Quantum tori assoc. w. sequences & their application II.

$\mathbf{i}$: arbitrary ~~red~~ seq $\mathfrak{g}$: KM

• Quantum group

$\mathfrak{g}$: KM alg $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ root lattice $Q^+ = \sum \mathbb{Z}_{\geq 0}\alpha_i$.

$Q^- = \sum \mathbb{Z}_{\leq 0}\alpha_i$

$U_q(\mathfrak{g})$ = quantum group over $\mathbb{Q}(q^{1/2})$ gen'd by e_i, f_i $(i \in I)$ & q^h

+ q-Serre rel.

$$\sum_{r=0}^{1-C_{ij}} (-1)^r f_i^{(1-C_{ij}-r)} f_j f_i^{(r)} = 0 \qquad \text{where} \quad f_i^{(n)} = f_i^n / [n]_i!$$

$$\sum_{r=0}^{1-C_{ij}} (-1)^r e_i^{(1-C_{ij}-r)} e_j e_i^{(r)} = 0 \qquad e_i^{(n)} = e_i^n / [n]_i! .$$

$\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}]$. $U_{\mathbb{A}}^+(\mathfrak{g})$ = $\mathbb{A}$-subalg of $U_q(\mathfrak{g})$ gen'd by $e_i^{(n)}$ $(i \in I, n \in \mathbb{Z}_{\geq 1})$

• Quantum unipotent coordinate ring. $A_q(\mathfrak{n})$

Set $A_q(\mathfrak{n}) = \bigoplus_{\beta \in Q^-} A_q(\mathfrak{n})_\beta$ where $A_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q^{1/2})}(U^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q^{1/2}))$.

Known • $A_q(\mathfrak{n})$ has also alg str.

· The $\mathbb{A}$-sub alg $A_{\mathbb{A}}(\mathfrak{n}) := \{\psi \in A_q(\mathfrak{n}) \mid \psi(U_{\mathbb{A}}^+(\mathfrak{g})) \subset \mathbb{A}\}$ has a unique $\mathbb{B}^{up}$

· $U_q^+(\mathfrak{g})$-acts on $A_q(\mathfrak{n})$ as $e_i\psi(x) = \psi(xe_i)$ $^{\forall} x \in U_q^+(\mathfrak{g})$, $\psi \in A_q(\mathfrak{n})$.

Let us choose $w \in W$, take $\mathbf{i} = (i_1 \ldots i_r) \in R(w)$ (red.seq)

For $\beta = \sum_i n_i\alpha_i \in Q^+$ with $r = \sum n_i$, set • $I^\beta = \{(i_1, \ldots i_r) \in I^r \mid \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r} = \beta\}$.

• $\Phi(w) = \{\beta_k^{\mathbf{i}} \mid 1 \leq k \leq r\} \subset \Phi^+$ & $\Phi(w) \cap (Q^+ \cap wQ^+) = \phi$.

$(^{\forall} \mathbf{i} \in R(w))$ $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) = \beta_k^{\mathbf{i}}$

• $A_{\mathcal{A}}(\mathfrak{n}(w)) := \mathrm{Span}_{\mathcal{A}}\{\psi \in A_{\mathcal{A}}(\mathfrak{n}) \mid e_{i_1}\cdots e_{i_r}\psi = 0 \quad \forall \beta \in (Q^+ \cap wQ^+)\setminus\{0\},\ \forall (i_1 \dots i_r) \in I^\beta\}$ has an alg str.

Thm [Kimura] $\mathbb{B}^{up}(w) := \mathbb{B}^{up} \cap A_{\mathcal{A}}(\mathfrak{n}(w))$ is an $\mathcal{A}$-basis of $A_{\mathcal{A}}(\mathfrak{n}(w))$

• i-boxes and unipotent quantum minors.

For $0 \le a \le b \le r$, we set $[a,b] = \{k \in \mathbb{Z} \mid a \le k \le b\}$ & call it interval.

