Quantum cluster mutation and 3D Integrability

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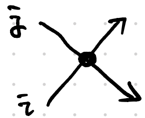
arXiv: 2310.14493, 2310.14529, 2401.**

Check my webpage!

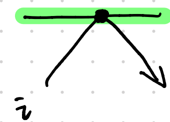
§ Introduction

- 2D (1+1)

R_{ij}

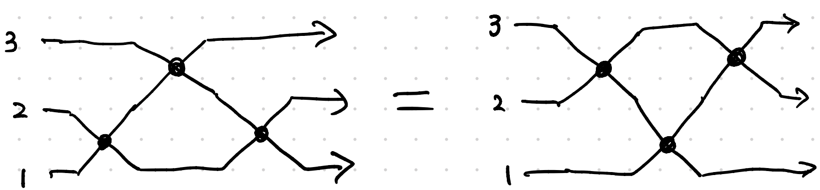


K_i



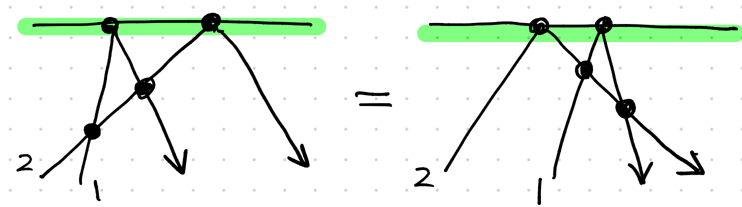
Yang-Baxter eq.

$$R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}$$

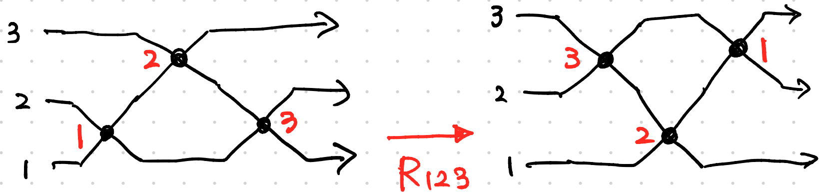


Reflection eq.

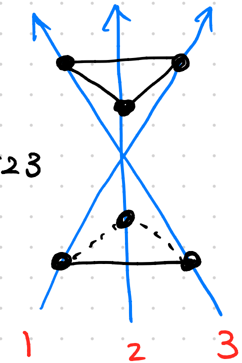
$$K_2 R_{21} K_1 R_{12} = R_{21} K_1 R_{12} K_2$$



- $3D\ (2+1)$

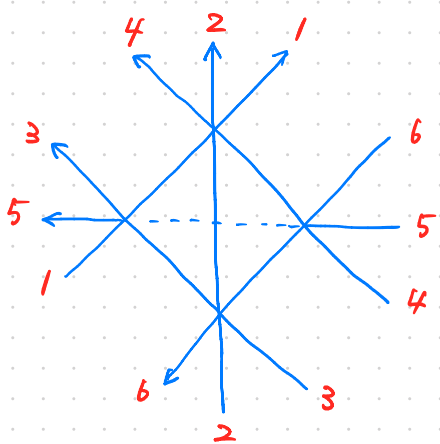


$\longleftrightarrow$ R_{123}



① <u>Tetrahedron eq.</u> [Zamolodchikov 1980]

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \curvearrowright V^{\otimes 6}$$



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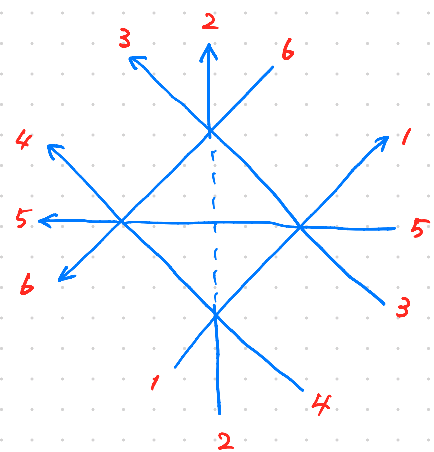
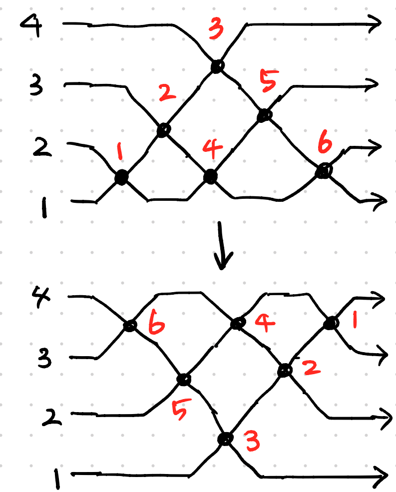
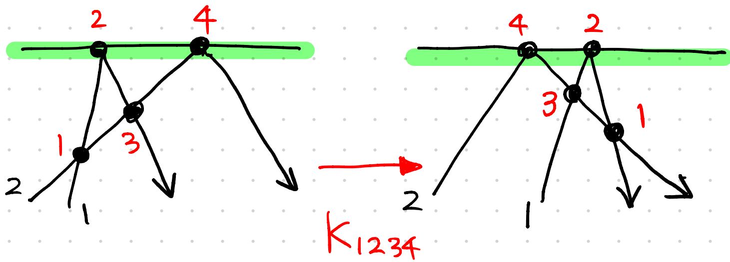


