Quantum tori assoc. w. sequences & their application.

toruses

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($\mathbb{Q}$: rational #)

rational fn field. z_i : vars $\quad$ $\tilde{z}_i$: vars

$\mathbb{Q}(z_1, \dots z_r)$ comm $\quad$ " $\mathbb{Q}(q^{1/2})(\tilde{z}_1, \dots \tilde{z}_r)$ "

$\cup$ $\qquad\qquad\qquad\qquad$ $\cup$

$\mathcal{A}$: cluster alg $\quad$ quantize $\rightsquigarrow$ $\quad$ $\mathcal{A}_q$: q. cluster alg

(Fomin-Zelevinsky) $\quad$ q : indeterminate $\quad$ (Berenstein - Zelevinski)

($q^{1/2}$: formal sq. root of q)

One of Goal of BFZ: studying the basis $\mathbb{B}^{up}$ of quantum gp $U_q(\mathfrak{g})$

(dual can / upper global) Lusztig / Kashiwara

1. Ingradients for quantum cluster alg.

K : index set w/ $K_{ex} \sqcup K_{fr} = K$ (exchangeable indices; frozen indices)

$L = (l_{ij})_{i,j \in K}$: $\mathbb{Z}$-matrix s.t $L = -L$ $\qquad$ ($\mathbb{Z}$: integer)

$\tilde{Z} = \{\tilde{z}_k\}_{k \in K}$ alg. ind. vars.

$*$ $\mathcal{A}_q$ is "related to" quantum torus determined by L.

Def [q.torus $\mathcal{T}(L)$] $\quad$ $\mathbb{Z}[q^{\pm 1/2}]$ - alg. gen. by $\{\tilde{z}_k^{\pm}\}$ s.t

$$\tilde{z}_k \tilde{z}_k^{-1} = \tilde{z}_k^{-1} \tilde{z}_k = 1, \quad \tilde{z}_k \tilde{z}_s = q^{l_{ks}} \tilde{z}_s \tilde{z}_k$$

① $\mathcal{T}(L)|_{q=1} \simeq \mathbb{Z}[z_k^{\pm 1}]_{k \in K}$ where $z_k := \tilde{z}_k|_{q=1}$

② $\mathcal{T}(L) \subset \mathbb{F}(L)$ field of fraction of $\mathcal{T}(L)$ $\qquad$ 1st ingradient

Def [Exchange matrix] An Ex mtx $\tilde{B} = (b_{ij})_{i \in K, j \in K_{ex}}$ is a $\mathbb{Z}$-mtx s.t $\qquad$ $K \times K_{ex}$

$B = (b_{ij})_{i,j \in K_{ex}}$ is skew-symm'ble i.e $\exists$ $D = \mathrm{diag}(d_i \geq 1)_{i \in K_{ex}}$ s.t

principal part

DB is skew-symmetric. i.e., $(DB)^t = -DB$.

2nd ing.

A pair $(L, \tilde{B})$ is said to be a compatible pair **CP** if

$$\sum_{k \in K} b_{ki} l_{kj} = 2 d_i \delta_{ij} \quad \text{for } i \in K_{ex},\ j \in K.$$

(we can define q. cluster when such $(L, \tilde{B})$ is given)

Q : How to construct or find such pairs ?

2. Kac Moody algebra $\mathfrak{g}$ and Index set I of $\mathfrak{g}$

Setting $\mathfrak{g}$: KM alg w/ $C = (c_{ij})_{i,j \in I}$ and "I" the index set

① $\{\alpha_i\}_{i \in I}$: simple roots ② $\{\varpi_i\}_{i \in I}$: f'tal wts

Assume we can choose them ① and ② s.t

$$\alpha_i = 2\varpi_i + \sum_{\substack{j \in I \\ c_{ji} < 0}} c_{ji} \varpi_j$$

③ $\Phi^{\pm}$: pos (neg) root $\Phi = \Phi^+ \sqcup \Phi^-$

④ $(,)$: wt pairing

⑤ W : Weyl gp of $\mathfrak{g}$ gen'd $\{s_i\}_{i \in I}$ s.t $s_i \varpi_j = \varpi_j - \delta_{ij} \alpha_i$

For $w \in W$, $R(w) := \{ (i_1 \dots i_r) \in I^r \mid s_{i_1} \cdots s_{i_r}$ is a red exp of $w \}$.

Combinatorics on sequence.

