Generalized Markov number and generalized cluster algebra

Yasuaki Gyoda (U. Tokyo)

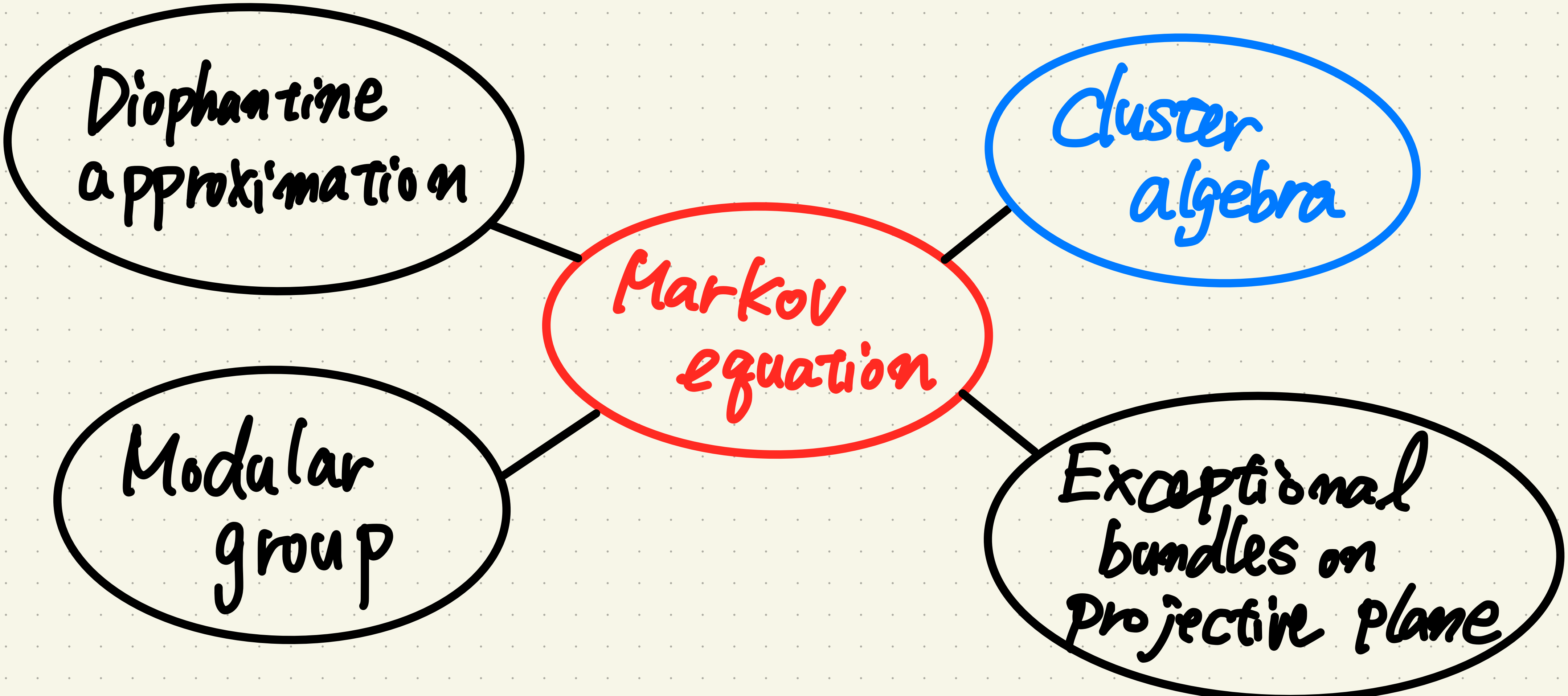
arXiv: 2109.09639
2201.10919 (j.w. with K. Matsushita)
2312.07329 (j.w. with S. Maruyama)

Contents

1. Introduction

Markov Equation (ME)

$$x^2 + y^2 + z^2 = 3xyz$$



$$\begin{array}{ccc} \boxed{\text{Cluster algebra}} & \longrightarrow & \boxed{\text{Markov equation}} \\ \cap & & \cap \\ \boxed{\text{generalized Cluster algebra}} & \longrightarrow & \boxed{\text{generalized Markov equation}} \end{array}$$

Generalized Markov Equation (GME)

$$x^2 + y^2 + z^2 + k_1 yz + k_2 zx + k_3 xy = (3 + k_1 + k_2 + k_3)\,xyz$$

2. Markov equation and its generalization

Markov Equation (ME)

$$x^2+y^2+z^2=3xyz$$

positive integer solutions to ME (Markov triple)

$(x,y,z)=(1,1,1),\ (1,1,2),\ (1,2,5),\ (1,5,13),\ \ldots$

The integers appearing in positive integer solutions to ME are called the Markov numbers.

