Cluster structures on q-Painlevé systems via toric geometry

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Advances in Cluster Algebras 2024
2024/03/11

Outline

1. Geometric interpretation of mutations due to Gross-Hacking-Keel
2. The cluster modular groupoid (symmetry of cluster theory)
3. q-Painlevé systems

Review: mutations

- The mutation formula

$$\mu_k^* : \operatorname{Frac}\mathbb{Z}[{x'}_1^{\pm 1}, \dots, {x'}_n^{\pm 1}] \to \operatorname{Frac}\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

$$\mu_k^*(x_i') := \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[b_{jk}]_+} + \prod_{j=1}^{n} x_j^{[-b_{jk}]_+}\right) & \text{if } i = k \\ x_i & \text{if } i \neq k \end{cases}$$

- How to prove that μ_k^* is well-defined?
- This reduces to prove the algebraic independence of $\mu_k^*(x_i')$.
- The most common explanation would be to use the fact that μ_k^* is an involution.
- But to be precise, can we use the property of μ_k^* before defining it?

Review: mutations

- We can actually prove that

$$\mu_k^* : \mathbb{Z}[{x'}_1^{\pm}, \ldots, {x'}_n^{\pm}]_{f'_k} \to \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]_{f_k}$$

  is well-defined, where this is a map between localizations associated with $f_k \coloneqq 1 + \hat{y}_k$ with $\hat{y}_k \coloneqq \prod_{j=1}^n x_j^{b_{jk}}$

  1. Literally, the mutation formula defines

$$\mu_k^* : \mathbb{Z}[{x'}_1^{\pm}, \ldots, {x'}_n^{\pm}] \to \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]_{f_k}$$

  2. To show that this induces a map from localization, we need to show that $\mu_k^*(f'_k)$ is a unit.
  3. This follows from $\mu_k^*(1 + \hat{y}'_k) = 1 + \hat{y}_k^{-1} = \hat{y}_k^{-1}(1 + \hat{y}_k)$.

- Then we obtain the map between the fraction fields by using the fact that a localization of a localization is a localization.

Review: mutations

- Geometrically, the mutation map

$$\mu_k^* : \mathbb{Z}[{x'}_1^{\pm}, \dots, {x'}_n^{\pm}]_{f'_k} \to \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]_{f_k}$$

  has the better interpretation than that between fraction fields.
- Recall that $\operatorname{Spec} \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]_{f_k} \cong (\mathbb{C}^\times)^n \setminus \{f_k = 0\}$.
- Thus we have a birational map

$$\mu_k : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$$

  that is an isomorphism on the open sets $(\mathbb{C}^\times)^n \setminus \{f_k = 0\}$ and $(\mathbb{C}^\times)^n \setminus \{f'_k = 0\}$
- We can glue two algebraic tori $(\mathbb{C}^\times)^n$ by μ_k along these open subsets.

Mutations due to GHK

- Gross-Hacking-Keel (2015) found another interpretation of the glued space as a blowup of a toric varieties.

Convention

- From now on, we work in GHK convention.
- We focus on cluster Poisson variables
  - y_i in Fomin-Zelevinsky
  - X_i in Fock-Goncharov
  - z^{e_i} in Gross-Hacking-Keel

Setting of cluster theory

- A **fixed data** consists of
  - a finite index set I
  - a free-abelian (or just a torsion-free abelian) group N
  - a skew-symmetric bilinear form $B : N \times N \to \mathbb{Z}$
- A **seed** $s = (e_i)_{i \in I}$ is a basis (or just a tuple) of N.
- A **seed mutation** $\mu_{k,\varepsilon} : s \to s'$ for $k \in I$, $\varepsilon : \mathrm{sign}$, is defined by

$$e'_i = \begin{cases} e_i + [\varepsilon B(e_i, e_k))]_+ e_k & \text{if } i \neq k \\ -e_k & \text{if } i = k \end{cases}$$