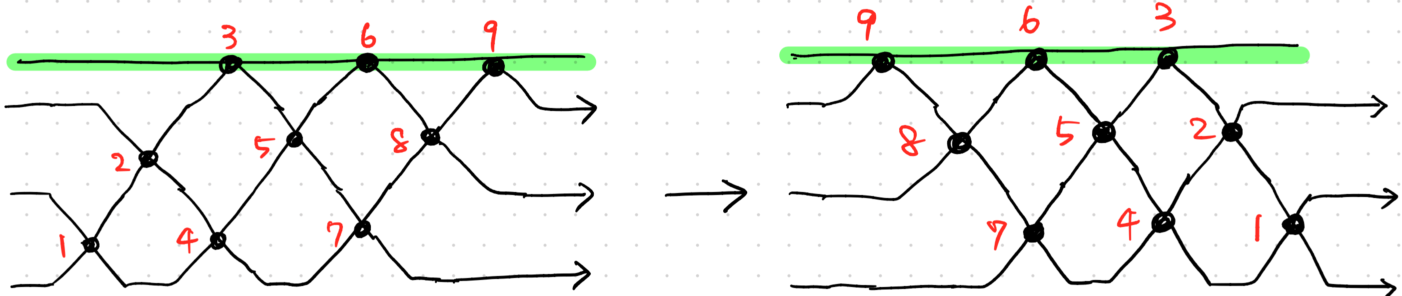




(Ref) §4, Kuniba's book
"Quantum groups in 3D integrability"

② 3D Reflection eq. [Isaev-Kulish 1997]

$R_{457}K_{4689}K_{2379}R_{258}R_{178}K_{1356}R_{124}$ = inverse order. $\curvearrowright (V\otimes V\otimes W)^{\otimes 3}$

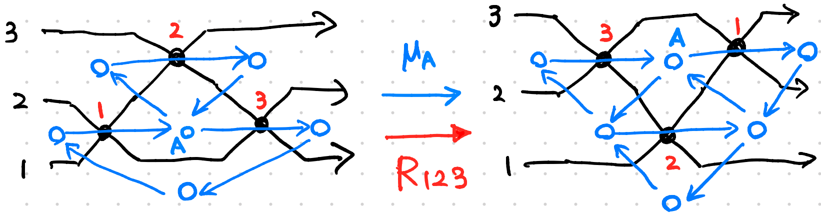


Idea

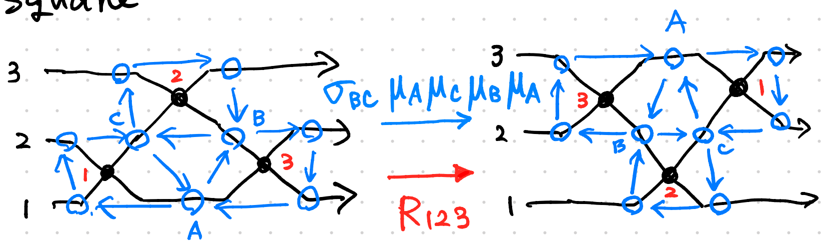
Place some quivers on the wiring diagrams,
and realize the transformations by mutating the quivers.

[Sun - Yagi 2022]

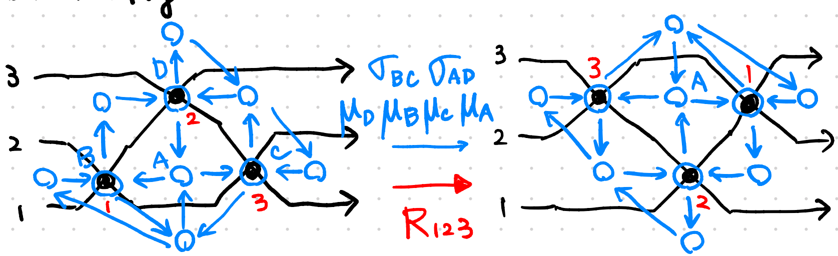
- triangle



- square



- butterfly

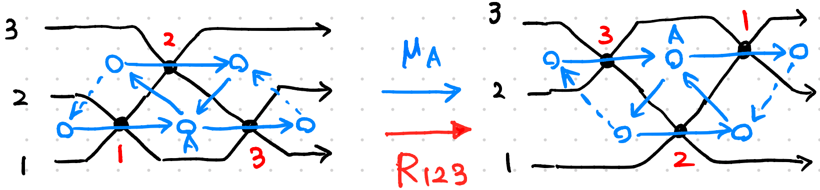


$\left\{ \begin{array}{l} \text{quantum } y\text{-variables} \\ \text{quantum-dilog. function} \\ q\text{-Weyl alg. on } \circ \end{array} \right.$

$\rightsquigarrow$ matrix elements of R

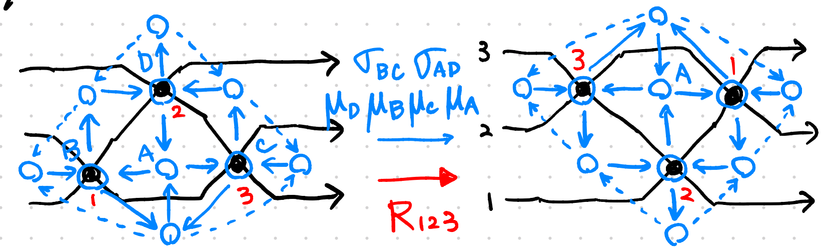
[I-Kuniba-Terashima 2023a,b] [I-K-Sun-T-Yagi 2024]

- Fock-Goncharov quiver ($\doteq$ triangle)



- square

- symmetric butterfly ($\doteq$ butterfly)



② K in terms of q-Weyl alg.