$\boxed{\text{Setting}}$ We have chosen $\mathbf{i} = (i_1 \cdots i_r) \in R(w)$ for $w \in W$.

- $K = [1,r]$ $\quad K = K_{ex} \sqcup K_{fr}$.
- An interval $[a,b] \subseteq [0,r]$ is called an i-box if $i_a = i_b$ or $a = 0$
- i-boxes $[a_1,b_1]$ & $[a_2,b_2]$ are said to be commutative if

  $a_1^- < a_2 \le b_2 < b_1^+$ or $a_2^- < a_1 \le b_1 < b_2^+$ (Recall $0^- = -1$)

For an i-box $[a,b]$ w/ $i_b \overset{i_a}{=} i$, we can associated an elt $D^{\mathbf{i}}[a,b] \in \mathbb{B}^{up}(w) \subseteq A_q(\mathfrak{n}(w))$ & called a unipotent q. minor with $\mathrm{wt}(D^{\mathbf{i}}[a,b]) = w_{\le b}\varpi_i - w_{\le a}\varpi_i \in Q^-$.
In particular, $D^{\mathbf{i}}[b^-,b]$ is an i-root vector of wt $-\beta^{\mathbf{i}}_k$ & $\{D^{\mathbf{i}}[k^-,k] \mid 1 \le k \le r\}$ generates $A_{\mathcal{A}}(\mathfrak{n}(w))$. dual PBW vector

For elts $x, y \in A_{\mathcal{A}}(\mathfrak{n})$, x,y are q-commutative if $xy = q^{\ell} yx$ for some $\ell \in \mathbb{Z}/2$.

• Quantum tori of $A_{\mathcal{A}}(\mathfrak{n}(w))$ c

Thm [BZ] If i-boxes $[a_1,b_1]$ & $[a_2,b_2]$ commute, then $D^{\mathbf{i}}[a_1,b_1]$ & $D^{\mathbf{i}}[a_2,b_2]$ q-commutative s.t $D^{\mathbf{i}}[a,b]\,D^{\mathbf{i}}[a',b'] \in q^{\mathbb{Z}}\mathbb{B}^{up}(w)$ and $\rightsquigarrow T(L^{\mathbf{i}})$

$$\ell = (w_{\le a_1}\varpi_{i_a} - w_{\le b_1}\varpi_{i_b},\ w_{\le a_2}\varpi_{i_{a_2}} + w_{\le b_2}\varpi_{i_{b_2}}) \text{ if } a_2^- < a_1 \le b_1 < b_2^+.$$

In particular $D^{\mathbf{i}} := \{D^{\mathbf{i}}[0,k] \mid 1 \le k \le r\}$ forms a q-commuting family, with

$$\ell_{st} = (\varpi_{i_s} - w_{\le s}\varpi_{i_s},\ \varpi_{i_t} + w_{\le t}\varpi_{i_t}) \text{ if } s \le t.$$

$\rightsquigarrow$ **we have a q. torus** of $A_q(\mathfrak{n}(w))$ and q. seed.

$$\mathcal{S}^{\mathbf{i}} := (L^{\mathbf{i}}, \widetilde{B}^{\mathbf{i}}, \{\widetilde{Z}_k = D^{\mathbf{i}}[0,k]\})$$

Thm (Geiß - Leclerc - Schröer, Goodreal - Yakimov, ...)

$$\underline{A_{\mathbb{A}}(\mathfrak{n}(w))} \simeq \mathcal{A}_q(\mathcal{S}^i)$$

q. cluster algebra.

• Elements in $\mathbb{B}^{up}$ & Conjectures

$b \in \mathbb{B}^{up}$ is real if $b^2 \in q^{\mathbb{Z}/2}\mathbb{B}^{up}$

prime if b does not have factorization $b = q^* b_1 b_2$ w/ $b_1, b_2 \in \mathbb{B}^{up}$.