- For any seed mutation $\mu_{k,\varepsilon} : s \to s'$, we define a birational map

$$\mathcal{X}(\mu_{k,\varepsilon}) : \mathcal{X}_s \to \mathcal{X}_{s'}$$

where $\mathcal{X}_s := \operatorname{Spec} \mathbb{C}[N]$ for any seed s, via the ring isomorphism

$$\mathcal{X}(\mu_{k,\varepsilon})^* : \mathbb{C}[N]_{1+z^{\varepsilon e_k}} \to \mathbb{C}[N]_{1+z^{\varepsilon e_k}}$$

$$\mathcal{X}(\mu_{k,\varepsilon})^*(z^n) := z^n(1 + z^{\varepsilon e_k})^{-B(n,e_k)}$$

Mutations due to GHK

- We will work over $\mathbb{C}$.
- We want to extends $\mu_{k,\varepsilon} : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ to a regular map.
- Choose a basis $u_1, \ldots, u_n$ of N such that $B(u_1, e_k) = 1$, $B(u_i, e_k) = 0$ $(i \geq 2)$, assuming this is possible.
- Set $x_i \coloneqq z^{u_i}$. (Remark: these are not cluster variables)
- Then $\mu_{k,\varepsilon}$ is expressed as

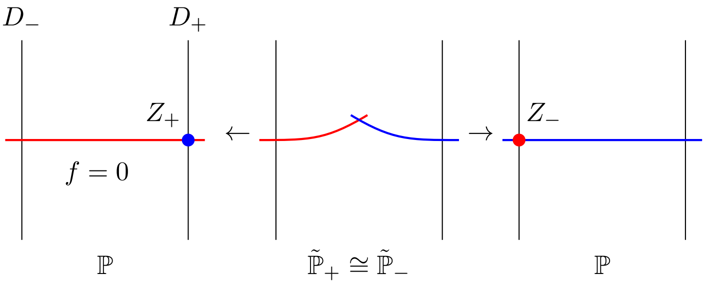
$$(x_1, \ldots, x_n) \mapsto (f(x_2, \ldots, x_n)^{-1} x_1, x_2, \ldots, x_n)$$

Mutations due to GHK (one direction case)

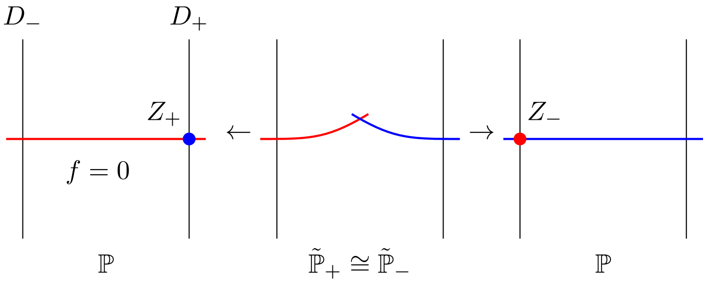
- Define $\mathbb{P} := \mathbb{P}^1_{x_1} \times (\mathbb{C}^\times)^{n-1}_{x_2,\dots,x_n}$, $D_+ := (x_1 = 0)$, and $D_- := (x_1 = \infty)$
- Define $Z_\pm := D_\pm \cap (f = 0)$, and let

$$b_\pm : \tilde{\mathbb{P}}_\pm \to \mathbb{P}$$

  be a blowup of $Z_\pm$.
- Then $\mu_{k,\varepsilon} : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ extends to an isomorphism $\mu_{k,\varepsilon} : \tilde{\mathbb{P}}_+ \to \tilde{\mathbb{P}}_-$.



Mutations due to GHK (one direction case)



proof. $\mu_{k,\varepsilon} : \mathbb{P} \dashrightarrow \mathbb{P}$ is given by

$$((x_1 : y_1), (x_2, \ldots, x_n)) \mapsto ((x_1 : f(x_2, \ldots, x_n)y_1), (x_2, \ldots, x_n))$$

The undifinedness along $Z_+ = (x_1 = f = 0)$ is resolved by a blowup of Z_+, and we get a map $\mu_{k,\varepsilon} : \tilde{\mathbb{P}}_+ \to \mathbb{P}$. This lifts to a $\mu_{k,\varepsilon} : \tilde{\mathbb{P}}_+ \to \tilde{\mathbb{P}}_-$ by the universality of blowup of Z_-. The inverse is given by the same argument for $\mu_{k,-\varepsilon}$.

