Skein and Cluster algebras of unpunctured surfaces for $\mathfrak{sp}_4$

joint work with Tsukasa Ishibashi

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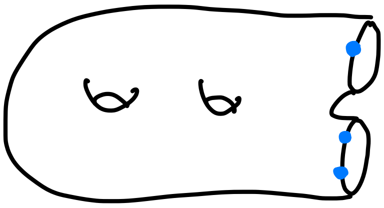
§ Introduction

unpunctured surface $\Sigma =$



$+ \; G = \mathrm{Lie}\, \mathfrak{g}$

$\mathcal{A}_{G,\Sigma}$ "the moduli space of decorated twisted local G-systems on Σ"

$\mathfrak{g}$-webs on Σ ↙ ↘ seeds S_Δ

$$\mathcal{S}^{q}_{\mathfrak{g},\Sigma} \overset{?}{\longleftrightarrow} \mathcal{A}^{q}_{\mathfrak{g},\Sigma}$$

skein algebra — quantum cluster algebra

$\# \mathbb{M} \geq 2$

<u>Theorem</u> (Muller '16) $\mathfrak{g} = \mathfrak{sl}_2$

$$\mathcal{S}^{q}_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathcal{A}^{q}_{\mathfrak{sl}_2,\Sigma} = \mathcal{U}^{q}_{\mathfrak{sl}_2,\Sigma}$$

$$\mathcal{A} \subset \mathcal{S} \subset \mathcal{U}$$

<u>Theorem</u> (Ishibashi-Y. '23) $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$

$$\mathcal{S}^{q}_{\mathfrak{g},\Sigma}[\partial^{-1}] \subset \mathcal{A}^{q}_{\mathfrak{g},\Sigma}$$

<u>Theorem</u> (Ishibashi-Oya-Shen) + Ishibashi-Y.

$$\mathcal{S}^{1}_{\mathfrak{g},\Sigma}[\partial^{-1}] = \mathcal{A}^{1}_{\mathfrak{g},\Sigma} = \mathcal{U}^{1}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$$

Conjecture $\mathcal{S}^{q}_{\mathfrak{g},\Sigma}[\partial^{-1}] = \mathcal{A}^{q}_{\mathfrak{g},\Sigma} = \mathcal{U}^{q}_{\mathfrak{g},\Sigma} \subset \mathrm{Frac}\,\mathcal{S}^{q}_{\mathfrak{g},\Sigma}$

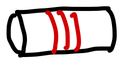
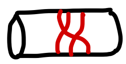
Conjecture $\{$ tree-type webs $\}$ = $\{$ cluster variables $\}$

- indecomposable
- immersion of a tree

(• not invariant under DT transformations)

Question Construct positive bases by using $\mathfrak{g}$-web.

e.g. $\mathfrak{g} = \mathfrak{sl}_2$:

- bangle basis 
- bracelet basis  1st $\leftrightarrow$ theta basis
- band basis  2nd

general $\mathfrak{g}$: We conjecture that these bases are constructed by substituting "Orbit functions"

Question Action of the mapping class groups

Generalize $\mathcal{S} = \mathcal{A}$ to 3-mfds. $\rightsquigarrow$ invariants of 3-mfds...?

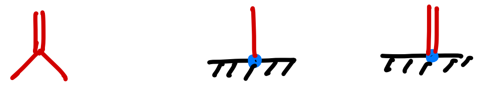
§ skein algebra for sp_4

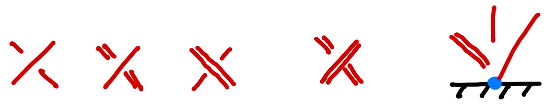
$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\mathcal{S}^{q}_{sp_4,\Sigma} = \mathbb{Z}_q[1/[2]]\,\{\text{tangled } sp_4\text{-graphs on } \Sigma\}\,/\,\text{skein rel.}$$

