Two deformations of a Markov Equation and related topics

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Contents

Introduction

In this talk, I will talk about two kinds of deformations of a Markov equation.
1st. kind of deformation (we call it t-defrmation) of Markov equation is related to Castling transformation of t-dimensional prehomogeneous vector spaces.
2nd.. kind of deformation is related to q-defrmation of rational numbers introduced by Morier-Genoud and Ovsienko, that is connected to knot theory, hyperbolic geometry , Cluster algebra.

What is Local Functional Equation(=LFE) for a pair of polynomials?

§ What is a local functional equation of pair of polynomials.

What is Local Functional Equation(=LFE) for a pair of polynomials?

Let (P, P^*) be a pair of homogeneous polynomials in n variable of degree d with real coefficients.
It is interesting problem both in Analysis and in Number Theory to find the following "Local Functional equation"=LFE (avrebiation):

For $\{\, x \in V \mid P(x) \neq 0\}_{\mathbb{R}} = \bigcup_{i=1}^{\nu} \Omega_i$:decomposition to connected components.

$$\widehat{|P(x)|_i^s}(= \text{Fourier tr. of } |P(x)|_i^s) = \sum_{j=1}^{\nu} \gamma_{ij}(s) |P^*(y)|_j^{-\frac{n}{d}-s} \quad (*)$$

where $d = \deg.P = \deg.P^*$, $|P(x)|_i := \begin{cases} |P(x)| & (x \in \Omega_i) \\ 0 & \text{otherwise} \end{cases}$

Remark

$\widehat{|P(x)|_i^s}$(= Fourier tr. of $|P(x)|_i^s$) $= \sum_{j=1}^{\nu} \gamma_{ij}(s)|P^*(y)|_j^{-\frac{n}{d}-s}$ (∗)

$\partial^m f = \frac{\partial^{m_1+\cdots+m_n} f}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}$ for $\forall m = (m_1, \ldots m_n) \in \mathbb{Z}_{\geq 0}^n$

$\varphi \in \mathcal{S}(\mathbb{R}^n) := \{f \mid \sup_x |Q(x)\partial^m f| < \infty \text{ for } \forall \text{polynomial } Q(x)\}$

$\widehat{\varphi}(y) = \int_{\mathbb{R}^n} \varphi(x)\exp(2\pi i\langle x, y\rangle)dx$:Fourier trans.form of φ

$$\int_* |P(x)|^s \hat{\varphi}(x)dx = \sum_{**} (\text{Gamma-factor}) \times \int_* |P^*(y)|^{-s-\frac{n}{d}} \varphi(y)dy$$

as a distribution

$\zeta(\varphi, s) = \int_* |P(x)|^s \varphi(x)dx$: local zeta function(zeta distribution)

(*) is also called FE of zeta distribution (local zeta function).

Classical examples

Example 1 : (FT of Positive def. quadratic forms)

$\widehat{(x_1^2+\cdots+x_n^2)^{s-\frac{n}{2}}} = \pi^{-2s+\frac{n}{2}}\Gamma(s)\Gamma(s-\frac{n-2}{2})(y_1^2+\cdots+y_n^2)^{-s}$

Example2 : (FT of Determinant)

$$\widehat{|\det X|^{s-n}} = (2\pi)^{-ns}(2\pi)^{\frac{n(n-1)}{2}}2^n\cos(\pi\frac{s}{2})\cdots\cos(\pi\frac{(s-n+1)}{2}) \times\Gamma(s)\Gamma(s-1)\cdots\Gamma(s-n+1)|\det Y|^{-s}$$

For the case of $n=1$, this is corresponds to Riemmann zeta function as follows:
$\zeta(1-s)=(2\pi)^{-s}\Gamma(s)2\cos(\frac{\pi s}{2})\zeta(s)$

Examples of LFE coming from PV-theory

These examples are coming from relative invarinats of prehomogeneous vector spaces. In simplest case, I will explain local functional equation coming from PV-theory.

Examples of LFE coming from PV-theory

Real PV $(GL(1,\mathbb{R}) \times SO(p,q), \Lambda_1, \mathbb{R}_{p,q})$ with a relative invariant
$P^* = P = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2$ has the following LFEs
(1)If $(p,q) = (n,0)$,
$\widehat{|P|^s} = -\pi^{2s+\frac{n}{2}+1}\Gamma(s+1)\Gamma(s+\frac{n}{2})\sin(s\pi)|P|^{-s-\frac{n}{2}}$

Example2 of LFE coming from PV-theory

(2)$(p,q)=(n-1,1)$,

$$\begin{bmatrix} \widehat{|P|^s_+} \\ \widehat{|P|^s_{-+}} \\ \widehat{|P|^s_{--}} \end{bmatrix} = \pi^{-2s-\frac{p+q}{2}-1}\Gamma(s+1)\Gamma(s+\tfrac{p+q}{2})$$

$$\times \begin{bmatrix} -\cos(s\pi) & -\cos(\frac{n\pi}{2}) & -\cos(\frac{n\pi}{2}) \\ \frac{1}{2} & \frac{1}{2}\mathbf{e}[-\frac{2s+n}{4}] & \frac{1}{2}\mathbf{e}[\frac{2s+n}{4}] \\ \frac{1}{2} & \frac{1}{2}\mathbf{e}[\frac{2s+n}{4}] & \frac{1}{2}\mathbf{e}[-\frac{2s+n}{4}] \end{bmatrix} \begin{bmatrix} |P|_+^{-s-\frac{p+q}{2}} \\ |P|_{-+}^{-s-\frac{p+q}{2}} \\ |P|_{--}^{-s-\frac{p+q}{2}} \end{bmatrix}$$