Thm (Markov).

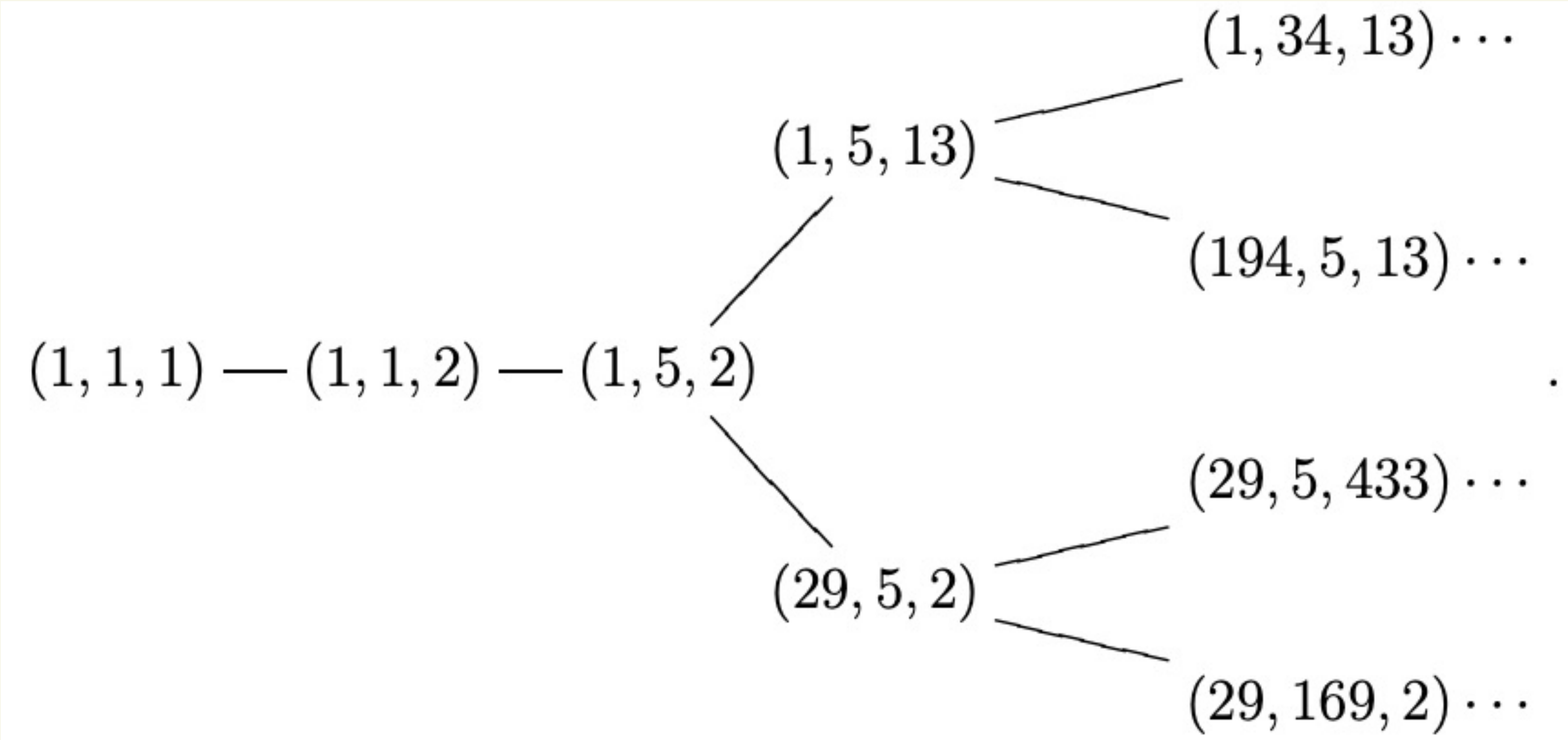
We consider

$$(a,b,c) \begin{cases} \left(\dfrac{b^2+c^2}{a},\ b,\ c\right) \\[2ex] \left(a,\ \dfrac{a^2+c^2}{b},\ c\right) \\[2ex] \left(a,\ b,\ \dfrac{a^2+b^2}{c}\right). \end{cases}$$

(1) If (a,b,c) is a positive integer solution to ME, then so are the right 3 triplets.

(2) All positive integer solutions to ME are obtained by applying this operation to $(1,1,1)$ repeatedly.