Conjecture. • {cluster monomials in $A_{\mathbb{A}}(\mathfrak{n}(w))$ ($= \mathcal{A}_q(\mathcal{S}^i)$)} $\overset{1-1}{\longleftrightarrow}$ {real elts in $\mathbb{B}^{up}$}

• {cluster variables in $A_{\mathbb{A}}(\mathfrak{n}(w))$} $\overset{1-1}{\longleftrightarrow}$ {prime real elt in $\mathbb{B}^{up}$}.

• Monoidal Categorification

(proposed by Hernandez - Leclerc for Conjectures)

$\mathcal{C}$: monoidal category with $\otimes$, auto-functor q. ("grading shift" ftor")

$\mathcal{C}$ provide a m. cat of a q. cluster alg $\mathcal{A}_q$ if

$\boxed{\text{Axioms for m. Cat}}$ ① $\mathcal{A}_q \simeq K(\mathcal{C})$, $\circlearrowleft \mathbb{Z}[q^{\pm 1}]$ ⓥ

② q. cluster monomials $\leftrightarrow$ real simple objs in $\mathcal{C}$

③ " vars $\leftrightarrow$ prime real simple objs in $\mathcal{C}$.

$\Rightarrow$ Categorifying cluster variables mutations

In this talk, $\mathcal{C}$ denotes a module Category.

Consequence. x : q. cluster var $\to$ M : simple

$$[M^d] \circ [M] = x = \frac{\sum_a P_a \tilde{z}^a}{\tilde{z}^d} = \frac{\sum_a P_a [M_1^{\otimes a_1} \otimes \cdots \otimes M_r^{\otimes a_r}]}{[M_1^{\otimes d_1} \otimes \cdots \otimes M_r^{\otimes d_r}]} = \frac{\sum P_a [M^a]}{\cancel{[M^d]}}$$

q. Laurent phenomena.

$\Rightarrow$ P_a : a "graded" decomp # of $[M^a]$ in $[M \circ M^d]$.

$\in \mathbb{Z}_{\geq 0}[q^{\pm 1}]$

• Quiver Hecke algebra. (Khovanov - Lauda, Rouquier).

Take $\beta = \sum n_i \alpha_i \in Q^+$ w/ $\sum n_i = r$ & recall $I^\beta \subset I^r$.

Fix symmetrizable KM-alg $\mathfrak{g}$

KL and R introduce a unital $\mathbb{Z}$-graded alg $R(\beta)$ gen'd by

x_k $(1 \le k \le r)$, τ_s $(1 \le s < r)$, $e(\nu)$ $(\nu = (\nu_1, \dots \nu_r) \in I^\beta)$,

subj to certain relations asso w/ $\mathfrak{g}$. idempotent $e(\nu)e(\mu) = \delta_{\nu\mu} e(\nu)$

Here $e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$ is a unit $R(\beta)$

$R(\beta)$-gmod := the category f.d graded $R(\beta)$-modules

$R\text{-gmod} = \bigoplus_{\beta \in Q^+} R(\beta)\text{-gmod}$ (block-decomposition)

$q : R\text{-gmod} \longrightarrow R\text{-gmod}$ auto-f4or s.t $(qM)_n = M_{n-1}$ for $M = \bigoplus_{k \in \mathbb{Z}} M_k \in R\text{-gmod}$.

• Convolution product (Monoidal structure)

For $M \in R(\beta)$-gmod, $N \in R(\gamma)$-gmod.

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N)$$

where $e(\beta,\gamma) = \sum_{\substack{\nu \in I^\beta \\ \mu \in I^\gamma}} e(\nu * \mu)$; via non-unital homo $R(\beta) \otimes R(\gamma) \hookrightarrow R(\beta+\gamma)$.