$\boxed{FG}$ ($\boxed{SB}$ in progress)

{ decomposition of quantum mutation
q-Weyl alg. on • → coordinate rep. / momentum rep.
q-dilog / modular (non compact) dilog.

⇓

① R in terms of q-Weyl alg.

$\boxed{FG}$ ••• [Maillard-Sergeev 1997]
[Bytsko-Volkov 2015]

$\boxed{Sq}$ ••• [Sergeev 1999]

$\boxed{SB}$ $\xrightarrow{\text{rep.}}$ [Kapranov-Voevodsky 1994]
[Bazhanov-Sergeev 2006]
[Bazhanov-Mangazeev-Sergeev 2010]
[Kuniba-Matsuike-Yoneyama 2023]

§ Fock-Goncharov quivers

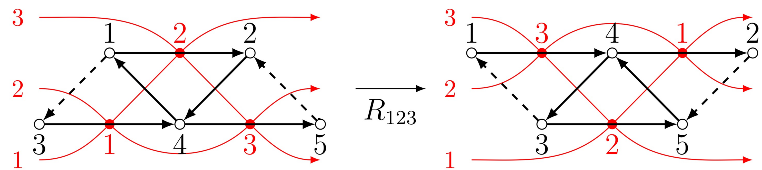
$\mathfrak{g}$: fin. dim. simple Lie alg. $\quad W(\mathfrak{g})$: Weyl grp $= \langle r_i \,;\, i=1,\dots,\mathrm{rank} \rangle$

$w_0 \in W(\mathfrak{g})$: longest $\quad r_{i_1} r_{i_2} \cdots r_{i_\ell}$: reduced expr. $\rightsquigarrow$ $\mathbf{i} = i_1 i_2 \cdots i_\ell$

$\mathbf{i} \longmapsto J_{\mathbf{i}}$: FG quiver [Fock-Goncharov 2006]