Example2 of LFE coming from PV-theory

(3) If $p, q \geq 2$,

$$\begin{bmatrix} \widehat{|P|_+^s} \\ \widehat{|P|_-^s} \end{bmatrix} = \pi^{-2s-\frac{p+q}{2}-1}\Gamma(s+1)\Gamma(s+\tfrac{p+q}{2})$$

$$\times \begin{bmatrix} -\sin\pi(s+\frac{q}{2}) & \sin(\frac{\pi p}{2}) \\ \sin(\frac{\pi q}{2}) & -\sin\pi(s+\frac{p}{2}) \end{bmatrix} \begin{bmatrix} |P|_+^{-s-\frac{p+q}{2}} \\ |P|_-^{-s-\frac{p+q}{2}} \end{bmatrix}$$

Tree of Castling trasnformations of PVs

§ Castling transform of Prehomogeneous vector spaces and t-Deformation of Markov triples.

Tree of Castling trasnformations of PVs

Grassmann duality

$$\wedge^k V \cong (\wedge^{n-k} V)^* \cong \wedge^{n-k}(V^*)$$

The transform coming from this Grassmann duality is called **Castling transform of vector spaces**

In particular, Castling transfroms preserve prehomogenety!

(Castling transform is introduced by Mikio Sato and Takuro Shintani.)

Tree of Castling trasnformations of PVs

Example

$(SO(2)\times GL(1)\times GL(1)\times GL(1), \rho\otimes\Lambda_1\otimes\Lambda_1\otimes\Lambda_1, V(3)\otimes V(1)\otimes V(1)\otimes V(1)) \leftrightarrow (3,1,1,1)$

$\Rightarrow$(castling transform)

$(SO(2)\times GL(2)\times GL(1)\times GL(1), \rho^*\otimes\Lambda_1\otimes\Lambda_1\otimes\Lambda_1, V(3)\otimes V(2)\otimes V(1)\otimes V(1)) \leftrightarrow (3,2,1,1)$

$\Rightarrow$(castling transform)

$(SO(2)\times GL(2)\times GL(5)\times GL(1), \rho\otimes\Lambda_1\otimes\Lambda_1\otimes\Lambda_1, V(3)\otimes V(2)\otimes V(5)\otimes V(1)) \leftrightarrow (3,2,5,1)$

There are two castling transforms for this.

One is $(3,2,5,1)\Rightarrow(3,13,5,1)$

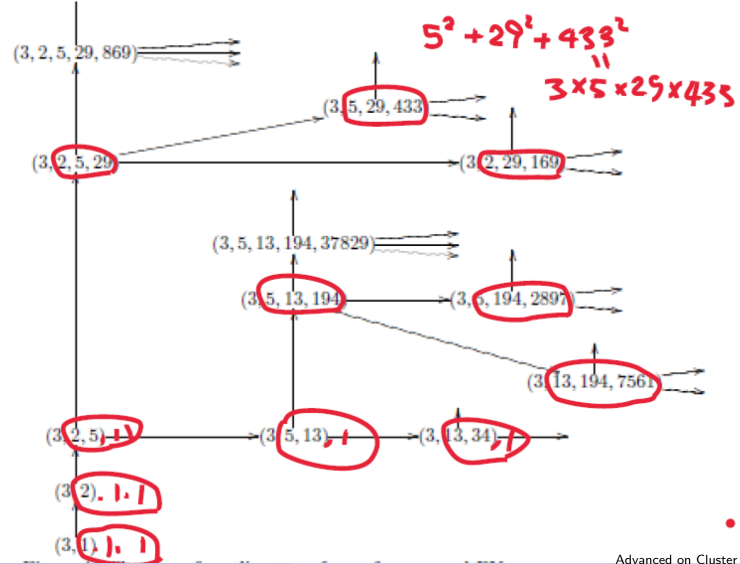
Another is $(3,2,5,1)\Rightarrow(3,2,5,29)$

Tree of Castling trasnformations of PVs

Here we explain the notation:
for example, 3-dimensional Prehomogeneous vector space
$(SO(3) \times GL(1) \times GL(1) \times GL(1), V(3) \otimes V(1) \otimes V(1) \otimes V(1))$ and
$(SL(2) \times GL(1) \times GL(1) \times GL(1), \mathrm{Sym}(2) \otimes V(1) \otimes V(1) \otimes V(1))$ corresponds to
$(3, 1, 1, 1)$
A diagram showing the tree growing from bottom to top with CT is on the next page.