Gluing tori vs blowup

- Let $U_k := \tilde{\mathbb{P}}_+ \setminus D_+$.
- Let X_k be a space obtained by gluing two $(\mathbb{C}^\times)^n$ by $\mu_{k,\varepsilon}$ along the open subsets $(\mathbb{C}^\times)^n \setminus \{1 + z^{e_k} \neq 0\}$.

Proposition [Gross-Haking-Keel (2015)]

We have an open immersion $X_k \hookrightarrow U_k$ whose image is of codimension two.

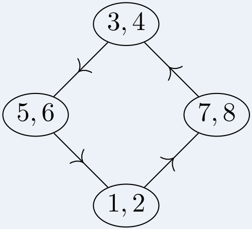
- Codimension two is "small", so the space U_k obtained by a blowup can be regarded as another realization of the glued space.

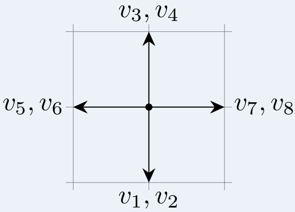
Mutations due to GHK (general case)

- In general, we need to blowup $Z_{i,+} = D_{i,+} \cap (1 + z^{e_k} = 0)$ for each $i \in I$.
- To define $Z_{i,+}$, we need to partially compactify $(\mathbb{C}^\times)^n$ to get the boundary component $D_{i,+}$.
- This is achieved by consider the toric variety associated with the fan generated by $v_i := \bar{e}_i \in \overline{N}$, where $\overline{N} := N/\operatorname{Ker} B$.

Example

Let $N \cong \mathbb{Z}^8$. Define B by the left quiver.





Mutations due to GHK (general case)

- Define a fan $\Sigma_+ \coloneqq \{0\} \cup \{\mathbb{R}_{\geq 0} v_k, \mathbb{R}_{\leq 0} v_k\} \cup \{\mathbb{R}_{\geq 0} v_i \mid i \neq k\}$
- Then we get the toric variety $\mathrm{TV}(\Sigma_+) \coloneqq \bigcup_{\sigma \in \Sigma_+} \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap N]$
- Let $D_{i,+} \subseteq \mathrm{TV}(\Sigma_+)$ be the divisor corresponding to v_i.
- Set $Z_{i,+} \coloneqq D_{i,+} \cap (f_i = 0)$, and let

$$b_+ : \widetilde{\mathrm{TV}}(\Sigma_+) \to \mathrm{TV}(\Sigma_+)$$

  be a blowup of $\bigcup_i Z_{i,+}$.
- The minus version is defined by the same argument for $s' = (e'_i)$.

Proposition [Gross-Haking-Keel (2015)]

$\mu_{k,\varepsilon} : (\mathbb{C}^\times)^n \dashrightarrow (\mathbb{C}^\times)^n$ extends to an isomorphism $\mu_{k,\varepsilon} : \widetilde{\mathrm{TV}}(\Sigma_+) \dashrightarrow \widetilde{\mathrm{TV}}(\Sigma_-)$ outside a codimension 2 subset.

- As a consequence, $\widetilde{\mathrm{TV}}(\Sigma_+)$ can be considered as a realization of the cluster variety up to codimension two.