$$\mathbb{Z}_q \subset \mathbb{Z}[q^{\pm 1/2}]$$

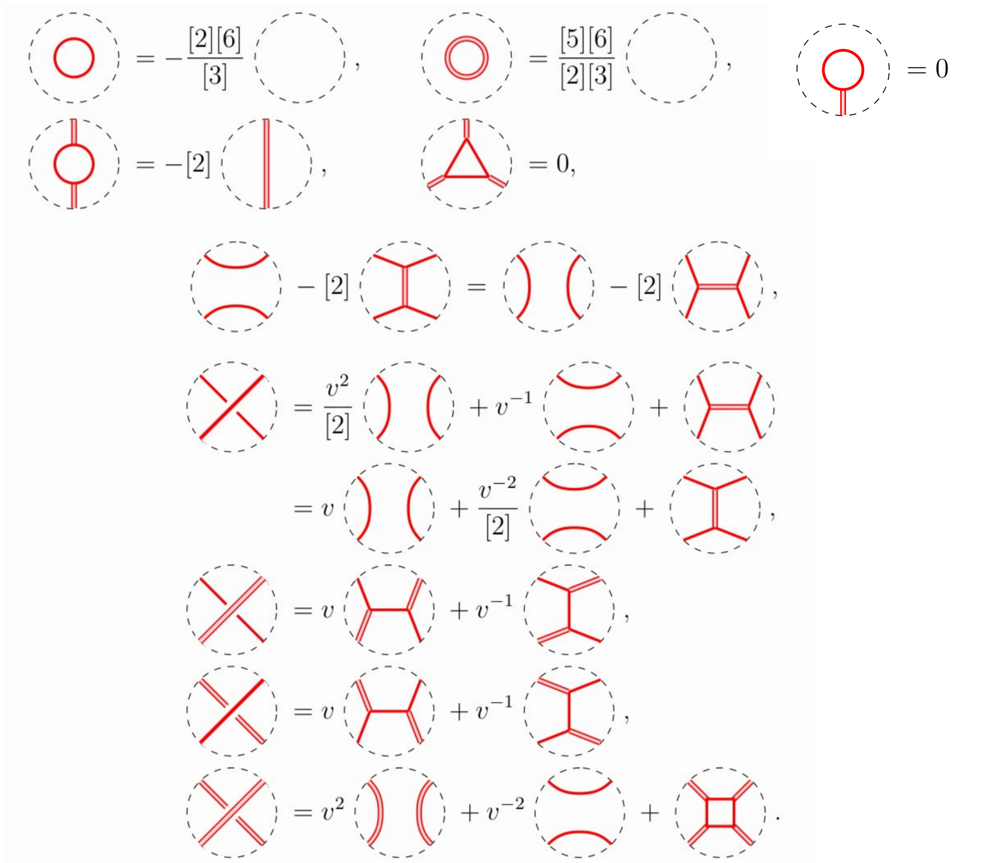
- sp_4-graphs : edges or 

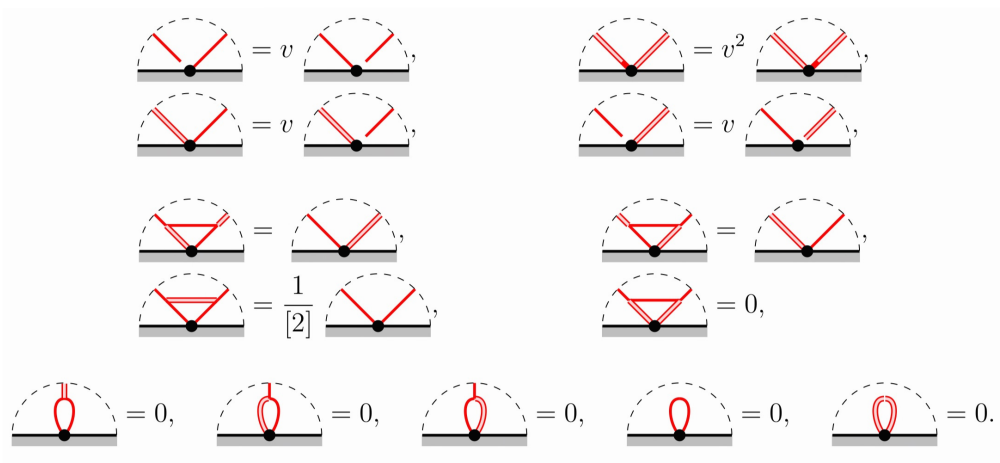
  vertices 

  crossings 

- skein relations

  $v = q$





• $\mathbb{Z}_q$ - form $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$

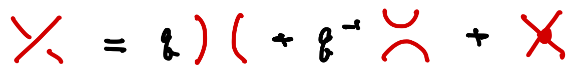
Def. crossroad

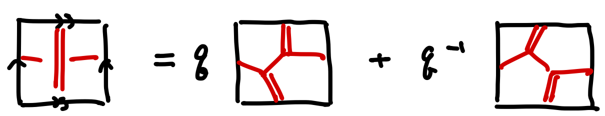
Thm. The set of basis webs BWeb is a $\mathbb{Z}_q[1/[2]]$ - basis of $\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$.