Solution tree of $x^2+y^2+z^2=3xyz$



Generalized Markov Equation (GME(k_1, k_2, k_3))

$$x^2+y^2+z^2+k_1yz+k_2zx+k_3xy=(3+k_1+k_2+k_3)xyz$$

for $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 0}$

$$k_1=k_2=k_3=0 \Rightarrow x^2+y^2+z^2=3xyz$$

$$k_1=k_2=k_3=1 \Rightarrow (x+y)^2+(y+z)^2+(z+x)^2=12xyz$$

Thm (G.-Matsushita, 2023)

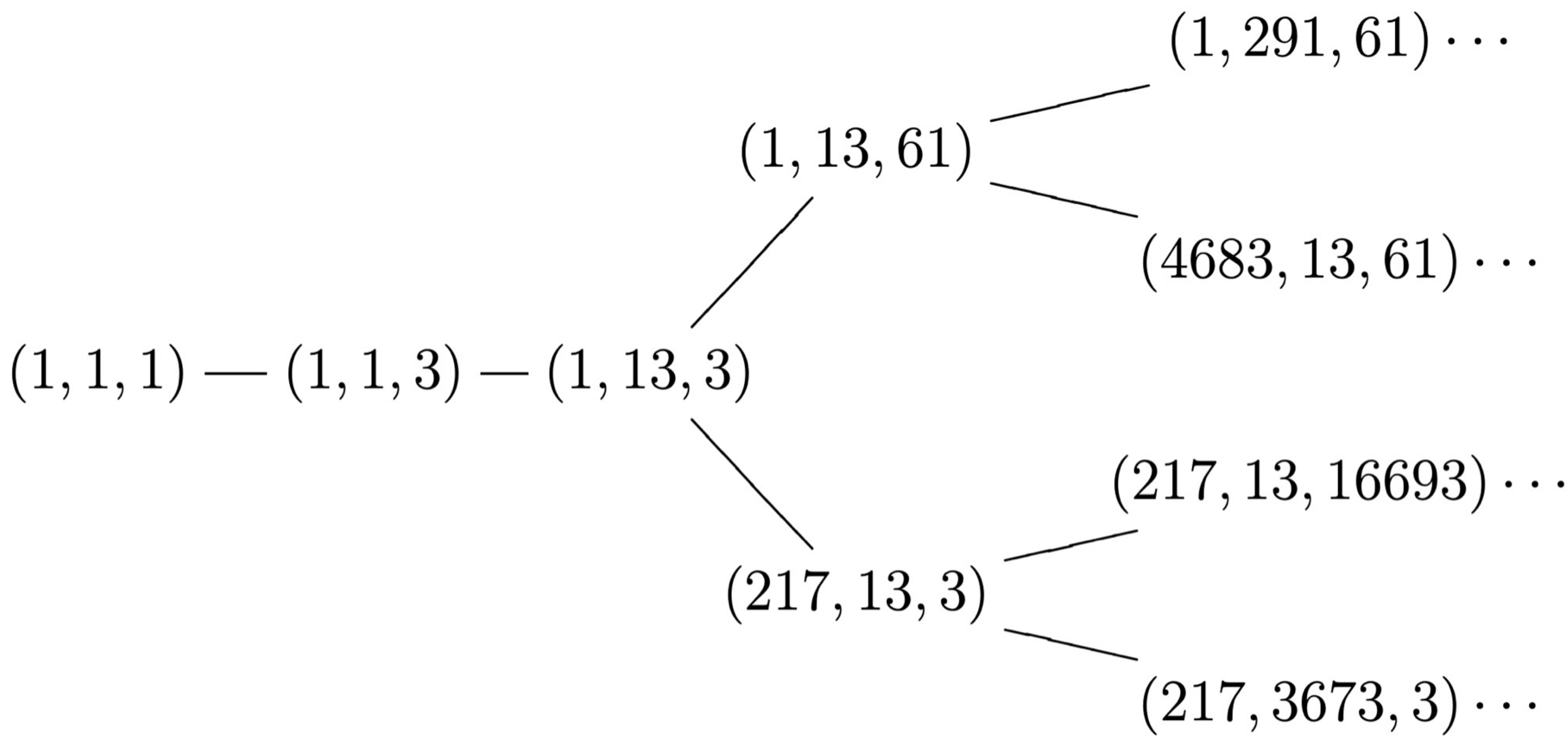
We consider

$$(a,b,c) \;\begin{array}{l} \nearrow \left(\dfrac{b^2+k_1bc+c^2}{a},\, b,\, c\right) \\ \to \left(a,\, \dfrac{a^2+k_2ac+c^2}{b},\, c\right) \\ \searrow \left(a,\, b,\, \dfrac{a^2+k_3ab+b^2}{c}\right) \end{array}$$

(1) If (a,b,c) is a positive integer solution to GME(k_1,k_2,k_3) then so are the right 3 triplets.