Tree of Castling trasnform. for 3-dim PV and Markov tree

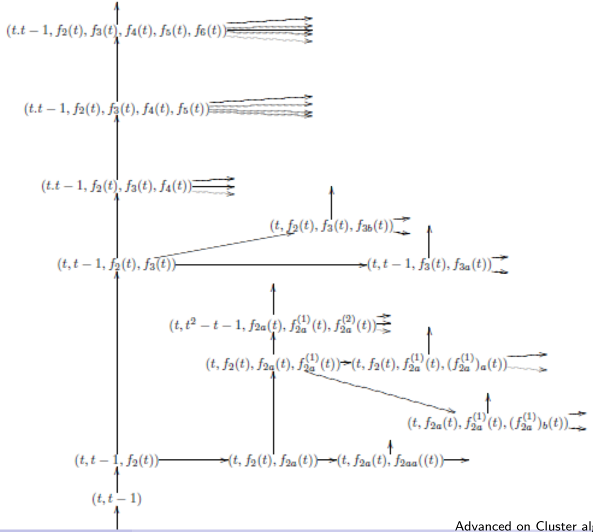
Remark(Markov number and Castling transform of PV)



Tree of Castling trasnformations of PVs

For the t-dimensional representation space, the diagram on the next page shows the tree that grows from bottom to top for Castling transformation.

Castling transform of prehomogeneous vector spaces and t-Deformation of Markov triples



Castling transform of prehomogeneous vector spaces and t-Deformation of Markov triples

where
$f_2(t)=t^2-t-1$,
$f_3(t)=t^4-2t^3+t-1$,
$f_4(t)=t^7-3t^6+t^5+2t^4+t^3-t^2-t-1$,
$f_5(t)=$
$t^{14}-6\,t^{13}+11\,t^{12}-2\,t^{11}-9\,t^{10}-4\,t^9+10\,t^8+7\,t^7-2\,t^6-7\,t^5-3\,t^4+t^3+2\,t^2+t-1$,
$f_6(t)=t^{28}-12\,t^{27}+58\,t^{26}-136\,t^{25}+127\,t^{24}+56\,t^{23}-126\,t^{22}-158\,t^{21}+$
$229\,t^{20}+196\,t^{19}-158\,t^{18}-314\,t^{17}+34\,t^{16}+294\,t^{15}+146\,t^{14}-142\,t^{13}-$
$213\,t^{12}-26\,t^{11}+116\,t^{10}+90\,t^9-9\,t^8-45\,t^7-23\,t^6+5\,t^5+9\,t^4+3\,t^3-t^2-t-1$,
$f_{3a}(t)=t^5-2t^4+2t+1, t^5-2t^4-t^2+2t+1$,
$f_{2a}^{(1)}(t)=t^6-2t^5-2t^4+4t^3-t^2-t-1$,
$(f_{2a}^{(1)})_a(t)=t^9-3t^8-t^7+8t^6-t^5-6t^4-2t^3+3t^2+3t-1$,
$(f_{2a}^{(1)})_b(t)=t^{10}-3t^9-2t^8+11t^7-t^6-12t^5+2t^4+3t^2+1$,
$f_{2a}^{(2)}(t)=t^{12}-4t^{11}+16t^9-10t^8-22t^7+15t^6+14t^5-5t^4-6t^3+t-1$,
$f_{3aa}(t)=t^4-t^3-3t^2+2t+1$,

Castling transform of prehomogeneous vector spaces and t-Deformation of Markov triples

In this diagram, Pick up subtree

$$(t, f_{w(a,b)}(t), f_{w(a,b)w'(a,b)}(t), f_{w'(a,b)}(t))$$

starting form

$$(t, f_a(t), f_{ab}(t), f_b(t)) := (t, t^2 - t - 1, t - 1),$$

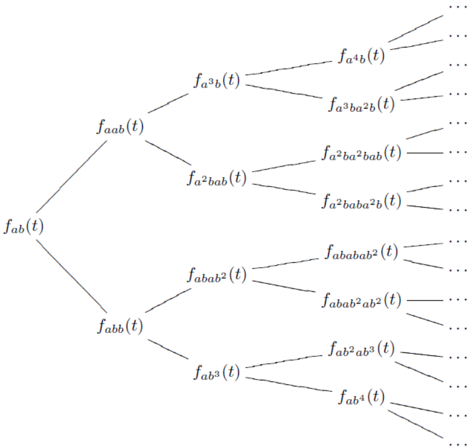
where Christoffel ab-words $w(a,b), w(a,b)w'(a,b)$, $w'(a,b)$ means triplet of t-polynomials.
Inother words,

$$(t, f_{w(a,b)}(t), f_{w(a,b)w'(a,b)}(t), f_{w'(a,b)}(t))$$

parametrized by Christoffel ab-words.
for example, Castling transform of $(t, f_a(t), f_{ab}(t), f_b(t))$ at $f_{ab}(t)$ is $(t, f_a(t), f_{ab}(t), tf_a(t)f_{ab}(t) - f_b(t))$ and we put $tf_a(t)f_{ab}(t) - f_b(t) = f_{a^2b}(t)$.
Thus we can consider a triplet $(f_a(t), f_{a^2b}(t), f_{ab}(t))$ parametrized by Christoffel ab-word (a, a^2b, ab).