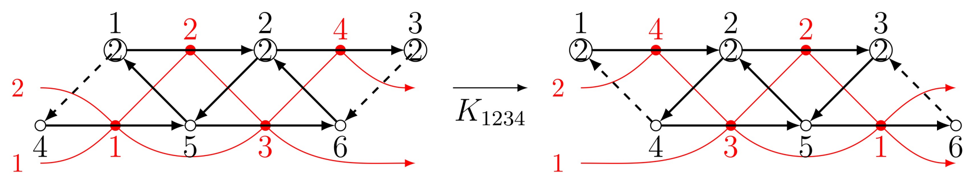
$\underline{\mathfrak{g} = A_2}$

$J_{121} \underset{\mu_4}{\longleftrightarrow} J_{212}$



$\underline{\mathfrak{g} = C_2}$

$J_{1212} \underset{\mu_2 \mu_5 \mu_2}{\longleftrightarrow} J_{2121}$

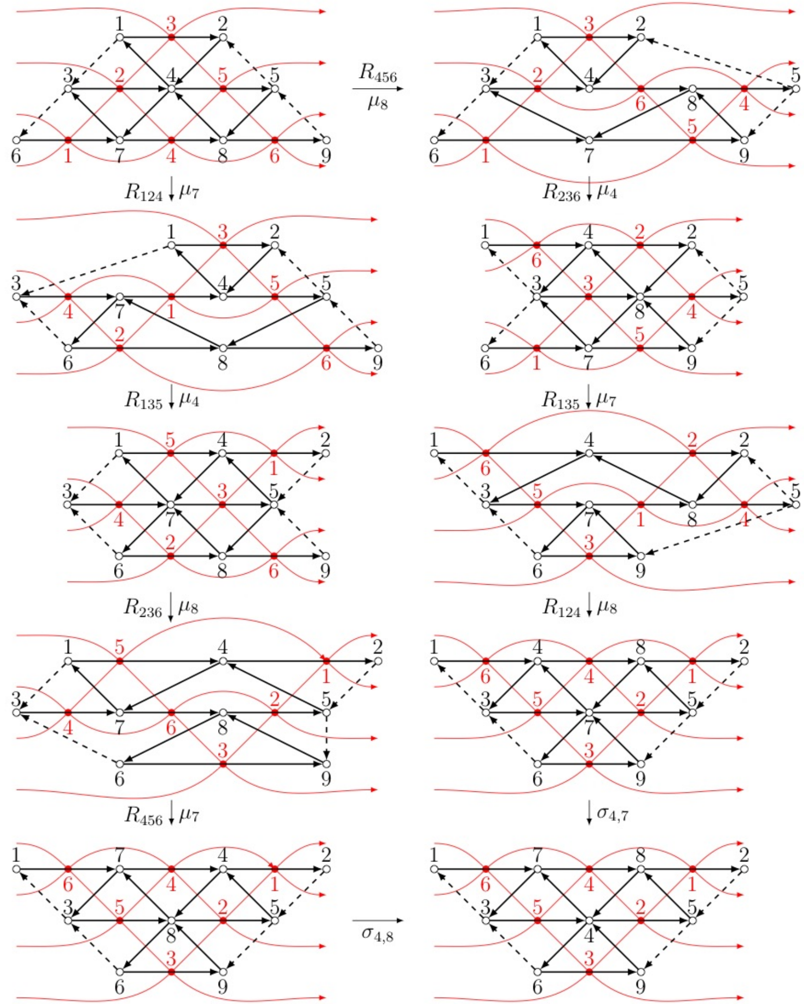


$\mathfrak{g} = A_3$

$$J_{123121} \longrightarrow J_{321323}$$

$$R_{456}R_{236}R_{135}R_{124} = R_{124}R_{135}R_{236}R_{456}.$$

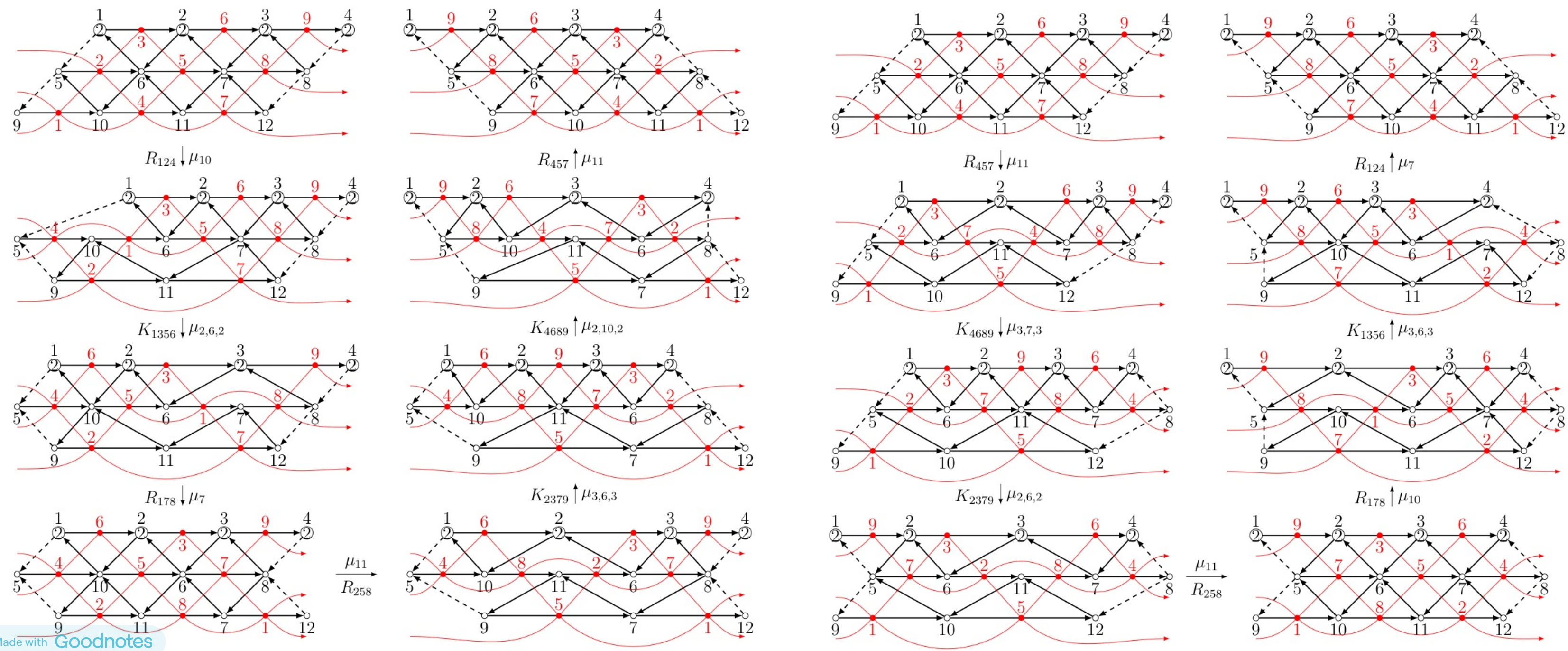
$$\sigma_{4,8}\mu_7\mu_8\mu_4\mu_7(J_{123121}, y) = \sigma_{4,7}\mu_8\mu_7\mu_4\mu_8(J_{123121}, y)$$



$\underline{\mathfrak{g} = C_3}$ $\quad J_{123123123} \longrightarrow J_{321321321}$

$$R_{457}K_{4689}K_{2379}R_{258}R_{178}K_{1356}R_{124} = R_{124}K_{1356}R_{178}R_{258}K_{2379}K_{4689}R_{457},$$

$$\underset{+}{\mu_{11}}\,\underset{-++}{\mu_{2,10,2}}\,\underset{-++}{\mu_{3,6,3}}\,\underset{+}{\mu_{11}}\,\underset{+}{\mu_{7}}\,\underset{-++}{\mu_{2,6,2}}\,\underset{+}{\mu_{10}}(J_{123123123}, y) = \underset{+}{\mu_{7}}\,\underset{-++}{\mu_{3,6,3}}\,\underset{+}{\mu_{10}}\,\underset{+}{\mu_{11}}\,\underset{-++}{\mu_{2,6,2}}\,\underset{-++}{\mu_{3,7,3}}\,\underset{+}{\mu_{11}}(J_{123123123}, y)$$



§ Quantum mutation [Fock-Goncharov 2009]

I : a finite set

$(B=(b_{ij})_{i,j\in I},\ Y=(Y_i)_{i\in I})$: a quantum y-seed $(b_{ij}\in\frac{\mathbb{Z}}{2})$

B : skew symmetrizable ; $\exists d=\mathrm{diag}(d_i)_{i\in I}\in(\mathbb{Z}_{>0})^I$, $\hat{B}:=Bd$ is skew sym.