(2) All positive integer solutions to GME(k_1,k_2,k_3) are obtained by applying this operation to $(1,1,1)$ repeatedly.

Solution tree of $x^2+y^2+z^2+yz+zx+xy=6xyz$



Solution tree of $x^2+y^2+z^2+1yz+2zx+0xy=6xyz$

- $(1,1,1)$
  - $(3,1,1)$
    - $(3,16,1)$
      - $(91,16,1)\cdots$
      - $(3,16,265)\cdots$
    - $(3,1,10)$
      - $(37,1,10)\cdots$
      - $(3,169,10)\cdots$
  - $(1,4,1)$
    - $(21,4,1)$
      - $(21,121,1)\cdots$
      - $(21,4,457)\cdots$
    - $(1,4,17)$
      - $(373,4,17)\cdots$
      - $(1,81,17)\cdots$
  - $(1,1,2)$
    - $(7,1,2)$
      - $(7,81,2)\cdots$
      - $(7,1,25)\cdots$
    - $(1,9,2)$
      - $(103,9,2)\cdots$
      - $(1,9,41)\cdots$

3. Generalized seed mutation

$\mathcal{F} := \mathbb{Q}(t_1, \dots, t_n)$

- $(\mathbf{x}, \mathbf{z}, B)$: seed $\mathbf{x}$: cluster, B: exchange matrix.

$:\Leftrightarrow$
- $\mathbf{x} = (x_1, \dots, x_n)$: free generating set of $\mathcal{F}$,
- $\mathbf{z} := (z_1, \dots, z_n)$:
  n-tuple of monic polynomials of one variables with strictly positive integer coefficients and constants 1.
- $B = (b_{ij})$: $n \times n$ skew-symmetrizable matrix
  i.e. $^{\exists} S = \mathrm{diag}(s_1, \dots, s_n), s_i > 0$, SB : skew-symmetric

Generalized seed mutation:

For $k \in [1, n]$ and $(\mathbf{x}, \mathbf{z}, B)$, we obtain a mutated seed $(\mathbf{x}', \mathbf{z}', B') := \mu_k(\mathbf{x}, \mathbf{z}, B)$ at k as follows:

$$
\begin{cases}
\circ\ z_i' = \begin{cases} u^{\deg z_i} \cdot z_i(u^{-1}) & (i = k) \\ z_i & (i \neq k) \end{cases} \\[2ex]
\circ\ b_{ij}' = \begin{cases} -b_{ij} & (i \text{ or } j = k) \\ b_{ij} + \deg z_k \cdot \left([b_{ik}]_+ b_{kj} + b_{ik} \underbrace{[-b_{kj}]_+}_{\max(-b_{kj}, 0)}\right) & (\text{otherwise}) \end{cases} \\[2ex]
\circ\ x_i' = \begin{cases} \dfrac{\left(\prod_j x_j^{[b_{jk}]_+}\right)^{\deg z_k} \cdot z_k\left(\prod_j x_j^{[-b_{jk}]_+}\right)}{x_k} & (i = k) \\ x_i & (i \neq k) \end{cases}
\end{cases}
$$

Ex.

$$\mathbf{x} = (x_1, x_2, x_3), \quad \mathbb{Z} = \begin{cases} Z_1 = 1 + u + u^2 \\ Z_2 = 1 + u \\ Z_3 = 1 + 2u + 1u^2, \end{cases} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$\Big\downarrow \mu_3$$

$$\mathbf{x}' = \left(x_1, x_2, \; x_3' = \frac{1(x_1')^2(x_2')^0 + 2(x_1')^1(x_2')^1 + 1(x_1')^0(x_2')^2}{x_3} \right) \quad \mathbb{Z}' = \begin{cases} Z_1' = 1 + u + u^2 \\ Z_2' = 1 + u \\ Z_3' = 1 + 2u + 1u^2 \end{cases} \quad B' = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\overset{u}{-B}$$

Rmk : When $\mathbb{Z} = (1+u, 1+u, \ldots, 1+u)$,

this operation coincides with the ordinary mutation.

Thm (G.-Matsushita, 2023)

For $(B, \mathbb{Z})$ in the below table, we set $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}$ as all clusters obtained from applying mutations to $(\mathbf{x}, \mathbb{Z}, B)$ repeatedly. Then $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n} \big|_{x_1=x_2=x_3=1}$ gives the set of positive integer solution to the corresponding equation.