Castling transform of prehomogeneous vector spaces and t-Deformation of Markov triples

Tree of
$\{(f_{w(a,b)}(t), f_{w(a,b)w'(a,b)}(t), f_{w'(a,b)}(t))\}_{(w(a,b),w(a,b)w'(a,b),w'(a,b))}$is a triple of Christoffel



t-Deformations of Markov triples

Theorem([K, arXiv2008.12913v3])
A triplet $(f_w(t), f_{ww'}(t), f_{w'}(t))$ of polynomials associated to Christoffel *ab*-word triple (w, ww', w'), then $(f_w(t), f_{ww'}(t), f_{w'}(t))$ is a solution of the following equation:

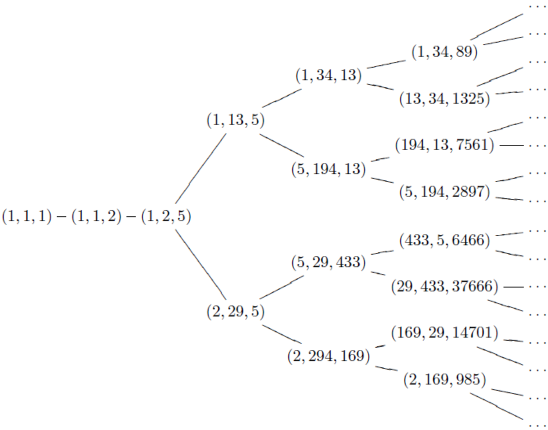
$$x^2 + y^2 + z^2 + (t-3) = txyz. \tag{1}$$

Elementary properties of Markov triples

§§ **Elementary properties of Markov triples**

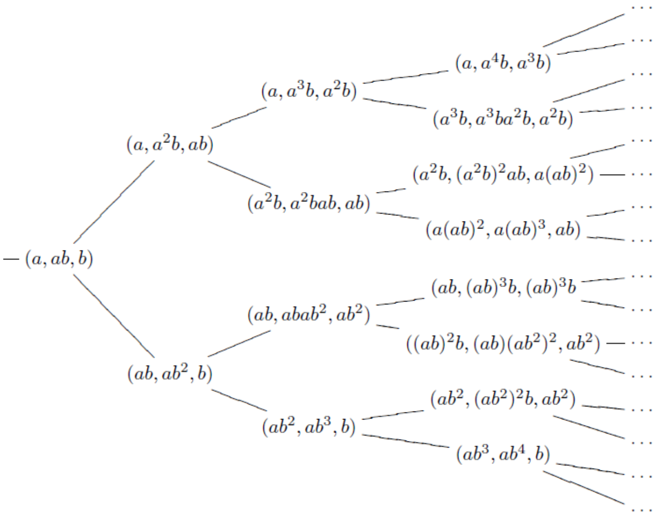
Markov triple, Markov tree

Integer-solution (x, y, z) of equation $x^2 + y^2 + z^2 = 3xyz$ is called Markov triple.



Remark: Markov conjecture: The maxima of each triplet are all different.

Christoffel *ab*-words and Markov triples



Christoffel *ab*-words and Markov triples

Theorem(Cohn cf. [Bombieri], [Aigner])
$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, B := \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$
$(w(A,B)_{1,2}, (w(A,B)w'(A,B))_{1,2}, w'(A,B)_{1,2}) =$
$(\frac{1}{3}\mathrm{tr}(w(A,B)), \frac{1}{3}\mathrm{tr}(w(A,B)w'(A,B)), \frac{1}{3}\mathrm{tr}(w'(A,B)))$
is a Markov triple for Christoffel *ab*-words $(w(a,b), w(a,b)w'(a,b), w'(a,b))$.

Continued fractions and their properties

§§ Continued fractions and their properties

Continued fractions and their properties

Well known properties for continued fractions :
For $\frac{r}{s} \in \mathbb{Q}$, we assume $\frac{r}{s} > 1$ and $\gcd(r, s) = 1$,

$$\frac{r}{s} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}}$$

$$= [[c_1, c_2, \ldots, c_k]] \qquad = [a_1, a_2, \ldots, a_{2m}]$$

Continued fractions and their properties

Example

$\frac{19}{7} = [2, 1, 2, 2] = [[3, 4, 2]]$

For a regular continued fraction, if $a_n \neq 1$,
$[a_1, \ldots, a_{n-1}, a_n] = [a_1, \ldots, a_{n-1}, a_n - 1, 1]$

We can asuume $n = \mathrm{even} = 2m$
Notations：
$N[a_1, \ldots, a_{2m}] = [a_1, \ldots, a_{2m}]$-Numerator
$D[a_1, \ldots, a_{2m}] = [a_1, \ldots, a_{2m}]$-Denominator
$N[[c_1, \ldots, c_k]] = [[c_1, \ldots, c_k]]$-Numerator
$D[[c_1, \ldots, c_k]] = [[c_1, \ldots, c_k]]$-Denominator

Continued fractions and their properties

Theorem1(Euler Continuants, 高木貞治「代数学講義」or Hardy-Wright)
Regular CF and negative CF satisfy the following **Euler's continuants**:

$$N[a_1,\ldots,a_{2m}] = \det\begin{pmatrix} a_1 & 1 & & & & & \\ -1 & a_2 & 1 & & & & \\ & -1 & a_3 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & a_{2m-1} & 1 \\ & & & & -1 & a_{2m} \end{pmatrix}$$