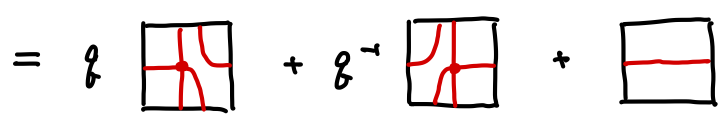
no crossings
no rungs  (have only crossroads)
no elliptic faces

Def. $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma} := \mathrm{span}_{\mathbb{Z}_q}$ BWeb : the $\mathbb{Z}_q$ - form

Lem. $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ is a $\mathbb{Z}_q$-subalgebra of $\mathscr{S}^{q}_{\mathfrak{sp}_4,\Sigma}$





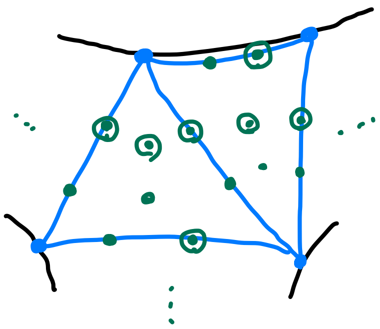


$\rightsquigarrow$ Compare $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ with $\mathscr{A}^{q}_{\mathfrak{sp}_4,\Sigma}$

$\uparrow$ $\mathbb{Z}_q$- alg.

§ quantum cluster algebra $\mathcal{A}^{q}_{\mathfrak{g},\Sigma}$

$\mathcal{F}$: a skew-field



$I = I_{uf}^{\Delta} \sqcup I_f^{\Delta}$: index set

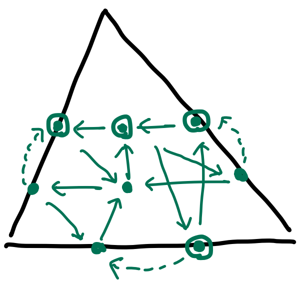
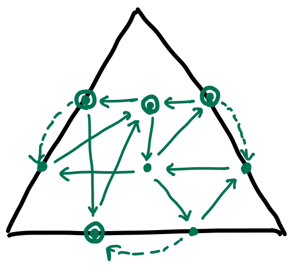
$D = \mathrm{diag}(d_i \mid i \in I)$

- quantum seeds $S^q = (B, \Pi, \mathring{\Lambda}, M)$

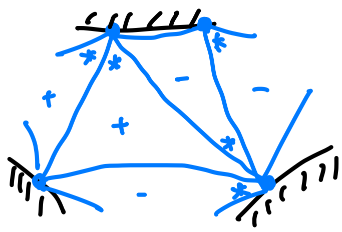
  - $B = (b_{ij})_{i,j \in I}$: skew-symmetrizable DB = skew-symmetric
  - $\Pi = (\pi_{ij})_{i,j \in I}$: skew-symmetric form on $\mathring{\Lambda}$
  - $\mathring{\Lambda} = \bigoplus_{i \in I} \mathbb{Z} f_i$ $\qquad \Pi(f_i, f_j) = \pi_{ij}$
  - $M : \mathring{\Lambda} \longrightarrow \mathcal{F} \setminus \{0\}$ s.t. $M(\alpha) M(\beta) = q^{\frac{\Pi(\alpha,\beta)}{2}} M(\alpha+\beta)$

$\rightsquigarrow \begin{cases} A_i := M(f_i) & \text{: cluster variable} \\ \{A_i\} & \text{: a cluster} \end{cases}$

$\mathfrak{g} = \mathfrak{sl}_4$

$\overset{*}{\underset{+}{\Delta}}$ =  $\qquad \overset{*}{\underset{-}{\Delta}}$ =  $\rightsquigarrow B$

- quantum seeds S^q_{Δ}



- mutation at $k \in I_{uf}$

$$(B, \Pi, \mathring{\Lambda}, M) \xleftrightarrow{\mu_k} (B', \Pi', \mathring{\Lambda}', M')$$

