito)'s braid group action S_i on $U_q(\mathfrak{g})$. (up to constant).

• **The action** $\{S_i\}$ used for the construction of PBW basis of $U_q^-(\mathfrak{g})$, $A_q(\mathfrak{n})$, $A_q(\mathfrak{n}(w))$.

• $A_q(\mathfrak{n}(w))$ is generated by dual root vectors of $\{\beta_k^{\mathbf{i}} \mid \mathbf{i} \in R(w)\}$, constructed by S_i's.

Thm [O-Park]. For an arbitrary sequence $\mathbf{i} = (i_1 \dots i_s)$ of I finite type $\mathfrak{g}$, $\exists$ a subalgebra $A_q(\mathbb{b})$ of $K_q(\mathfrak{g})$ by using the construction of its PBW-basis. Here $\mathbb{b}$ is an elt of Braid group $B_{\mathfrak{g}}$ s.t $\mathbb{b} = r_{i_1} \cdots r_{i_s}$.

Moreprecisely, define

$$F_k^{\mathbf{i}} := T_{i_1} \cdots T_{i_{k-1}} (f_{i_k,0}) \quad \forall\ 1 \leq k \leq s.$$

and certain non-deg pairing

$\langle\!\langle\ ,\ \rangle\!\rangle$ on $K_q(\mathfrak{g})$. ← (generalizing $(\,,)_L$ & $(\,,)_K$)

Then we have • $F_k \cdot F_\ell = \sum_{\mathbf{c} \text{ between } k \,\&\, \ell} A_{\mathbf{c}} F^{\mathbf{c}}$ LS formula.

• each PBW-basis $P_{\mathbf{j}}$ is orthogonal to $\langle\!\langle\ \rangle\!\rangle$ for $r_{\mathbf{j}} = \mathbb{b}$.

• $\langle\!\langle x, y \rangle\!\rangle = \langle\!\langle T_i(x), T_i(y) \rangle\!\rangle$.

Ultimate goal : For any $\mathbf{i}$ and $\mathfrak{g}$ ....