$e(\beta),\ e(\gamma) \longmapsto e(\beta,\gamma)$

not unit $R(\beta+\gamma)$

$\rightsquigarrow$ $(R\text{-gmod}, \circ)$: monoidal category $\Rightarrow$ $K(R\text{-gmod})$ has $\mathbb{Z}[q^{\pm 1}]$-alg str.

Thm [Khovanov–Lauda, Rouquier, ...]

① $\exists$ $\mathbb{A}$-alg iso $A_{\mathbb{A}}(n(w_0))$ $A_{\mathbb{A}}(n(w))$

$$\Omega : \mathbb{A} \otimes_{\mathbb{Z}[q^{\pm 1}]} K(R\text{-gmod}) \xrightarrow{\sim} A_{\mathbb{A}}(n)$$

(As far as I know $A_{\mathbb{A}}(n)$ does not have q. cluster alg str)

② For each $\mathbf{i} = (i_1, \dots i_r) \in R(w)$ for $w \in W$ and an $\mathbf{i}$-box $[a,b] \subset [0,r]$, $\exists$ prime real simple module $M^{\mathbf{i}}[a,b] \in R(w_{\le a}\varpi_{i_a} - w_{\le b}\varpi_{i_b})$-gmod s.t

$$\Omega(M^{\mathbf{i}}[a,b]) = D^{\mathbf{i}}[a,b]$$

In particular we set $S_k^{\mathbf{i}} = M^{\mathbf{i}}[k^-, k]$ for $1 \le k \le r$.

$D_k^{\mathbf{i}}$

Def (Subcategory $\mathcal{C}_w$ of R-gmod)

$\mathcal{C}_w$ is the smallest monoidal full subcat of R-gmod

① Stable under taking subquot, ext & grading shift

② containing $\{ S_k^{\mathbf{i}} \mid 1 \le k \le r \}$.

$\rightsquigarrow$ $\Omega(\mathbb{A}\otimes_{\mathbb{Z}[q^{\pm 1/2}]} K(\mathscr{C}_w)) \simeq A_{\mathbb{A}}(\mathfrak{n}(w))$ (note $A_{\mathbb{A}}(\mathfrak{n}(w))$ has q-cluster alg str).

Here the def of $\mathscr{C}_w$ does not depend on the choice of $\mathbf{i} \in R(w)$.

Note For the def of $R(\beta)$, we need to choose set of "polys".
But we skip it and take it in [KL].

From now on, we consider $\mathfrak{g}$ of symmetric type

R-matrix, Λ-inv For $M, N \in R$-gmod $\exists$ a non-zero R-module homo

$$r_{M,N} : q^a M \circ N \longrightarrow N \circ M.$$

We call $r_{M,N}$ the R-matrix & denote by $\Lambda(M,N) = a$ the degree of $r_{M,N}$

prop [Brundan–Kleshchev–McNamara (finite), Tingley–Webster (general)] (Unmixed property)

For $1 \le k \le l \le r$,

$$\Lambda(S_l^{\mathbf{i}}, S_k^{\mathbf{i}}) = \boxed{-\delta(k \neq l)(\beta_l^{\mathbf{i}}, \beta_k^{\mathbf{i}})} = N^{\mathbf{i}}(l,k)$$

Rmk The existence of R-matrix for non-symmetric $\mathfrak{g}$ is quite hard problem

d-invariant For $M, N \in R$-gmod, $d(M,N) := \frac{1}{2}(\Lambda(M,N) + \Lambda(N,M))$.

$d(M,N) \in \mathbb{Z}$, $\Lambda(N,M) = -\Lambda(M,N)$ $T(\Lambda^{k})$

Thm (Kang–Kashiwara–Kim–O) M, N: simple R-module s.t. one of them is real.