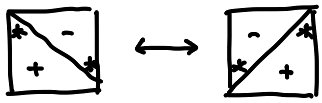
$$A_k \longleftrightarrow A_k'$$

$$A_k A_k' = q^{\bullet} \prod_{j \in I} A_j^{[b_{jk}]_+} + q^{\bullet} \prod_{j \in I} A_j^{[-b_{jk}]_+}$$

quantum exchange relation

Fact Δ, Δ' : decorated triangulations

$S_\Delta \leftrightarrow \cdots\cdots \leftrightarrow S_{\Delta'}$
$\exists$ mutation seq

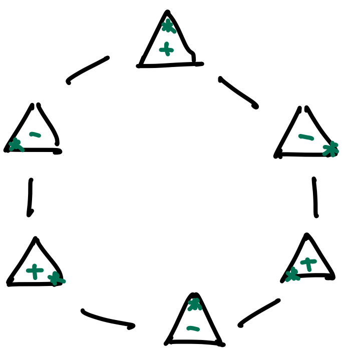
In fact a flip  is realized by 8 mutations.

Def $\mathcal{A}^q_{sp_4,\Sigma}$ = $\mathbb{Z}_q$-subalgebra of $\mathcal{F}$ generated by all clusters related to S_Δ

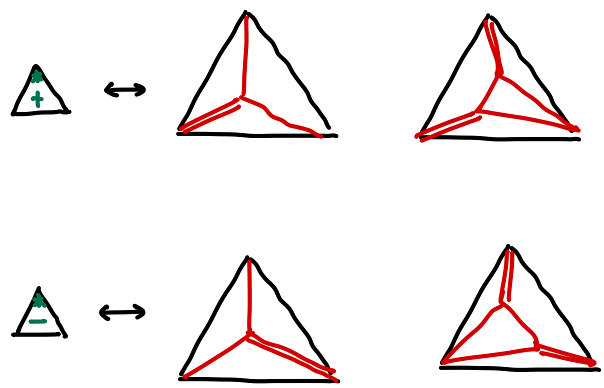
§ Examples : $\mathcal{S}^{\mathbb{Z}_q}_{sp_4,\Sigma}[\partial^{-1}] \subset \mathcal{A}^q_{sp_4,\Sigma}$

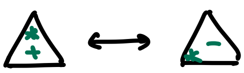
- Σ = triangle $\mathcal{S}^{\mathbb{Z}_q}_{sp_4,\Delta}[\partial^{-1}] = \mathcal{A}^q_{sp_4,\Delta}$