$\mathbf{i} = (i_1 \cdots i_r)$ "any seq" of I, $1 \le k \le r$, $j \in I$

$k_{\mathbf{i}}^+(j) = \min(\{u \mid u > k \ \ i_u = j\} \sqcup \{r+1\})$ $\quad k_{\mathbf{i}}^+ = k_{\mathbf{i}}^+(i_k)$

$k_{\mathbf{i}}^-(j) = \max(\{u \mid u < k \ \ i_u = j\} \sqcup \{0\})$ $\quad k_{\mathbf{i}}^- = k_{\mathbf{i}}^-(i_k)$

$k_{\mathbf{i}}^{\min} = \min\{u \mid 1 \le k \le r,\ i_k = i_u\}$

$W \ni w_{\le k}^{\mathbf{i}} = s_{i_1} \cdots s_{i_k}$ (for $k \le r$). $\boxed{0^- := -1.}$

We set $K = [1, r] = \{1, 2, \dots, r\} \subseteq \mathbb{Z}$, $K_{fr} = \{k \in K \mid k^+ = r+1\}$ $K_{ex} = K \setminus K_{fr}$.

↳ the index set for $(L, \tilde{B})$

(Ex) $\mathfrak{g} = A_3$ $\mathbf{i} = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 & i_9 & i_{10} & i_{11} \\ 1 & 2 & 1 & 2 & 1 & 3 & 3 & 1 & 2 & 3 & 2 \end{pmatrix} \Rightarrow$

$K = [1, 11]$
$4^+ = 9$
$4^- = 2$
$4^+(1) = 5$ $4^-(3) = 0$
$K_{fr} = \{8, 10, 11\}$
$w_{\le 4} = s_1 s_2 s_1 s_2$

Let us fix a sequence (any seq)

$\mathbf{i} = (i_1, \dots, i_r)$

until the end of today talk, and we skip it frequently in notations!

Def\Thm (BZ, ..., Fujita-Hernandez-O-OYa) $(L^{\mathbf{i}}, \tilde{B}^{\mathbf{i}})$ defined below is a **CP**.

$$\tilde{B}^{\mathbf{i}} = (b^{\mathbf{i}}_{st}) \text{ s.t } b^{\mathbf{i}}_{st} = \begin{cases} \pm 1 & \text{if } s = t^{\mp} \\ c_{i_s, i_t} & \text{if } s < t < s^+ < t^+ \\ -c_{i_s, i_t} & \text{if } t < s < t^+ < s^+ \\ 0 & \text{o.w} \end{cases}$$

$$L^{\mathbf{i}} = (\ell^{\mathbf{i}}_{st}) \text{ s.t } \ell_{st} = \underbrace{(\varpi_{i_s} - w^{\mathbf{i}}_{\le s}\varpi_{i_s}, \varpi_{i_t} + w^{\mathbf{i}}_{\le t}\varpi_{i_t})}_{\text{complicated!!}} \text{ for } s \le t.$$

$T(L^{\mathbf{i}})$

3. Isomorphism of two quantum tori. $T(\Lambda^{\mathbf{i}})$

Set $\beta^{\mathbf{i}}_k := \underline{s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})} \in \Phi$ for $1 \le k \le r$. Define a skew-sym pairing $N^{\mathbf{i}}$ on $K \times K$ $(K = [1, r] = \{1 \dots r\})$

$$N^{\mathbf{i}}(a,b) = \underline{(-1)^{\delta(a>b)} \delta(a \ne b) (\beta_a, \beta_b)}$$

Here for statement P,
$$\delta(P) = \begin{cases} 1 & \text{if } P \text{ is true.} \\ 0 & \text{o.w} \end{cases}$$

Let $\{\tilde{X}_k\}_{1 \le k \le r}$: other alg. ind. vars.

Def [q. torus $T(\Lambda^{\mathbf{i}})$] $\mathbb{Z}[q^{\pm 1/2}]$-alg gen'd by $\{\tilde{X}_k^{\pm 1}\}$ s.t

$$\tilde{X}_k \tilde{X}_k^{-1} = 1 = \tilde{X}_k^{-1}\tilde{X}_k, \quad \underline{\tilde{X}_k \tilde{X}_\ell = q^{-N^{\mathbf{i}}(k,\ell)} \tilde{X}_\ell \tilde{X}_k}.$$

Set $-\mathrm{wt}_{\mathbf{i}}(\tilde{X}_k) = \underline{\beta^{\mathbf{i}}_k}$ $\quad (\Leftrightarrow \mathrm{wt}_{\mathbf{i}}(\tilde{X}_k) = \underline{-\beta^{\mathbf{i}}_k})$