① $d(M,N) \geq 0$ (MoM simple)

② $d(M,N) = 0 \iff q^{*} M \circ N \simeq N \circ M$ simple $* = \Lambda(M,N)$

③ $\mathrm{hd}(M \circ N)$ is simple
(head)

Note For commuting $\mathbf{i}$-boxes $[a,b]$ & $[a',b']$, $d(M^{\mathbf{i}}[a,b], M^{\mathbf{i}}[a',b']) = 0$) &

$$\Lambda(M^{\mathbf{i}}[a,b], M^{\mathbf{i}}[a',b']) = -(w_{\le a}\varpi_{i_a} - w_{\le b}\varpi_{i_b}, w_{\le a'}\varpi_{i_{a'}} + w_{\le b'}\varpi_{i_{b'}}).$$

Comparison $A_{\mathbb{A}}(\mathfrak{n}(w)) \longleftrightarrow \mathscr{C}_w$
$\cup$ $\cup$

- cluster var are homo $\longleftrightarrow$ simple modules $M \in R(\beta)$-gmod.

- $D[a,b] \longleftrightarrow M[a,b]$
- $D[a,b]\,D[a',b'] = q^{l} D[a',b']\,D[a,b] \longleftrightarrow l = -\Lambda(M^i[a,b], M^i[a',b'])$
- $N^i \longleftrightarrow \Lambda$-inv among $\{S_k^i\}$.
- exchange matrix $\longleftrightarrow$?

$l(M^i[0,k] \circ M) = 2$

Thm (KKKO) For $k \in K_{ex}$, $\exists$ real simple module M' s.t. • "$d(M^i[0,k], M') = 1$"

- $0 \to \underset{b_{ik}>0}{\circ} M[0,i]^{\circ b_{ik}} \to M[0,k] \circ M' \to \underset{b_{ki}>0}{\circ} M[0,i]^{\circ b_{ki}} \to 0$,
  ① ②

where ①, ② are simples (categorifying initial ex matrices and mutation rule)
and comm / with $M[0,i]$ ($i \neq k$).

Thm (KKKO) C_w provides a m. cat of the q-clusters alg $A_{q}(n(w))$.
In particular, every cluster monomial corresponds to a real simple module $M \in C_w$,
is elt in B^{up} $\longleftrightarrow$

Additional information (Maximal commuting family : Suggested by Kimura as the notation (maximal strongly compatible subset in $B^{up}(w)$).

(Kashiwara-Kim)

Cor X simple in R-gmod and $\{M_k\}$ a set of real simple module categorifying a cluster $\{\tilde{Z}_k\}$. If $X \circ M_k \simeq q^{*} M_k \circ X$ $\forall$ $1 \le k \le r$, $\exists\, p \in \mathbb{Z}^k_{\ge 0}$
s.t. $X \simeq M_1^{\circ p_1} \circ M_2^{\circ p_2} \circ \cdots \circ M_r^{\circ p_r}$

pf By Laurent phenomenon, $\exists\, a \in \mathbb{Z}^k_{\ge 0}$ s.t.

$$[X] \circ [M^{a}] = \sum_s q^{c_s} [M^{b_s}] \qquad \text{☆}$$

By ☆, RHS $= [M^{b}]$, i.e. $[X \circ M^{a}] = M^{b}$

- $d(M_k', X \circ M^{a}) = d(M_k', X) + a_k$
  $\quad \| \qquad\quad \cup\!|$
  $d(M_k', M^{b}) = b_k \qquad \Rightarrow b_k \ge a_k$

Then $X \simeq M^{b-a}$

Cor For every simple $M \in C_w$, $[M]$ is (co)-pointed.

(pf) $[M] \circ [M^{a}] = \sum q^{c_s} [M^{b_s}]$ $\qquad b_1 \Updownarrow$ head $\qquad b_s \Updownarrow$ socle

$g \Leftrightarrow b_1 \succneqq \cdots \succneqq b_s$ " g " $\qquad b_k \succ b_s \Leftrightarrow \tilde{Z}^{b_k} / \tilde{Z}^{b_s} =$ products $\tilde{y}_k$