Symmetry of cluster theory

- A symmetry of the cluster theory is well described by a groupoid.
- It is generated by the seed mutation and seed isomorphisms.
- A **seed isomorphism** $\sigma : s \to s'$ consists of
  - a bijection $\sigma^\sharp : I \to I$
  - an isomorphism $\sigma^\flat : N \to N$ of abelian groups preserving the skew-symmetric form
  - such that $\sigma^\flat(e_i) = e'_{\sigma^\sharp(i)}$ for all $i \in I$

$$\begin{array}{ccc} I & \xrightarrow{\sigma^\sharp} & I \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ N & \xrightarrow[\sigma^\flat]{} & N \end{array}$$

- For a seed isomorphism $\varepsilon\sigma : \mathbf{i} \to \mathbf{i}'$, we define an isomorphism:

$$\mathcal{X}(\mu_{k,\varepsilon}) : \mathcal{X}_s \to \mathcal{X}_{s'}$$

$$\mathcal{X}(\mu_{k,\varepsilon})^* : \mathbb{C}[N] \to \mathbb{C}[N], \quad z^{\sigma^\flat(n)} \mapsto z^n.$$

Cluster modular groupoid

- The **cluster modular groupoid**, denoted by $\mathbf{Seed}$, is a category generated by seed mutations and seed isomorphisms, modulo the relation

$$\forall \mu, \mu' : s \to s', \quad \mu \sim \mu' \quad \text{if} \quad \mathcal{X}(\mu) = \mathcal{X}(\mu')$$

  - Objects: seeds
  - Morphisms: formal compositions of seed mutations and seed isomorphisms, modulo cluster relations
- The inverse of $\mu_{k,\varepsilon}$ is $\mu_{k,-\varepsilon}$.
- We have a functor $\mathcal{X} : \mathbf{Seed} \to \mathbf{AlgTorus}_{\mathbf{Birat}}$
- $\mathrm{Aut}_{\mathbf{Seed}}(s)$ is called the **cluster modular group** at s.
- Remark: these definitions are slightly modified version of those in [Fock-Goncharov (2009)].

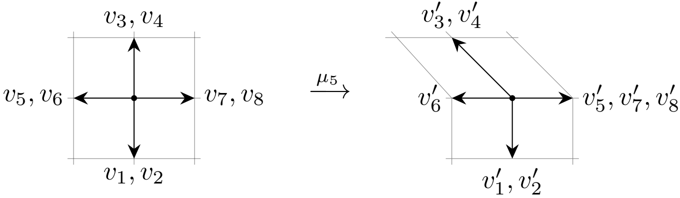
Let $s = (e_1, e_2)$ be a seed in $N = \mathbb{Z}^2$. Define $B(e_1, e_2) = -B(e_2, e_1) = 1$. Then

$$\mathrm{Aut}_{\mathbf{Seed}}(s) = \langle \sigma \circ \mu_{1,+} \rangle \cong \mathbb{Z}/5\mathbb{Z}$$

where $\sigma : (e_1, e_2) \mapsto (-e_2, e_1)$.

Reflections

- These are special elements in $\mathrm{Aut}_{\mathbf{Seed}}(s)$ that have order 2, which we call **reflections**.
- If $v_i = v_j$, then we have a seed isomorphism $(i,j) \in \mathrm{Aut}_{\mathbf{Seed}}(s)$.
- More generally, there exists $\mu : s \to s'$ such that $v'_i = v'_j$, we have a seed isomorphism $\mu^{-1} \circ (i,j) \circ \mu \in \mathrm{Aut}_{\mathbf{Seed}}(s)$.



Roots

- Define the set of **roots**

$$\Delta_s := \{\alpha \in \operatorname{Ker} B \mid \exists\, s',\ \exists\, \mu : s \to s',\ \exists\, i, j \in I,\ \alpha = e'_j - e'_i\}$$

- For $\alpha \in \Delta_s$, we define the **reflection** $r^*_\alpha \in \operatorname{Aut}_{\mathbf{Seed}}(s)$ by

$$r^*_\alpha := \mu^{-1} \circ (i, j) \circ \mu$$

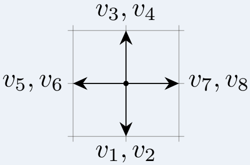
  by choosing s', μ, i, j.
- r^*_α satisfies
  - $r^*_\alpha \circ r^*_\alpha = \mathrm{id}$
  - $r_\alpha(\alpha) = -\alpha$, where $r_\alpha = r^{**}_\alpha : \operatorname{Ker} B \to \operatorname{Ker} B$ is the action given by the functor $\mathcal{X}$.