↑ technical reason. we put −

Thm (Hernandez-Leclerc, Fujita-O, Kashiwara-O, Fujita-Hernadez-O-OYA)

$$\begin{array}{cccc} & T(L^{\mathbf{i}}) & \xrightarrow{\sim} & T(\Lambda^{\mathbf{i}}) \\ e_k - e_{k^-} & \underline{\tilde{Z}_k / \tilde{Z}_{k^-}} & \longmapsto & \underline{\tilde{X}_k} & (1 \le k \le r) \\ (\text{equivalently} & \underline{\tilde{Z}_k} & \longmapsto & \underline{\tilde{X}_k \tilde{X}_{k^-} \cdots X_{k^{-m(k)}}}) \end{array}$$

4. Definition of (quantum) cluster algebra.

For a while, let us forget $\mathbf{i}$-combinatorics!

$(L,\tilde{B})$: compatible pair, $\{\tilde{Z}_k\}_{k\in K}$ = q-comm family controlled by L.

The triple $\mathcal{S} = (L, \tilde{B}, \{\tilde{Z}_k\})$ called a q.seed & $\{\tilde{Z}_k\}$ a q. cluster

clust var.

Set $|K| = r$. For $a = (a_k) \in \mathbb{Z}^K$, $\tilde{Z}^a := q^{\frac{1}{2}\sum_{i>j} a_i a_j l_{ij}} \tilde{Z}_1^{a_1} \cdots \tilde{Z}_r^{a_r}$.

• Mutation at $k \in K_{ex}$.

Set $[a]_+ = \max(a, 0)$ for $a \in \mathbb{Z}$.

For $k \in K_{ex}$ and $(L = (l_{ij}), \tilde{B} = (b_{ij}))$ CP

$$\mu_k(\tilde{B}) = \tilde{B}' = (b'_{ij}) \text{ where } b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}[b_{kj}]_+ + [-b_{ik}]_+ b_{kj} & \text{if } i, j \neq k \end{cases}$$

$$\mu_k(L) = L' = (l'_{ij}) \text{ where } l'_{ij} = \begin{cases} 0 & \text{if } i = j \\ -l_{kj} + \sum_{t\in K} [-b_{tk}]_+ l_{tj} & \text{if } i = k, j \neq k \\ -l_{ik} + \sum_{t\in K} [-b_{tk}]_+ l_{it} & \text{if } i \neq k, j = k \\ l_{ij} & \text{o.w.} \end{cases}$$

We define $a'_i = \begin{cases} -1 & \text{if } i = k \\ [b_{ik}]_+ & \text{if } i \neq k \end{cases}$ $a''_i = \begin{cases} -1 & \text{if } i = k \\ [-b_{ik}]_+ & \text{if } i \neq k \end{cases}$. and $a' = (a'_i)$, $a'' = (a''_i)$ $\in \mathbb{Z}^K$.

We set $\tilde{Z}'_i = \begin{cases} \tilde{Z}_i & \text{if } i \neq k \\ \tilde{Z}^{a'} + \tilde{Z}^{a''} & \text{if } i = k. \end{cases}$

$\mu_k(\mathcal{S}) = (\mu_k(L), \mu_k(\tilde{B}), \mu_k(\{\tilde{Z}_i\}) := \{\tilde{Z}'_i\})$ mutation of $\mathcal{S}$ at $k \in K_{ex}$.

Thm (BZ, FZ) $\mu_k(\mathcal{S})$ is also a q.seed. i.e

① $(\mu_k(L), \mu_k(\tilde{B}))$ CP, $\{\tilde{Z}'_k\}$ = alg ind var

② $\tilde{Z}'_k \tilde{Z}'_s = q^{l'_{ks}} \tilde{Z}'_s \tilde{Z}'_k$.

We call $\{\tilde{Z}'_k\}$ also a q cluster. A monomial $\tilde{Z}'^a_k$ in a cluster $\{\tilde{Z}'_k\}$ a cluster monomial.

q. cluster var.

Finally, the def of q. cluster alg: The q. cluster alg $A_q(\mathcal{S})$ asso. w/ q. seed $\mathcal{S}$ is $\mathbb{Z}[q^{\pm 1/2}]$-alg gen'd by all q. clus vars in q. seed obtained from $\mathcal{S}$ by any seq of mutations. $A_q(\mathcal{S})|_{q=1} = A(\mathcal{S})$ cluster alg.