Continued fractions and their properties

$$N[[c_1,\dots,c_k]]=\det\begin{pmatrix} c_1 & 1 & & & & \\ 1 & c_2 & 1 & & & \\ & 1 & c_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & c_{k-1} & 1 \\ & & & & 1 & c_k \end{pmatrix}$$

If $a_1 \neq 0$, $D[a_1,\dots,a_{2m}] = \frac{\partial}{\partial a_1} N[a_1,\dots,a_{2m}]$
If $c_1 \neq 0$, $D[[c_1,\dots,c_k]] = \frac{\partial}{\partial c_1} N[[c_1,\dots,c_k]]$

Continued fractions and their properties

Theorem2

For $\frac{r}{s} \in \mathbb{Q}_{>0}$,
$\frac{r}{s} = [a_1, a_2, \ldots, a_{2m}] = [[c_1, c_2, \ldots, c_k]]$,

$$M^+(a_1, a_2, \ldots, a_{2m}) := \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2m} & 1 \\ 1 & 0 \end{pmatrix}$$

$$M(c_1, c_2, \ldots, c_k) := \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_k & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$

$M^+(a_1, \ldots, a_{2m}) = \begin{pmatrix} r & r'_{2m-1} \\ s & s'_{2m-1} \end{pmatrix}, M(c_1, \ldots, c_k) = \begin{pmatrix} r & -r_{k-1} \\ s & -s_{k-1} \end{pmatrix}$

$\frac{r'_{2m-1}}{s'_{2m-1}} = [a_1, a_2, \ldots, a_{2m-1}], \frac{r_{k-1}}{s_{k-1}} = [[c_1, \ldots, c_{k-1}]]$

Continued fractions and their properties

Corollary of Theorem2

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(R, S) and (L, S) is the standard choice of generators of the group $SL(2, \mathbb{Z})$ the above matrices are as follows:
$M^{+}(a_1, \ldots, a_{2m}) = R^{a_1} L^{a_2} R^{a_3} L^{a_4} \cdots R^{a_{2m-1}} L^{a_{2m}}$
$M(c_1, \ldots, c_k) = R^{c_1} S R^{c_2} S R^{c_3} S \cdots R^{c_k} S$

Geometric invariants coming from a graph related to continued fractions

[LS]:K.Lee and R.Schiffler , Cluster algebras and Jones polynomials. . Selecta Math. (N.S.) 25 (2019), no. 4, Paper No. 58, 41 pp.
[KW]:T.Kogiso, and M. Wakui, A bridge between Conway-Coxeter friezes and rational tangles through the Kauffman bracket polynomials. J. Knot Theory Ramifications 28 (2019), no. 14, 1950083, 40 pp.
[NT]:W.Nagai and Y.Terashima,Cluster variables, ancestral triangles and Alexander polynomials, Adv. Math. 363 (2020), 106965, 37 pp.
[MO]:S. Morier-Genoud, V. Ovsienko, q-deformed rationals and q-continued fractions, Forum Math. Sigma 8 (2020), No.e13, 55pp.

[LS] $\Rightarrow$ Jones polynomials of 2-bridge links by using Snake graph and F-polynomials .
[KW] $\Rightarrow$ Kauffman bracket polynomials of 2-bridge links by using Ancestral triangles and Conway-Coxeter frieze.
[NT] $\Rightarrow$ Alexander polynomials and Jones polynomials of 2-bridge links by using Ancestral triangles and F-Polynomials.
[MO] $\Rightarrow$ Jones polynomials of 2-bridge links by using Fraey Boats.

q-Continued fractions and their properties

§§ q-Deformations of continued fractions

q-Continued fractions and their properties

Sophie Morier-Genord and Valentin Ovsienko,
q-deformed rationals and q-continued fractions,
Forum Math. Sigma 8 (2020), Paper No. e13, 55 pp.

q-Continued fractions and their properties

Definition 1.1

(a) $$[a_1, a_2, \ldots, a_{2m}]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{[a_4]_{q^{-1}} + \cfrac{q^{-a_4}}{\ddots \cfrac{}{[a_{2m-1}]_q + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}}$$

q-Continued fractions and their properties

(b) $$[[c_1, c_2, \ldots, c_k]]_q = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{[c_3]_q - \cfrac{q^{c_3-1}}{[c_4]_q - \cfrac{q^{c_4-1}}{\cfrac{\ddots}{[c_{k-1}]_q - \cfrac{q^{c_{k-1}-1}}{[c_k]_q}}}}}}$$

q-Continued fractions and their properties

Theorem 1(M-J and O)

$\frac{r}{s} = [a_1, \ldots, a_{2m}] = [[c_1, \ldots, c_k]]$
$\Rightarrow$
$[a_1, \ldots, a_{2m}]_q = [[c_1, \ldots, c_k]]_q$
then $[a_1, \ldots, a_{2m}]_q = [[c_1, \ldots, c_k]]_q =: [\frac{r}{s}]_q$