$(B,d)\underset{1:1}{\longleftrightarrow}$ weighted quiver ; $\mathrm{wt}(i)=d_i$

- $\mathcal{Y}(B)$: a skew field gen. by Y_i ; $Y_iY_j=q^{2\hat{b}_{ij}}Y_jY_i$

  $\cup$

  $\mathcal{T}(B)$: a quantum torus alg. gen. by $\mathbf{Y}^{e_i}=Y_i$; $e_i\in\mathbb{Z}^I$

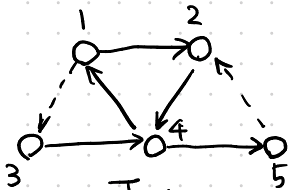
$$\mathbf{Y}^{e_i+e_j}=q^{\hat{b}_{ji}}\mathbf{Y}^{e_i}\mathbf{Y}^{e_j}$$

- $\Phi_q(y):=\dfrac{1}{\prod_{k=0}^{\infty}(1+q^{2k+1}y)}$: quantum dilogarithm funct.

$$\begin{cases}\Phi_q(q^2y)=(1+qy)\Phi_q(y)\\ UW=q^2WU \Rightarrow \Phi_q(U)\Phi_q(W)=\Phi_q(W)\Phi_q(q^{-1}UW)\Phi_q(U)\ \text{: pentagon id.}\end{cases}$$

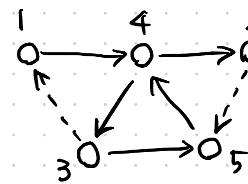
• $T_{k,\pm}$

(Ex)



J_{121}

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \xrightarrow{\mu_4}$$



J_{212}

$$\begin{bmatrix} Y_1(1+qY_4) \\ Y_2(1+qY_4^{-1})^{-1} \\ Y_3(1+qY_4^{-1})^{-1} \\ Y_4^{-1} \\ Y_5(1+qY_4) \end{bmatrix}$$

$$\mu_4^* : \mathcal{Y}(J_{212}) \longrightarrow \mathcal{Y}(J_{121}) \ ; \ (Y_i')$$

a morphism of skewfields

$$(Y_i') \xmapsto{T_{4,+}} \begin{bmatrix} Y_1 \\ q^{-1}Y_2Y_4 \\ q^{-1}Y_3Y_4 \\ Y_4^{-1} \\ Y_5 \end{bmatrix} \xmapsto{\mathrm{Ad}\,\Phi_q(Y_4)} \begin{bmatrix} Y_1(1+qY_4) \\ Y_2(1+qY_4^{-1})^{-1} \\ Y_3(1+qY_4^{-1})^{-1} \\ Y_4^{-1} \\ Y_5(1+qY_4) \end{bmatrix}$$

$$(Y_i') \xmapsto{T_{4,-}} \begin{bmatrix} qY_1Y_4 \\ Y_2 \\ Y_3 \\ Y_4^{-1} \\ qY_1Y_5 \end{bmatrix} \xmapsto{\mathrm{Ad}\,\Phi_q(Y_4^{-1})^{-1}} \begin{bmatrix} Y_1(1+qY_4) \\ Y_2(1+qY_4^{-1})^{-1} \\ Y_3(1+qY_4^{-1})^{-1} \\ Y_4^{-1} \\ Y_5(1+qY_4) \end{bmatrix}$$

Def.

$\varepsilon \in \{+,-\}$, $k \in I$ unfrozen

$$T_{k,\varepsilon} : \mathcal{Y}(B') \longrightarrow \mathcal{Y}(B) \ ; \ Y_i' \longmapsto \begin{cases} Y_k^{-1} & i=k \\ q^{-b_{ik}[\varepsilon b_{ik}]_+} Y_i Y_k^{[\varepsilon b_{ik}]_+} & \text{ow} \end{cases}$$

Fact [Keller 11] (cf. [Kashaev–Nakanishi 11])

① $\mathrm{Ad}(\Phi_{q_k}(Y_k)) \circ T_{k,+} = \mathrm{Ad}(\Phi_{q_k}(Y_k^{-1})^{-1}) \circ T_{k,-} = \mu_k^* \quad ; \quad q_k = q^{d_k}$

② $(B, Y) \xrightarrow[\mu_{i_1}]{} (B(2), Y(2)) \xrightarrow[\mu_{i_2}]{} \cdots \xrightarrow[\mu_{i_L}]{} (B(L+1), Y(L+1)) \xrightarrow[\sigma \in \mathfrak{S}_I]{} (B, Y)$ (1)

α_k : the c-vector of $Y(k)_{i_k}$

ε_k : the tropical sign of α_k

$\Longrightarrow \begin{cases} T_{i_1,\varepsilon_1} T_{i_2,\varepsilon_2} \cdots T_{i_L,\varepsilon_L} \circ \sigma = \mathrm{id} \quad (\Leftarrow \text{periodicity of trop. } y) \\ \Phi_{q_{i_1}}(\mathbf{Y}^{\alpha_1\varepsilon_1})^{\varepsilon_1} \Phi_{q_{i_2}}(\mathbf{Y}^{\alpha_2\varepsilon_2})^{\varepsilon_2} \cdots \Phi_{q_{i_L}}(\mathbf{Y}^{\alpha_L\varepsilon_L})^{\varepsilon_L} = 1 \end{cases}$ (2)