	Equation	B	$\mathbf{Z}$
ME	$x^2+y^2+z^2=3xyz$	$\begin{bmatrix}0&2&-2\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$
GME$(0,0,k_3)$	$x^2+y^2+z^2+k_3xy=(3+k_3)xyz$	$\begin{bmatrix}0&2&-1\\-2&0&1\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+k_3u+u^2\end{cases}$
GME$(k_1,0,k_3)$	$x^2+y^2+z^2+k_3xy+k_1yz$ $=(3+k_3+k_1)xyz$	$\begin{bmatrix}0&2&-1\\-1&0&1\\1&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_1u+u^2\\Z_2(u)=1+u\\Z_3(u)=1+k_3u+u^2\end{cases}$
GME(k_1,k_2,k_3)	$x^2+y^2+z^2+k_3xy+k_1yz+k_2zx$ $=(3+k_1+k_2+k_3)xyz$	$\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+k_1u+u^2\\Z_2(u)=1+k_2u+u^2\\Z_3(u)=1+k_3u+u^2\end{cases}$

4. Generalized uniqueness conjecture

Markov uniqueness conjecture.

For a Markov number c, there is a unique positive integer solution $\{x, y, z\} = \{a, b, c\}$ to ME up to order such that $a, b \leq c$.

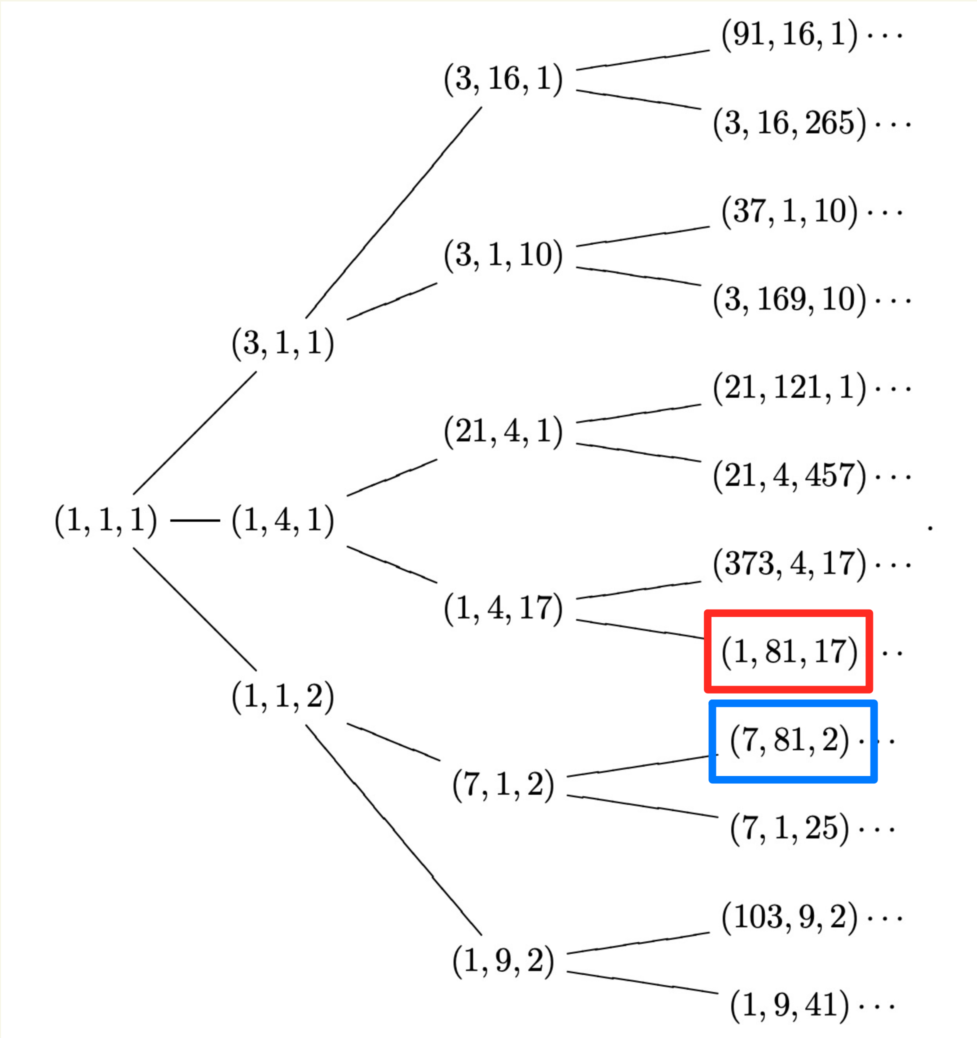
For example, for $c = 5$, positive integer solution $\{a, b, 5\}$ to ME which satisfies $a, b \leq 5$ is only $\{1, 2, 5\}$.