Example

$[\frac{5}{2}]_q = [[3,2]]_q = [2,2]_q = \frac{1+2q+q^2+q^3}{1+q}$
$[\frac{5}{3}]_q = [[2,3]]_q = [1,1,1,1]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$
$[\frac{7}{3}]_q = [[3,2,2]]_q = [2,3]_q = \frac{1+2q+2q^2+q^3+q^4}{1+q+q^2}$
$[\frac{7}{4}]_q = [[2,4]]_q = [1,1,2,1]_q = \frac{1+q+2q^2+2q^3+q^4}{1+q+q^2+q^3}$
$[\frac{7}{5}]_q = [[2,2,3]]_q = [1,1,2,1]_q = \frac{1+q+2q^2+2q^3+q^4}{1+q+2q^2+q^3}$

for the case of denominator $[2]_q$,

(c)$[\frac{2m+1}{2}]_q = \frac{1+2q+2q^2+\cdots+2q^{m-1}+q^m+q^{m+1}}{1+q}$

(d)$[\frac{3m+1}{3}]_q = \frac{1+2q+3q^2+3q^3+\cdots+3q^{m-1}+2q^m+q^{m+1}+q^{m+2}}{1+q+q^2}$

$[\frac{3m+2}{3}]_q = \frac{1+2q+3q^2+3q^3+\cdots+3q^{m-1}+2q^m+2q^{m+1}+q^{m+2}}{1+q+q^2}$

q-Continued fractions and their properties

Theorem(Morier-Genoud and V.Ovsienko, 2019)

$$M_q^+(a_1,\dots,a_{2m}) :=$$

$$\begin{pmatrix}[a_1]_q & q^{a_1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}[a_2]_{q^{-1}} & q^{-a_2}\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}[a_{2m-1}]_q & q^{a_{2m-1}}\\ 1 & 0\end{pmatrix}\begin{pmatrix}[a_{2m}]_{q^{-1}} & -q^{-a_{2m}}\\ 1 & 0\end{pmatrix}$$

$$M_q(c_1,\dots,c_k) :=$$

$$\begin{pmatrix}[c_1]_q & -q^{c_1-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}[c_2]_q & -q^{c_2-1}\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}[c_{k-1}]_q & -q^{c_{k-1}-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix}[c_k]_q & -q^{c_k-1}\\ 1 & 0\end{pmatrix}$$

$\Rightarrow$

q-Continued fractions and their properties

$$(\mathrm{i})M_q^+(a_1,\dots,a_{2m})=\begin{pmatrix} q\mathcal{R} & \mathcal{R}'_{2m-1}\\ q\mathcal{S} & \mathcal{S}'_{2m-1}\end{pmatrix}$$

where $\frac{\mathcal{R}(q)}{\mathcal{S}(q)}=[a_1,a_2,\dots,a_{2m}]_q,\quad \frac{\mathcal{R}'_{2m-1}(q)}{\mathcal{S}'_{2m-1}(q)}=[a_1,\dots,a_{2m-1}]_q$

$$(\mathrm{ii})M_q(c_1,\dots,c_k)=\begin{pmatrix} \mathcal{R} & -q^{c_k-1}\mathcal{R}_{k-1}\\ \mathcal{S} & -q^{c_k-1}\mathcal{S}'_{k-1}\end{pmatrix}$$

where $\frac{\mathcal{R}(q)}{\mathcal{S}(q)}=[[c_1,\dots,c_k]]_q,\quad \frac{\mathcal{R}_{k-1}(q)}{\mathcal{S}_{k-1}(q)}=[c_1,\dots,c_{k-1}]_q$

$$(\mathrm{iii})R_q:=\begin{pmatrix} q & 1\\ 0 & 1\end{pmatrix},\ L_q:=\begin{pmatrix} 1 & 0\\ 1 & q^{-1}\end{pmatrix},\ S_q:=\begin{pmatrix} 0 & -q^{-1}\\ 1 & 0\end{pmatrix}$$

$\Rightarrow$

$$M_q^+(a_1,\dots,a_{2m})=R_q^{a_1}L_q^{a_2}\cdots R_q^{a_{2m-1}}L_q^{a_{2m}}$$
$$M_q^+(c_1,\dots,c_k)=R_q^{c_1}S_qR_q^{c_2}S_q\cdots S_qR_q^{c_k}S_q$$

q-Continued fractions and their properties

(iv) $K_{2m}^{+}(a_1,\ldots,a_{2m})_q :=$

$$\det\begin{pmatrix} [a_1]_q & q^{a_1} & & & & & \\ -1 & [a_2]_{q^{-1}} & q^{-a_2} & & & & \\ & -1 & [a_3]_q & q^{a_3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & [a_{2m-1}]_q & q^{a_{2m-1}} \\ & & & & -1 & [a_{2m}]_{q^{-1}} \end{pmatrix}$$

$$K_k(c_1,\ldots,c_k)_q := \det\begin{pmatrix} [c_1]_q & q^{c_1-1} & & & & \\ 1 & [c_2]_q & q^{c_2-1} & & & \\ & 1 & [c_3]_q & q^{c_3-1} & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & [c_{k-1}]_q & q^{c_{k-1}-1} \\ & & & & 1 & [c_k]_q \end{pmatrix}$$