□

Remark

① $\mathbf{Y}^{\alpha_k} = T_{i_1,\varepsilon_1} \cdots T_{i_{k-1},\varepsilon_{k-1}}(\mathbf{Y}(k)_{i_k}) \in \mathcal{T}(B)$

$Y(k)_i = \mu_{i_1}^* \cdots \mu_{i_{k-1}}^*(\mathbf{Y}(k)_i) \in \mathcal{Y}(B)$

② $\mathbb{A}(B)$: a non-comm. alg gen. by Y_i

$\hat{\mathbb{A}}(B)$: a completion of $\mathbb{A}(B)$ by ideals gen. by Y_i

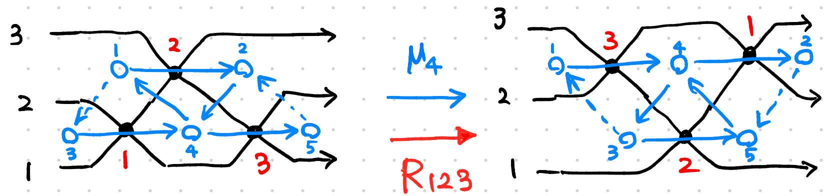
③ the $\alpha_k \varepsilon_k$ in (2) are positive $(\alpha_1\varepsilon_1 = e_{i_1})$

$\mathbf{Y}(k)_i$: the gen. of $\mathcal{T}(k)$

Remark

$\Phi_q^{\pm 1}(\mathbf{Y}^\beta) \in \hat{\mathbb{A}}(B)$ if β is positive.

§ Tetrahedron eq. for $\boxed{FG}$



$i = 1, 2, 3$

(p_i, u_i) : canonical pairs

$$\begin{cases} [p_i, u_j] = \hbar \delta_{ij} \\ [p_i, p_j] = [u_i, u_j] = 0 \end{cases}$$

$\lambda_i \in \mathbb{C}$: spectral parameters

<u>Def.</u>

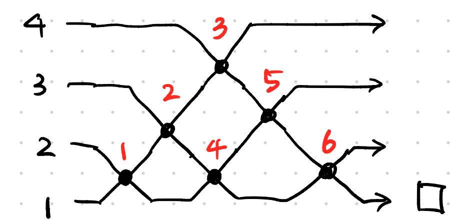
$W(A_2) := \langle e^{\pm p_i}, e^{\pm u_i} ; i = 1, 2, 3 \rangle$: q-Weyl alg.

$$e^{p_i} e^{u_i} = q\, e^{u_i} e^{p_i} \quad (q := e^{\hbar})$$

$$\mathcal{R}_{123} := \Phi_q\left(e^{p_1 + u_1 + p_3 - u_3 - p_2 + \lambda_1 - \lambda_3}\right) P_{123}$$

$$P_{123} := \rho_{23}\, e^{\frac{1}{\hbar} p_1 (u_3 - u_2)}\, e^{\frac{\lambda_2 - \lambda_3}{\hbar}(u_3 - u_1)}$$

$\rightsquigarrow$ $\mathcal{R}_{ijk}$, P_{ijk} for



$\square$

<u>Thm</u> [IKT 2023a]

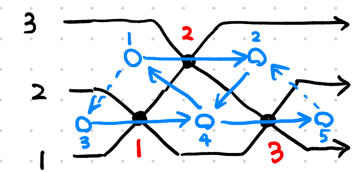
① P_{ijk} satisfy the tetrahedron eq.,

$$P_{456} P_{236} P_{135} P_{124} = P_{124} P_{135} P_{236} P_{456}.$$

② $\mathcal{R}_{ijk}$ also satisfy the tetrahedron eq.

$$\mathcal{R}_{456} \mathcal{R}_{236} \mathcal{R}_{135} \mathcal{R}_{124} = \mathcal{R}_{124} \mathcal{R}_{135} \mathcal{R}_{236} \mathcal{R}_{456}. \quad \square$$

How to get $\mathcal{R}$



(a) Define morphisms of skewfields:

$$\phi : \mathcal{Y}(J_{121}) \hookrightarrow \operatorname{Frac} W(A_2)$$
$$\phi' : \mathcal{Y}(J_{212}) \nearrow$$

and automorphism: $\pi_{123} \curvearrowright W(A_2)$

s.t.

$$\begin{tikzcd}
\mathcal{Y}(J_{212}) \arrow[r, "\phi'"] \arrow[d, "T_{4,+}"'] \arrow[dr, phantom, "\circlearrowright"] & \operatorname{Frac} W(A_2) \arrow[d, "\pi_{123}"] \\
\mathcal{Y}(J_{121}) \arrow[r, "\phi"'] & \operatorname{Frac} W(A_2).
\end{tikzcd}$$

ϕ : $Y_1 \mapsto e^{p_2-u_2-p_1-\lambda_2}$

$Y_2 \mapsto e^{p_2+u_2-p_3+\lambda_2}$

$Y_3 \mapsto e^{p_1-u_1-\lambda_1}$

$Y_4 \mapsto e^{p_1+u_1+p_3-u_3-p_2+\lambda_1-\lambda_3}$

$Y_5 \mapsto e^{p_3+u_3+\lambda_3}$

π_{123} : $p_1 \mapsto p_1+\lambda_2-\lambda_3$

$p_2 \mapsto p_1+p_3$

$p_3 \mapsto p_2-p_1-\lambda_2+\lambda_3$

$u_1 \mapsto u_1+u_2-u_3$

$u_2 \mapsto u_3$

$u_3 \mapsto u_2$

(b) Construct $\mathbb{P}_{123}$ s.t. $\pi_{123} = \operatorname{Ad} \mathbb{P}_{123}$.