How about GME version?

Thm (G.-Matsushita, 2023)

There is a counterexample of GME version of the conjecture if k_1, k_2, k_3 are not the same number.

When $k_1=1, k_2=2, k_3=0$, $(1,81,17)$ and $(7,81,2)$ are both solutions to GME which have 81 as maximal.



Question (open problem!)

How about the case $k_1 = k_2 = k_3$?

From now on, $k := k_1 = k_2 = k_3$

(a, b, c) : k-gen. Markov triple

:$\Leftrightarrow$ (a, b, c) is positive integer solution to GME(k, k, k).

Thm (G.-Maruyama, 2023)

If c : k-gen. Markov number with $c = p$ or $2p$ (p: prime),

then $\exists ! (a, b, c)$: k-gen. Markov triple with $a \leq b \leq c$.

Rmk. $k = 0$ case:

Baragar (1996), Schmitz (1996), Button (1998), Zhang (2006),

Lang-Tang (2007)

5. k-generalized Cohn matrix

To avoid duplications of solutions, we redefine the solution tree:

k-Generalized Markov tree $M\Pi(k)$

- Root $(1, 1, 1)$
- Generation rule: (a, b, c)

$$
\begin{tikzcd}
 & (a, b, c) \arrow[dl, dash] \arrow[dr, dash] & \\
\left(a, \dfrac{a^2+kab+b^2}{c}, b\right) & & \left(b, \dfrac{b^2+kbc+c^2}{a}, c\right)
\end{tikzcd}
$$

(k=1)

(omit) (omit)

$(1,1,1) - (1,\boxed{3},1) - (1,\boxed{13},3)$

$(1,\boxed{13},3) - (13,\boxed{217},3)$, $(13,\boxed{217},3) - (217,\boxed{3673},3)$, $(13,\boxed{217},3) - (13,\boxed{16693},217)$

$(1,\boxed{13},3) - (1,\boxed{61},13)$, $(1,\boxed{61},13) - (61,\boxed{4683},13)$, $(1,\boxed{61},13) - (1,\boxed{291},61)$

$\boxed{}$: new number.

Prop.

Let $k \in \mathbb{Z}_{\geq 0}$.

1. Each vertex in the k-gen. Markov tree is k-gen. Markov triple,
2. For a k-gen. Markov triple (a, b, c) with $b \geq a, c$, $\exists!\, v$: vertex in the k-gen. Markov tree s.t. $v = (a, b, c)$

Prop.

k-gen. Markov Conjecture is true

$\Leftrightarrow$ the 2nd elements of vertices in $\boxed{}$ are distinct.

"Matrixize" k-generalized Markov numbers, triples, tree:

Def (k-gen. Cohn matrix).

$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in M(2, \mathbb{Z})$: k-gen. Cohn matrix

$:\Leftrightarrow \begin{cases} \circ\ P \in SL(2, \mathbb{Z}) \\ \circ\ p_{12} : k\text{-gen. Markov number} \\ \circ\ \operatorname{tr} P = (3+3k)p_{12} - k \end{cases}$

Ex.

13: 1-generalized Markov number $\to \begin{pmatrix} 3 & 13 \\ 17 & 74 \end{pmatrix}$: (one of) corresponding matrix

Def (k-gen. Cohn triple)

(P, Q, R) : k-gen. Cohn triple

$$:\Leftrightarrow \begin{cases} P, Q, R : k\text{-gen. Cohn matrices} \\ (p_{12}, q_{12}, r_{12}) : k\text{-gen. Markov triple} \\ Q = PR - \begin{bmatrix} k & 0 \\ 3k^2+3k & k \end{bmatrix} =: S_k \end{cases}$$

Ex. 1-gen. cohn triple with $(p_{12}, q_{12}, r_{12}) = (1, 61, 13)$

$$\begin{pmatrix} -1 & 1 \\ -7 & 6 \end{pmatrix}, \begin{pmatrix} 13 & 61 \\ 75 & 352 \end{pmatrix}, \begin{pmatrix} 3 & 13 \\ 17 & 74 \end{pmatrix}$$

Def (k-gen. Cohn tree)

- Root: k-gen. Cohn triple with $(p_{12}, q_{12}, r_{12}) = (1, 1, 1)$
- Generation rule: (P, Q, R)

$$(P, PQ-S, Q) \qquad\qquad (Q, QR-S, R)$$

Thm (G.-Maruyama, 2023)

(1) Each vertex in k-gen. Cohn tree is a k-gen. Cohn triple.

(2) $(P, Q, R) \mapsto (p_{12}, q_{12}, r_{12})$ induces the graph iso. from k-gen. Cohn tree to k-gen. Markov tree.