$\Rightarrow$

q-Continued fractions and their properties

$$M_q^+(a_1,\dots,a_{2m})=\begin{pmatrix}K_{2m}^+(a_1,\dots,a_{2m})_q & q^{a_{2m}}K_{2m-1}^+(a_1,\dots,a_{2m-1})_{q^{-1}}\\ K_{2m-1}^+(a_2,\dots,a_{2m})_q & q^{a_{2m}}K_{2m-2}^+(a_2,\dots,a_{2m-1})_{q^{-1}}\end{pmatrix}$$

$$M_q(c_1,\dots,c_k)=\begin{pmatrix}K_k(c_1,\dots,c_k)_q & -q^{c_k-1}K_{k-1}(c_1,\dots,c_{k-1})_q\\ K_{k-1}(c_2,\dots,c_k)_q & -q^{c_k-1}K_{k-2}(c_2,\dots,c_{k-1})_q\end{pmatrix}$$

(v) For $[\frac{r}{s}]_q=[a_1,a_2,\dots,a_{2m}]_q=[[c_1,\dots,c_k]]_q=\frac{\mathcal{R}(q)}{\mathcal{S}(q)}$,

$$\mathcal{R}(q)=K_k(c_1,\dots,c_k)_q=q^{a_2+a_4+\cdots+a_{2m-1}}K_{2m}^+(a_1,a_2,\dots,a_{2m})_q$$

$$\mathcal{S}(q)=K_{k-1}(c_1,\dots,c_k)_q=q^{a_2+a_4+\cdots+a_{2m}-1}K_{2m-1}^+(a_2,\dots,a_{2m})_q$$

(vi) $K_k(c_1,\dots,c_k)_q=q^{c_1+c_2+\cdots+c_c-k}K_k(c_k,\dots,c_1)_{q^{-1}}$

q-Continued fractions and their properties

(vii) If $(c_1, \ldots, c_k)$ is quiddity sequence of a triangulated n-gon one has $K_k(c_1, \ldots, c_k)_q = K_{n-k-2}(c_{k+2}, \ldots, c_n)_q$

(viii) (q-Ptolemy relation)
$K^q_{i,j} := K_{j-i-1}(c_{i+1}, \ldots . c_{j-1})_q$ ($K^q_{i,i} = 0,\ K^q_{i,i+1} = 1$)
$\Rightarrow$
$K^q_{i,j}K^q_{j,\ell} = q^{c_j+\cdots+c_{k-1}-(k-j)}K^1_{i,j}K^q_{k,\ell} + K^q_{j,k}K^q_{i,\ell}$ $(1 \leq i < j < k < \ell \leq n)$
(Ptolemy-relation)

q-Continued fractions and their properties

Example

$\frac{5}{3} = 1 + \frac{2}{3} = 1 + \frac{1}{1+\frac{1}{2}} = [1,1,2] = [1,1,1,1] = 1 - \frac{1}{3} = [[2,3]]$

and

(i)$M_q(2,3) = \begin{pmatrix} [2]_q & -q^{2-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [3]_q & -q^{3-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{R} & -q^2\mathcal{R}'_{k-1} \\ \mathcal{S} & -q^2\mathcal{S}'_{k-1} \end{pmatrix}$

$M_q^+(1,1,1,1) = \begin{pmatrix} [1]_q & q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [1]_{q^{-1}} & q^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [1]_q & q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [1]_{q^{-1}} & q^{-1} \\ 1 & 0 \end{pmatrix} =$
$\begin{pmatrix} q^{-1}\mathcal{R} & q^{-2}\mathcal{R}_{2m-1} \\ q^{-1}\mathcal{S} & q^{-2}\mathcal{S}_{2m-1} \end{pmatrix}$

$\frac{M_q(2,3)(1,1)}{M_q(2,3)(2,1)} = \frac{1+q+2q^2+q^3}{1+q+q^2} = \frac{M_q^+(1,1,1,1)(1,1)}{M_q^+(1,1,1,1)(2,1)}$

q-Continued fractions and their properties

$$R_qL_qR_qL_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix}\begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & q^{-1} \end{pmatrix}$$

$$R_q^2S_qR_q^3S_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}^2\begin{pmatrix} 0 & q^{-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}^3\begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix} = M_q(3,2) =$$
$$\begin{pmatrix} [2]_q & -q^{2-1} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} [3]_q & -q^{3-1} \\ 1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} [2]_q & q^{2-1} \\ 1 & [3]_q \end{pmatrix} = [2]_q[3]_q - q = N[\tfrac{3}{2}]_q = q^3 \det\begin{pmatrix} [1]_q & q & & \\ -1 & [1]_{q^{-1}} & q^{-1} & \\ & -1 & [1]_q & q \\ & & -1 & [1]_{q^{-1}} \end{pmatrix}$$

$$[\tfrac{5}{3}]_q = [1,1,1,1]_q = [[2,3]]_q = \frac{\mathcal{R}(q)}{\mathcal{S}(q)}$$
$$\mathcal{R}(q) = K_k(c_1,\dots,c_k) = q^{a_2+\cdots a_{2n}-1}K^+_{2m}(a_1,\dots,a_{2m})_q$$
$$\mathcal{S}(q) = K_{k-1}(c_2,\dots,c_k) = q^{a_2+\cdots a_{2n}-1}K^+_{2m-1}(a_2,\dots,a_{2m})_q$$

q-Deformation of Markov triples and Markov equations

§ q-Deformation of Markov triples and Markov equations

q-Deformations of Markov triples

Theorem1([K], arXiv2008.12913v3)