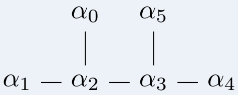
Conjecture

r^*_α does not depend on these choices.

Roots

Example





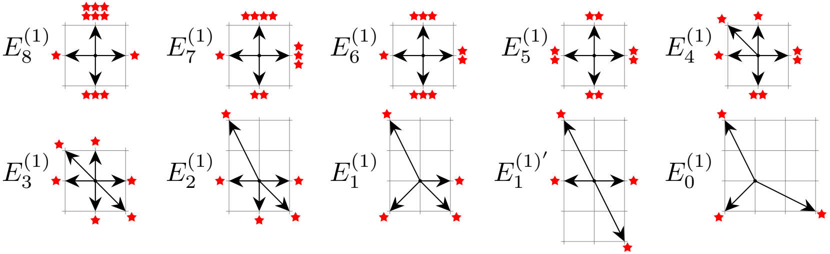
$$\Delta_s = \text{the set of real roots of type } D_5^{(1)}$$

$$\alpha_0 = e_2 - e_1, \quad \alpha_1 = e_4 - e_3, \quad \alpha_2 = e_1 + e_3,$$
$$\alpha_3 = e_5 + e_7, \quad \alpha_4 = e_6 - e_5, \quad \alpha_5 = e_8 - e_7$$

$$r_0^* = (1,2), \quad r_1^* = (3,4), \quad r_2^* = \mu_{1,-} \circ (1,3) \circ \mu_{1,+},$$
$$r_3^* = \mu_{5,-} \circ (5,7) \circ \mu_{5,+}, \quad r_4^* = (3,4), \quad r_5^* = (7,8).$$

q-Painlevé systems

- When B is of rank two (in the sense of the linear algebra), then by GHK interpretation, the cluster Poisson variety is a family of blowups of toric surfaces.
- The intersection form on a surface induces the symmetric bilinear form on $\mathrm{Ker}\, B$ [GHK 2015].
- r_α in this setting is genuinely a reflection with respect to this form.
- If the roots system Δ_s is of "affine type", we have a nice theory.
- In fact, this is the theory of q-Painlevé systems, developed by Sakai from geometric viewpoint.



q-Painlevé systems

- Example: q-$\mathrm{P_{VI}}$ [Jimbo-Sakai 1996]:

$$f(qt)f(t) = b_7 b_8 \frac{g(qt) - qb_1 t}{g(qt) - b_3} \frac{g(qt) - qb_2 t}{g(qt) - b_4}$$

$$g(qt)g(t) = b_3 b_4 \frac{f(t) - b_5 t}{f(t) - b_7} \frac{f(t) - b_6 t}{f(t) - b_8}$$

- $b_1, \ldots, b_8$ are constants satisfying $q = b_3 b_4 b_5 b_6 / b_1 b_2 b_7 b_8$.
- They derived q-$\mathrm{P_{VI}}$ from the connection preserving deformation of a linear q-difference equation.
- Recently, physicists and mathematicians are actively studying relation to q-deformed conformal blocks, topological strings,...

From cluster algebras

- The time evolution is given by a birational map

$$\left(\begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{array} ;\ f, g \right) \xmapsto{q\text{-}\mathsf{P}_{\mathrm{VI}}} \left(\begin{array}{cccc} qb_1 & qb_2 & b_3 & b_4 \\ qb_5 & qb_6 & b_7 & b_8 \end{array} ;\ \bar{f}, \bar{g} \right)$$