(c) The trop. signs of mutation seq.,

$$\sigma_{4,8}\mu_7\mu_8\mu_4\mu_7(J_{123121}, y) = \sigma_{4,7}\mu_8\mu_7\mu_4\mu_8(J_{123121}, y)$$

are all +

$$\Longrightarrow \quad T_{7+}T_{4+}T_{8+}T_{7+}\sigma_{4,8} = T_{8+}T_{4+}T_{7+}T_{8+}\sigma_{4,7}$$

$$\Phi_q(\mathbf{Y}^{e_7})\,\Phi(\mathbf{Y}^{e_4+e_7})\,\Phi_q(\mathbf{Y}^{e_8})\,\Phi_q(\mathbf{Y}^{e_4}) = \Phi_q(\mathbf{Y}^{e_8})\,\Phi_q(\mathbf{Y}^{e_4})\,\Phi_q(\mathbf{Y}^{e_7+e_8})\,\Phi_q(\mathbf{Y}^{e_7})$$

(d) Proof of Thm :

① ($P_{\bar{i}\bar{j}k}$) direct computation using Baker-Campell-Hausdorff formula.

② ($\mathcal{R}_{\bar{i}\bar{j}k}$) due to ① and (c) with the map ϕ.

$$\left(\Phi\Phi\Phi\Phi PPPP = \Phi P \Phi P \Phi P \Phi P = \mathcal{R}\mathcal{R}\mathcal{R}\mathcal{R} \right). \square$$

Remark

① ϕ brings spectral parameters !

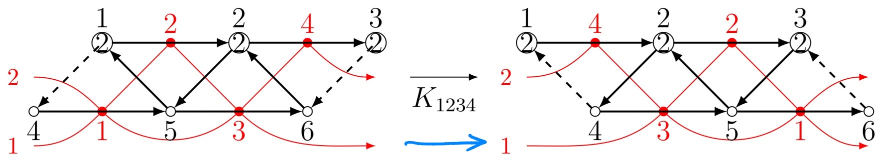
$$\mathcal{R}_{123} := \Phi_q\left(e^{p_1+u_1+p_3-u_3-p_2+\lambda_1-\lambda_3}\right) P_{123}$$

$$P_{123} := \rho_{23}\, e^{\frac{1}{\hbar}p_1(u_3-u_2)}\, e^{\frac{\lambda_2-\lambda_3}{\hbar}(u_3-u_1)}$$

w.o. spectral parameter

② μ_4^* and π_{123} appear in [Bytsko-Volkov 2015].

§ 3D Reflection eq. for $\boxed{FG}$



$\mu_2\mu_5\mu_2$
$-\ +\ +$

$i=1,2,3,4$

(p_i, u_i) : $\begin{cases} [p_i, u_j] = \begin{cases} \hbar\delta_{ij} & (i=1,3) \\ 2\hbar\delta_{ij} & (i=2,4) \end{cases} \\ [p_i, p_j] = [u_i, u_j] = 0 \end{cases}$

$\lambda_i \in \mathbb{C}$: spectral parameters

Def.

$$W(C_2) := \langle e^{\pm p_i}, e^{\pm u_i} ; i=1,2,3,4 \rangle$$

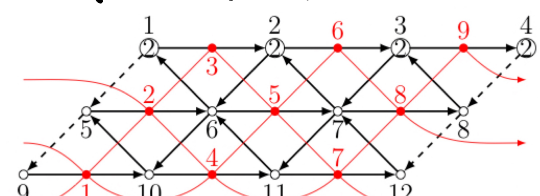
$$e^{p_i} e^{u_i} = \begin{cases} q\, e^{u_i} e^{p_i} & (i=1,3) \\ q^2 e^{u_i} e^{p_i} & (i=2,4) \end{cases}$$

corresponding to $T_{2,+}\, T_{5,+}\, T_{2,-}$

$$K_{1234} := \Phi_{q^2}\left(e^{p_2+u_2+p_4-u_4-2p_3+\lambda_2-\lambda_4}\right) \Phi_q\left(e^{p_1+u_1+p_3-u_3-p_2+\lambda_1-\lambda_3}\right) \Phi_{q^2}\left(\text{—}\right)^{-1} P^K_{1234}$$

$$P^K_{1234} := \rho_{24}\, e^{\frac{1}{\hbar} p_i (u_4 - u_2)}\, e^{\frac{\lambda_2-\lambda_4}{2\hbar}(2u_3 - 2u_1 + u_4 - u_2)}$$

$\rightsquigarrow$ K_{ijkl}, R_{ijk} for



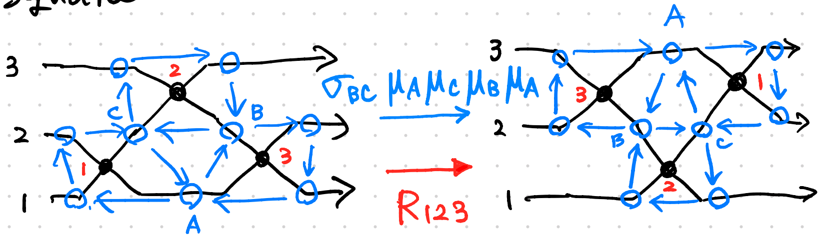
Thm [IKT 2023a]

① P_{ijk} and P^K_{ijkl} satisfy the 3D reflection eq.