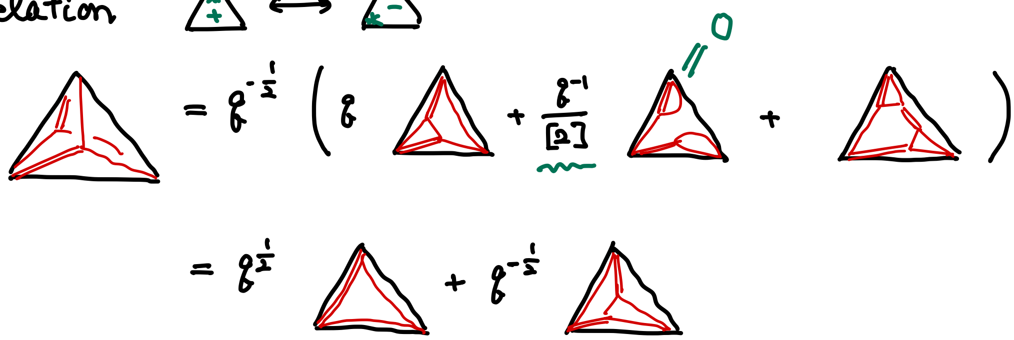
quantum seeds



web cluster



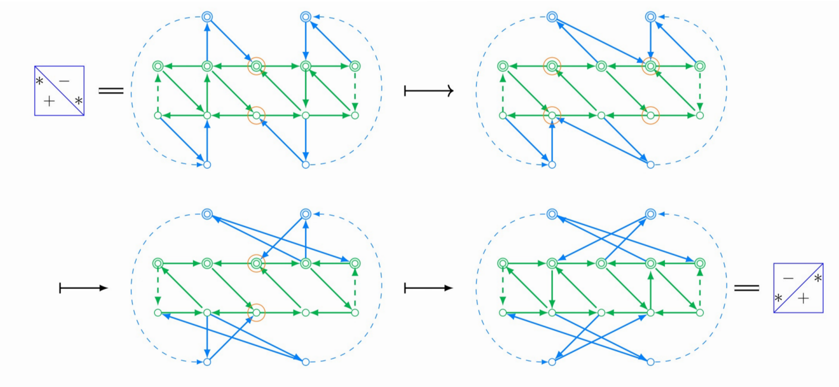
exchange relation 

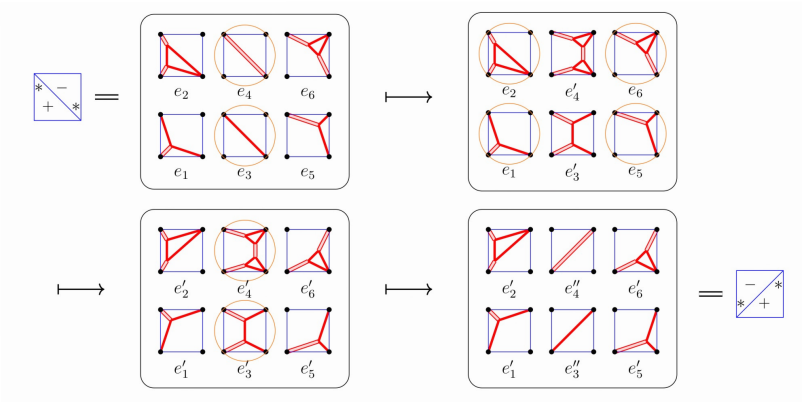


⊛ Σ = quadrilateral, $\mathcal{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathcal{A}^q_{\mathfrak{sp}_4,\Sigma}$

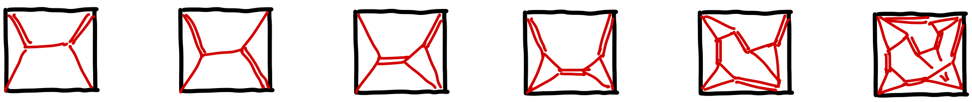
$\exists$ ∞-many clusters

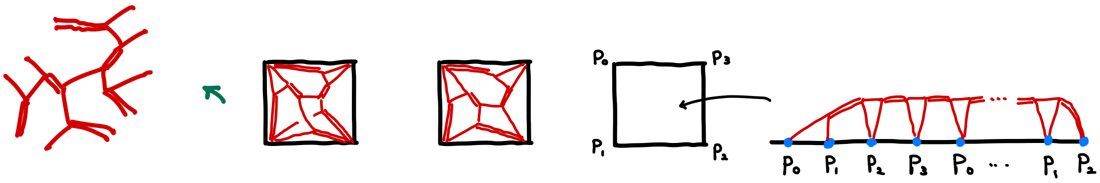
- a flip : 8-mutations





e.g. cluster variables





etc.

<u>Conjecture</u> {tree-type webs} = {cluster variables}

§ Strategy to prove $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^q_{\mathfrak{sp}_4,\Sigma}$

$\mathcal{F} := \operatorname{Frac} \mathscr{S}^q_{\mathfrak{sp}_4,\Sigma}$

(I) Construct quantum seeds $\{S_\Delta\}$ and adjacent seeds in $\mathcal{F}$ $\rightsquigarrow \mathscr{A}^q_{\mathfrak{sp}_4,\Sigma} \subset \mathcal{F}$

q-Laurent phenomenon

clasped skein algera

$\cong$ (reduced) stated skein algebra

(II) $\mathscr{S}^q_{\mathfrak{sp}_4,\Sigma}$ is an Ore domain $\rightsquigarrow \mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma} \subset \mathcal{F}$

state-clasp correspondence [IY] $\mathscr{S}^{st}(\Sigma) \hookrightarrow \otimes \mathscr{S}^{st}(T)$