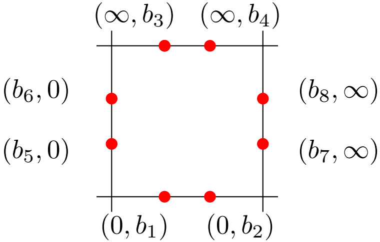
$$\bar{f} = \frac{b_7 b_8}{f} \frac{\bar{g} - qb_1}{\bar{g} - b_3} \frac{\bar{g} - qb_2}{\bar{g} - b_4}, \quad \bar{g} = \frac{b_3 b_4}{g} \frac{f - b_5}{f - b_7} \frac{f - b_6}{f - b_8}$$

q-Painlevé systems

Proposition [Sakai 2001]

q-P_{VI} gives an isomorphism $X_b \cong X_{\bar{b}}$ between algebraic surfaces

- X_b is called the space of initial values for q-P_{VI}, and obtained by an 8-points blowup from $\mathbb{P}^1 \times \mathbb{P}^1$.



- The isomorphism q-$\mathrm{P}_{\mathrm{VI}} : X_b \cong X_{\bar{b}}$ can be realized by a sequence of simpler isomorphisms.

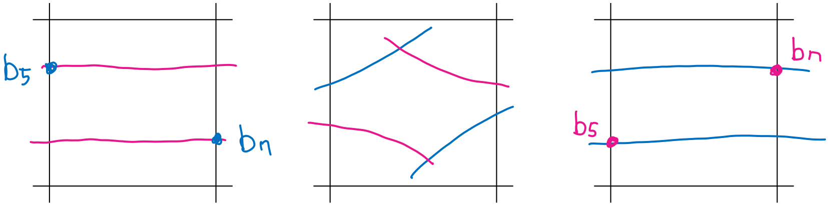
$$q\text{-}\mathrm{P}_{\mathrm{VI}} = \sigma_2 \circ r_2 \circ r_1 \circ r_0 \circ r_2 \circ \sigma_1 \circ r_3 \circ r_5 \circ r_4 \circ r_3$$

q-Painlevé systems

- $r_0, \ldots, r_5, \sigma_1, \sigma_2$ gives an action of the extended affine Weyl group of type $D_5^{(1)}$

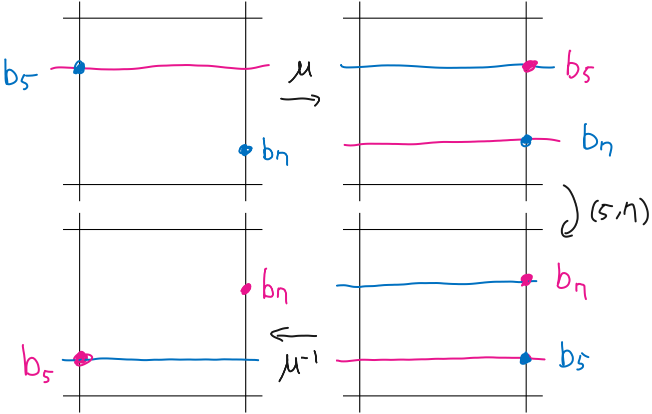
$$\begin{array}{ccccccc} & & \alpha_0 & & \alpha_5 & & \\ & & | & & | & & \\ \alpha_1 & — & \alpha_2 & — & \alpha_3 & — & \alpha_4 \end{array}$$

- r_0, r_1, r_4, r_5 are just permutations (e.g. $r_0 = (b_1 \leftrightarrow b_2)$)
- r_3: blowup of $(f, g) = (b_5, 0), (b_7, \infty)$, and then blowdown the strict transforms of $(f - b_5 = 0)$ and $(f - b_7 = 0)$ (similarly for r_2)



q-Painlevé systems

- In other words, we have a decomposition $s_3 = \mu^{-1} \circ (5,7) \circ \mu$.



- The map μ is a mutation!

Historical remark

- Okubo (2013): some elements of q-Painlevé systems (non-factorized form, that is, q-P_{VI} itself for instance) can be realized by mutation sequences
- Bershtein-Gavrylenko-Marshakov (2018): all symmetries of q-P_{VI} can be realized by mutation sequences. They derivation of quivers is from cluster integrable systems [Goncharov-Kenyon 2013].
- M (2024): revealing geometric origin of these quivers, and clarifying the relation to Sakai's framework.