In the def, we can "only" guarantee that "$A_q(\mathcal{S}) \subset \mathbb{F}(L)$" !!

Thm [BZ, FZ : **quantum Laurent phenomenon**] $\underline{\mathcal{A}_q(\mathcal{S}) \subset \mathcal{T}(L)}$ Indeed!

$\Rightarrow$ Hence we can understand $\mathcal{A}_q$ as a $\underline{\mathbb{Z}[q^{\pm 1/2}]\text{-subalg}}$ of $\underline{\mathcal{T}(L)}$

<u>Conjecture</u> [**quantum Laurent positivity conjecture**]. Moreover, every coefficient of a q-cluster var resides in $\underline{\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]}$

up to now, $\{\widetilde{Z}_k\}$, $\{\widetilde{X}_k\}$ appear. ($\widetilde{Y}$?)

For $\forall k \in K_{ex}$, $\underline{\widetilde{y}_k} := \underline{\widetilde{Z}^{\widetilde{B}e_k}}$ where $\{e_k\}$ is a natural basis of $\mathbb{Z}^{K_{ex}}$.

<u>(Y-variable)</u>

Let us comback to i-combinatorics.

$$\mathcal{S}^{\mathbf{i}} = (L^{\mathbf{i}}, \widetilde{B}^{\mathbf{i}}, \{\widetilde{Z}_k\}).$$

Recall $\mathcal{T}(L^{\mathbf{i}}) \longrightarrow \mathcal{T}(\Lambda^{\mathbf{i}})$

$\widetilde{Z}_k \longrightarrow \widetilde{x}_k \widetilde{x}_{k^-} \cdots \widetilde{x}_{k^{\min}}$ Then $\mathrm{wt}_{\mathbf{i}}(\widetilde{Z}_k) = \beta_k + \beta_{k^-} + \cdots + \beta_{k^{\min}}$.

$\boxed{\text{proposition}}$ $\mathrm{wt}_{\mathbf{i}}(\widetilde{y}_k) = \underline{0}$

(pf) $\boxed{1}$ $u \in [1,r]$ with $u^- \in [1,r]$ & $i = i_u$

$$\text{“}w_{\leq u}\varpi_i - w_{\leq u^-}\varpi_i\text{”} = w_{\leq u}(\varpi_i - s_i\varpi_i) = w_{\leq u}(\alpha_i) \qquad (\because s_j\varpi_i = \varpi_i - \delta_{ij}\alpha_i)$$

$$= w_{\leq u}\Big(2\varpi_i + \sum_j c_{ji}\varpi_j\Big) = \text{“}2w_{\leq u}\varpi_i + \sum_j c_{ji}\, w_{\leq u^-(j)}\varpi_j\text{”}$$

$\Rightarrow$ $$\underline{w_{\leq u}\varpi_i + w_{\leq u^-}\varpi_i + \sum_j c_{ji}\, w_{\leq u^-(j)}\varpi_j = 0} \text{ —— “⊛”} \quad ⊛$$

⊛ – twice

$$w_{\leq u^+}\varpi_i - w_{\leq u^-}\varpi_i + \sum_{j \sim i} c_{ji}\big(w_{\leq (u^+)^-(j)}\varpi_j - w_{\leq u^-(j)}\varpi_j\big) = 0 \text{ —— ⊠}$$

$j \sim i \Leftrightarrow c_{ji} < 0$

$\boxed{2}$ By def of $\widetilde{B}^{\mathbf{i}}$, $\widetilde{y}_u = \widetilde{Z}_{u^-} \cdot \widetilde{Z}_{u^+}^{-1} \cdot \prod_{j \sim i}\big(\widetilde{Z}_{u^-(j)} \cdot \widetilde{Z}_{(u^+)^-(j)}^{-1}\big)^{c_{ji}}$

$$\Rightarrow \mathrm{wt}_{\mathbf{i}}(\widetilde{y}_u) = \mathrm{wt}_{\mathbf{i}}(\widetilde{Z}_{u^-}) - \mathrm{wt}_{\mathbf{i}}(\widetilde{Z}_{u^+}) + \sum_{j \sim i} c_{ji}\big(\mathrm{wt}_{\mathbf{i}}(\widetilde{Z}_{u^-(j)}) - \mathrm{wt}_{\mathbf{i}}(\widetilde{Z}_{(u^+)^-(j)})\big)$$

Then the assertion follow from $wt_{\mathbf{i}}(\tilde{z}_k) = -(\overline{w}_{ik} - w_{\le k}\,\overline{w}_{ik})$.