(3) 2nd component of all vertices in k-gen. Cohn tree are distinct.

$CT(k,-k)$

$(k=1)$

$LCT(k,-k)$

$$\begin{pmatrix}-1 & 1\\ 7 & 6\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}\cdots$$

$$\begin{pmatrix}-1 & 1\\ 7 & 6\end{pmatrix}\boxed{\begin{pmatrix}1 & 3\\ 5 & 16\end{pmatrix}}\begin{pmatrix}1 & 1\\ 3 & 4\end{pmatrix}\cdots$$

$$\begin{pmatrix}-1 & 1\\ 7 & 6\end{pmatrix}\boxed{\begin{pmatrix}3 & 13\\ 17 & 74\end{pmatrix}}\begin{pmatrix}1 & 3\\ 5 & 16\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\ 7 & 6\end{pmatrix}\boxed{\begin{pmatrix}13 & 61\\ 75 & 352\end{pmatrix}}\begin{pmatrix}3 & 13\\ 17 & 74\end{pmatrix} \qquad \begin{pmatrix}3 & 13\\ 17 & 74\end{pmatrix}\boxed{\begin{pmatrix}67 & 217\\ 381 & 1234\end{pmatrix}}\begin{pmatrix}1 & 3\\ 5 & 16\end{pmatrix}\cdots$$

$$\begin{pmatrix}-1 & 1\\ 7 & 6\end{pmatrix}\boxed{\begin{pmatrix}61 & 291\\ 353 & 1684\end{pmatrix}}\begin{pmatrix}13 & 61\\ 75 & 352\end{pmatrix} \qquad \begin{pmatrix}13 & 61\\ 75 & 352\end{pmatrix}\boxed{\begin{pmatrix}1075 & 4683\\ 6203 & 29022\end{pmatrix}}\begin{pmatrix}3 & 13\\ 17 & 74\end{pmatrix}$$

$\boxed{}$: New matrix

Sufficient condition of Conjecture

For a k-gen. Markov number c, if
$\exists! M = \begin{pmatrix} * & c \\ * & * \end{pmatrix}$: k-gen. Cohn matrix in $LC\Pi(k,-k)$,
then $\exists! (a, b, c)$: k-gen. Markov triple with
$a \leq b \leq c$.

To determine M, it suffices to determine $(1,1)$-entry because M is bounded by tr. and det.

Thm (G.-Maruyama , 2023)

For a k-gen. Cohn matrix $\begin{pmatrix} u & c \\ * & * \end{pmatrix}$ in $LC\Pi(k,-k)$,
u is a solution to

$$\begin{cases} x^2+kx+1 \equiv 0 \mod c \\ 0 < x < \frac{c}{2} \end{cases}$$

Ex. $c=3 \Rightarrow u=1$, $1^2+1\cdot1+1 \equiv 0 \mod 3$, $1<\frac{3}{2}$

$c=13 \Rightarrow u=3$, $3^2+1\cdot3+1=13 \equiv 0 \mod 13$, $3<\frac{13}{2}$

$c=217 \Rightarrow u=67$, $67^2+1\cdot67+1=4557 \equiv 0 \mod 217$, $67<\frac{217}{2}$

Thm (G.-Maruyama, 2023)

If $c = p$ or $2p$ (p: prime), the solution to

$$\begin{cases} x^2 + kx + 1 \equiv 0 \mod c \\ 0 < x < \frac{c}{2} \end{cases}$$

is unique.

Thm (G.-Maruyama, 2023).

If c : k-gen. Markov number with $c = p$ or $2p$ (p: prime),
then $\exists! (a, b, c)$: k-gen. Markov triple with $a \leq b \leq c$.

How many prime k-gen. Markov numbers are there?

infinitely many? $\longrightarrow$ open problem even if $k=0$!

The list of the numbers of prime k-gen. Markov numbers appearing up to depth 12 in the k-gen. Markov tree: (1024 k-gen. Markov numbers)

$k=0$: 93 $\quad$ $k=1$: 61 $\quad$ $k=2$: 0 $\quad$ $k=3$: 55

$k=4$: 49 $\quad$ $k=5$: 38 $\quad$ $k=6$: 37 $\quad$ $k=7$: 34

$k=8$: 31 $\quad$ $k=9$: 28 $\quad$ $k=10$: 34

6. Questions

- Can k-gen. Cohn matrices be constructed from the information of the generalized cluster algebras?
  - In particular, u in $\begin{pmatrix} u & m \\ * & * \end{pmatrix} \in LM\Pi(k,-k)$ can be expressed in term of generalized cluster algebras?
  - Can "the mutation of k-gen. Cohn triple" be defined?

$$(P,Q,R) \begin{cases} (P',Q,R) \\ (P,Q',R) \\ (P,Q,R') \end{cases}$$

$Q' = PR - S$?
$\quad\ \ RP - S$?