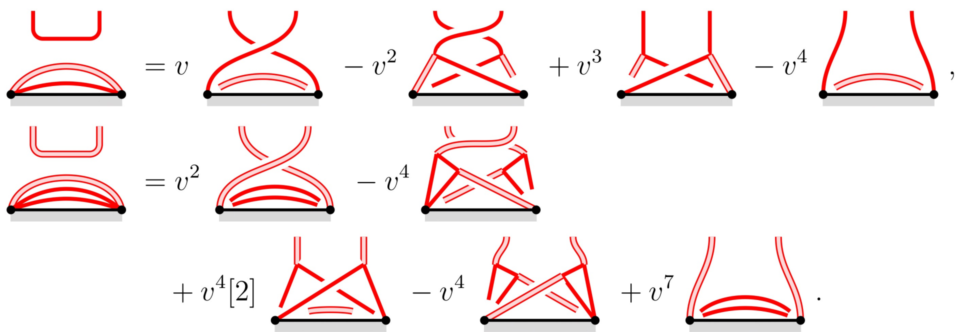
stated skein

triangle

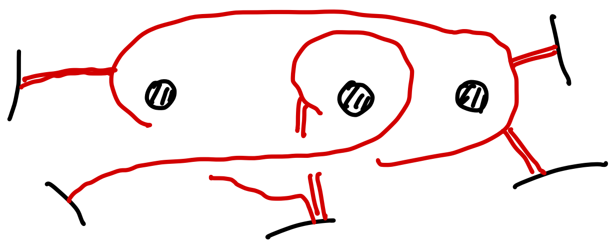
(III) $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}[\partial^{-1}] \subset \mathscr{A}^q_{\mathfrak{sp}_4,\Sigma}$ in $\mathcal{F}$

sticking trick

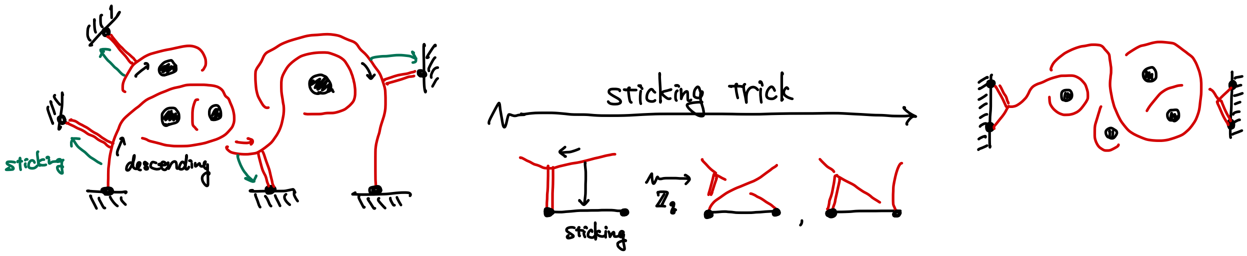
§ sticking trick

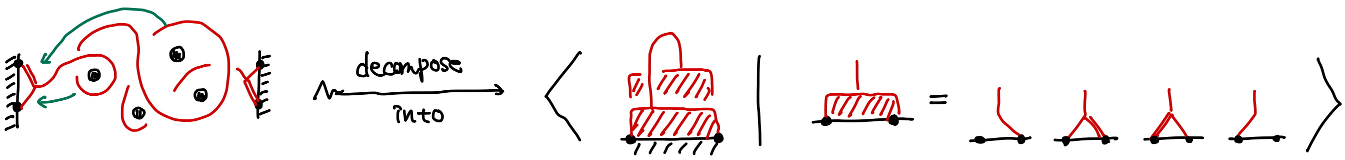


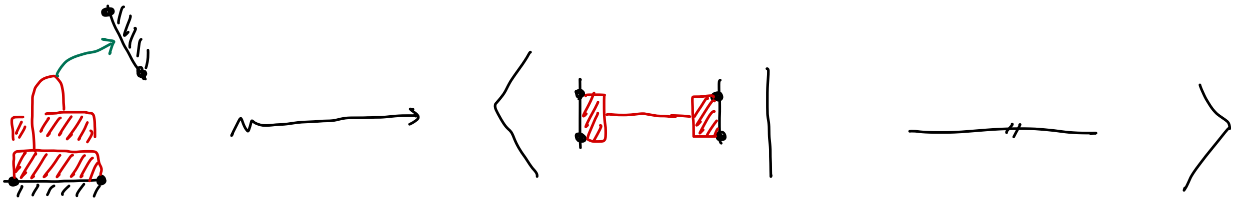
Lem $\mathscr{S}^{\mathbb{Z}_q}_{\mathfrak{sp}_4,\Sigma}$ is generated by "descending" webs



proof. of $\mathscr{S}[\partial^{-1}] \subset \mathscr{A}$







cluster variables in a quadrilateral

Fin.

- Other works

  Lê-Sikora, Lê-Yu : stated skein alg. & quantum trace $\mathfrak{g}=\mathfrak{sl}_n$ for SL_n

  Ishibasi-Kano-Y. : skein & cluster with coefficents.

  Ishibasi-Sun-Y. : bounded $\mathfrak{sp}_4$-lamination etc.