5. pointed & co-pointed elts in $T(L)$

Def $x \in T(L)$ is pointed if it is of the form

$$x = q^a \tilde{z}^{\bar{g}} + \sum_{c \in \mathbb{Z}_{\ge 0}^{k_{ex}} \setminus \{0\}} P_c \tilde{z}^{\bar{g} + \tilde{B}c} \qquad \tilde{z}^{\tilde{B}e_k} = \tilde{y}_k$$

$$= q^a \tilde{z}^{\bar{g}} \left(1 + \sum_c P_c \tilde{z}^{\tilde{B}c} \right) \quad \text{for some } a \in \mathbb{Z}/2,\ P_c \in \mathbb{Z}[q^{\pm 1/2}] \text{ \& } \bar{g} \in \mathbb{Z}^k.$$

we call "$\bar{g}$" the g-vector of a pointed elt $x \in T(L)$.

$x \in T(L)$ is copointed if it is of the form

$$x = q^a \tilde{z}^{\underline{g}} + \sum_{d \in \mathbb{Z}_{\le 0}^{k_{ex}} \setminus \{0\}} P_d \tilde{z}^{\underline{g} + \tilde{B}d} \qquad \tilde{y}_k^{-1}$$

$$= q^a \tilde{z}^{\underline{g}} \left(1 + \sum_d P_d \tilde{z}^{\tilde{B}d} \right) \quad \text{for some } a \in \mathbb{Z}/2,\ P_d \in \mathbb{Z}[q^{\pm 1/2}] \text{ \& } \underline{g} \in \mathbb{Z}^k.$$

we call $\underline{g}$ the dual g-vector of a copointed elt $x \in T(L)$.

Consequence Every (co)-pointed elt can be written as

$$q^a \tilde{z}^{\bar{g}} (1 + \text{poly}(\tilde{y}_k)) \quad (\text{resp } q^a \tilde{z}^{\underline{g}} \cdot (1 + \text{poly}\ \tilde{y}_k^{-1})) \qquad (\text{not Laurent!!})$$

Thm [Tran, FZ] Every q-cluster var of $A_q(S)$ asso. w/ $S = (L, \tilde{B}, \{\tilde{z}_k\})$ is pointed w.r.t $T(L)$.

$\Rightarrow$ Mul. str of A_q is determined by the ones among $\tilde{z}_k$ & $\tilde{y}_k$

Corollary Every q-cluster var of $A_q(S^{\mathbf{i}})$ is homo w.r.t $wt_{\mathbf{i}}$

Recall • $k_{\mathbf{i}}^+(j)$, $k_{\mathbf{i}}^+$, $k_{\mathbf{i}}^-(j)$, $k_{\mathbf{i}}^-$, Hence the mul. str. of $A_q(S^{\mathbf{i}})$ is determined by the ones among $\tilde{x}_k$ & $\tilde{y}_k$.

<u>Lemma</u> The <u>mul. str among $\star$</u> is given as follows: (cf <u>Hernandez, Kashiwara-O</u>)

$$\tilde{x}_p \tilde{y}_s = \begin{cases} q^{\underline{-(\alpha_{i_p}, \alpha_{i_p})}} \tilde{y}_s \tilde{x}_p & \text{if } \underline{p = s} \\ q^{\underline{(\alpha_{i_p}, \alpha_{i_p})}} \tilde{y}_s \tilde{x}_p & \text{if } \underline{p = s^+} \\ \underline{\tilde{y}_s \tilde{x}_p} & \text{o.w} \end{cases}$$

$$\tilde{y}_t \tilde{y}_u = \begin{cases} q^{\underline{\pm(\alpha_{i_u}, \alpha_{i_u})}} \tilde{y}_u \tilde{y}_t & \text{if } \underline{t = u^{\mp}}. \\ q^{\underline{2(\alpha_{i_u}, \alpha_{i_t})}} \tilde{y}_u \tilde{y}_t & \text{if } \underline{t = u^-(i_t), u = (t^+)^-(i_u)} \quad \checkmark \\ q^{\underline{-2(\alpha_{i_u}, \alpha_{i_t})}} \tilde{y}_u \tilde{y}_t & \text{if } \underline{u = t^-(i_u), t = (u^+)^-(i_t)} \quad \checkmark \\ \underline{\tilde{y}_u \tilde{y}_t} & \underline{\text{o.w}}. \end{cases}$$

<u>DAY I</u>