Additional topics (1)

There are other equations whose positive integer solutions have a cluster structure.

Thm (G.-Matsushita, 2023)

For $(B, \mathbf{Z})$ in the below table, we set $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}$ all clusters obtained from applying mutations to $(\mathbf{x}, \mathbf{Z}, B)$ repeatedly. Then $\{\mathbf{x}_t\}_{t\in\mathbb{T}_n}\big|_{x_1=x_2=x_3=1}$ gives the set of positive integer solution to the corresponding equation.

Lampe's equation →

Equation	B	$\mathbf{Z}$	D
$x^2+y^4+z^4+2xy^2+2z^2x=7xy^2z^2$	$\begin{bmatrix}0&1&-1\\-4&0&2\\4&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+u\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$
$x^2+y^4+z^4+2xy^2+ky^2z^2+2z^2x$ $=(7+k)xy^2z^2$	$\begin{bmatrix}0&1&-1\\-2&0&2\\2&-2&0\end{bmatrix}$	$\begin{cases}Z_1(u)=1+ku+u^2\\Z_2(u)=1+u\\Z_3(u)=1+u\end{cases}$	$\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Rank 2 Case [G.-Matsushita, 2023, Chen-Li, 2024]

$x^2y + xy^2 + x^2 + y^2 + 2x + 2y + 1 = 9xy \longleftrightarrow$ A_2 type

$y^4 + x^2y^2 + 2y^2 + x^2 + 2x + 1 = 8xy^2 \longleftrightarrow$ B_2 type

$y^4 + xy^3 + y^3 + x^2y + 2xy + x^2 + y + 2x + 1 = 11xy^2 \longleftrightarrow$ G_2 type

$x^2 + y^2 + 1 = 3xy \longleftrightarrow$ $A_2^{(1)}$ type

$y^4 + x^2 + 2x + 1 = 5xy^2 \longleftrightarrow$ $A_2^{(2)}$ type

$x^2 + y^2 + kx + 1 = (3+k)xy \longleftrightarrow$

$x^2 + y^2 + k_1 y + k_2 y + 1 = (3+k_1+k_2)xy \longleftrightarrow$

sp. of gen. Markov case

$y^4 + x^2 + ky^2 + 2x + 1 = (5+k)xy \longleftrightarrow$ sp. of gen. Lampe case

Thm (Chen-Li, 2024).

> There is no equation corresponding to non-finite and non-affine (ordinary) cluster algebra of rank 2.

Question

- Can the above theorem be generalized to generalized cluster alg.?
- Is there an equation corresponding to a (generalized) cluster alg. of rank ≥ 4?

Additional topic (2)

Thm (G.-Maruyama, 2023)

$\boxed{\text{If } c : k\text{-gen. Markov number with } c = p \text{ or } 2p \ (p:\text{prime}), \text{ then } \exists! (a, b, c) : k\text{-gen. Markov triple with } a \le b \le c.}$

How about other k-gen. Markov numbers?

If $k = 0$, then the above theorem also holds when $c = p^m$ or $2p^m$, where $m \ge 2$ (proved by Aigner etc).

Thm (G. - Maruyama, 2023)

We assume the pair (k, p) satisfies one of the following conditions:

(1) $k = 2$,

(2) $k \geq 4$ is even, and both $\frac{k}{2}+1$ and $\frac{k}{2}-1$ are not divided by p^2,

(3) k is odd, and both $k+2$ and $k-2$ are not divided by p^2.

If c is a k-gen. Markov number with $c = p^m$ or $2p^m$, then $\exists! (a, b, c)$: k-gen. Markov triple with $a \leq b \leq c$.

This condition is derived from Hensel's lemma:

Hensel's lemma

Let p be a prime. For a polynomial $f(x)$ with integer coefficient, if $\exists x_k \in \mathbb{Z}$ s.t.

$$f(x_k) \equiv 0 \bmod p^k \text{ and } f'(x_k) \not\equiv 0 \bmod p,$$

then $\exists x_{k+1} \in \mathbb{Z}$ s.t.

$$x_{k+1} \equiv x_k \bmod p^k \text{ and } f(x_{k+1}) \equiv 0 \bmod p^{k+1}$$

When we lift the solution $\underbrace{x^2+kx+1}_{f(x)} \equiv 0 \bmod p$ to $\bmod p^k$, we use this lemma.

Question

Can the condition of the previous theorem be removed?