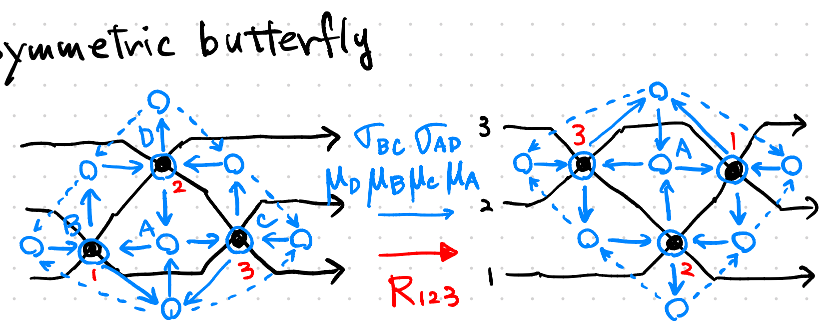
② So do R_{ijk} and K_{ijkl}. □

§ About $\boxed{Sq}$ and $\boxed{SB}$

- square



- symmetric butterfly



$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$$

tropical signs are not uniform

(Ex) $\boxed{Sq}$

LHS : $++--$ $++++$ $++++$ $++++$

RHS : $++-+$ $-+++$ $++++$ $++++$

Want a uniform solution $\mathcal{R}_{ijk}$

$$\begin{cases} \Pi_{ijk} \text{ satisfy tetrahedron eq.} \\ \exists\, P_{ijk} \text{ s.t. } \Pi_{ijk} = \mathrm{Ad}\, P_{ijk} \end{cases}$$

$\rightsquigarrow$

[Sergeev 1999] $\to$ $\mathcal{R}_{--++}$, $\mathcal{R}_{+-+-}$

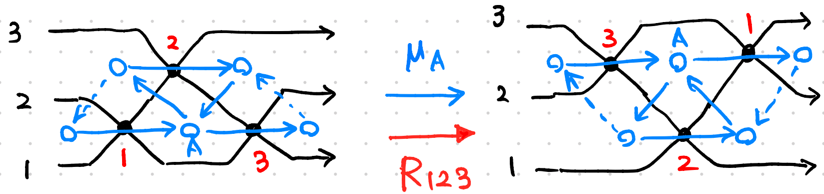
(Need additional thought to justify $\mathcal{R}_{ijk}$)

Remark

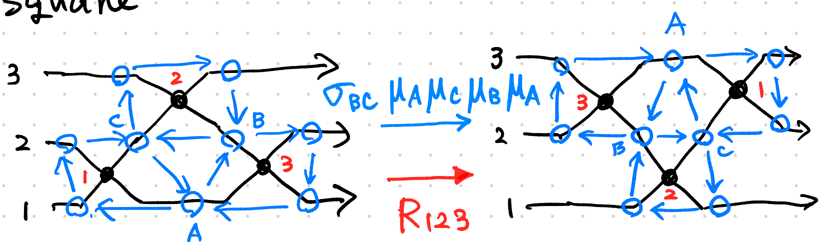
We have "good choices of signs", $I_{k,+}$ and $I_{k,-}$, besides the tropical signs.

§ Summary

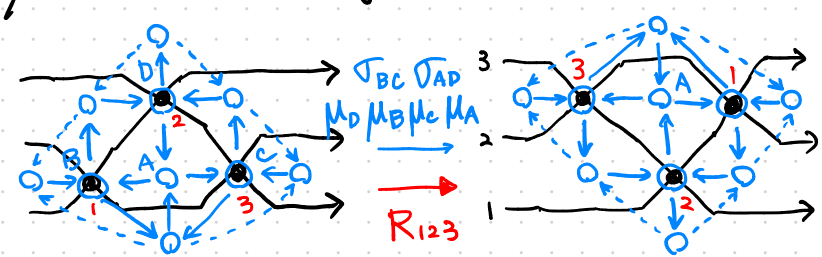
- Fock–Goncharov quiver



- square



- symmetric butterfly



① R in terms of q-Weyl alg.

$\boxed{\text{FG}}$ ••• [Maillard–Sergeev 1997]
[Bytsko–Volkov 2015]

$\boxed{\text{Sq}}$ ••• [Sergeev 1999]

$\boxed{\text{SB}}$ $\xrightarrow{\text{rep.}}$ [Kapranov–Voevodsky 1994]
[Bazhanov–Sergeev 2006]
[Bazhanov–Mangazeev–Sergeev 2010]
[Kuniba–Matsuike–Yoneyama 2023]

② K in terms of q-Weyl alg.

$\boxed{\text{FG}}$ ($\boxed{\text{SB}}$ in progress)

Thank you!