Put $A_q := \begin{pmatrix} q+1 & q^{-1} \\ 1 & q^{-1} \end{pmatrix}$, $B_q := \begin{pmatrix} \frac{q^3+q^2+2q+1}{q} & \frac{q+1}{q^2} \\ \frac{q+1}{q} & q^{-2} \end{pmatrix} \in SL(2, \mathbb{Z}[q, q^{-1}])$,

$\Rightarrow$

$(x, y, z) =$
$(\mathrm{tr} w(A_q, B_q)/[3]_q, \{\mathrm{tr} w(A_q, B_q) w'(A_q, B_q)\}/[3]_q, \mathrm{tr}(w'(A_q, B_q)/[3]_q) \in$
$\mathbb{Z}[q, q^{-1}]^3$

is a solution of

$$x^2 + y^2 + z^2 + \frac{(q-1)^2}{q^3} = [3]_q xyz$$

for a Christoffel ab-words $(w(a,b), w(a,b)w'(a,b), w'(a,b))$.

q-Deformations of Markov triples

Theorem2([K], arXiv2008.12913v3.)
If $(x, y, z) = (a_q, b_q, c_q)$ is a solution of

$$(**q) \qquad x^2 + y^2 + z^2 + \frac{(q-1)^2}{q^3} = [3]_q xyz$$

$\Rightarrow$
$(\tilde{x}, y, z) = ([3]_q b_q c_q - a_q, b_q, c_q), (x, \tilde{y}, z) = (a_q, [3]_q a_q c_q - b_q, c_q), (x, y, \tilde{z}) = (a_q, b_q, [3]_q a_q b_q - c_q)$
is also a solution of $(**q)$.

q-Deformations of Markov triples

Example1

$$\left(\frac{\mathrm{tr}(A_q^3B_q)}{[3]_q}\right)^2+\left(\frac{\mathrm{tr}(A_q^3B_qA_q^2B_q)}{[3]_q}\right)^2+\left(\frac{\mathrm{tr}(A_q^2B_q)}{[3]_q}\right)^2+\frac{(q-1)^2}{q^3}$$

$$=\left\{\frac{\left(q^2+1\right)\left(q^6+3q^5+3q^4+3q^3+3q^2+3q+1\right)}{q^5}\right\}^2+$$

$$\left\{\frac{\left(q^4+q^3+q^2+q+1\right)\left(q^{12}+5q^{11}+12q^{10}+22q^9+32q^8+39q^7+43q^6+39q^5+32q^4+22q^3+12q^2+5q+1\right)}{q^9}\right\}^2+$$

$$\left\{\frac{q^4+q^3+q^2+q+1}{q^3}\right\}^2+\frac{(q-1)^2}{q^3}$$

$$=[3]_q\left\{\frac{\left(q^2+1\right)\left(q^6+3q^5+3q^4+3q^3+3q^2+3q+1\right)}{q^5}\right\}$$

$$\left\{\frac{\left(q^4+q^3+q^2+q+1\right)\left(q^{12}+5q^{11}+12q^{10}+22q^9+32q^8+39q^7+43q^6+39q^5+32q^4+22q^3+12q^2+5q+1\right)}{q^9}\right\}$$

$$\left\{\frac{q^4+q^3+q^2+q+1}{q^3}\right\}$$

$$=[3]_q\frac{\mathrm{tr}(A_q^3B_q)}{[3]_q}\frac{\mathrm{tr}(A_q^3B_qA_q^2B_q)}{[3]_q}\frac{\mathrm{tr}(A_q^2B_q)}{[3]_q}$$

Castling transform and t-Deformations of Markov triples

Tree of
$(\{(f_{w(a,b)}(t), f_{w(a,b)w'(a,b)}(t), f_{w'(a,b)}(t))\}_{(w(a,b),w(a,b)w'(a,b),w'(a,b))\text{is a triple of Christoffel}}$

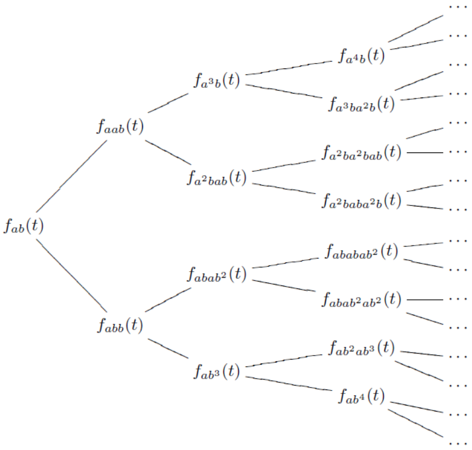


Figure:

t-Deformations of Markov triples

Theorem([K], arXiv2008.12913v3)
For Christoffel ab-word triple (w, ww', w') ,
$(f_w(t), f_{ww'}(t), f_{w'}(t))$
is a solution of

$$x^2 + y^2 + z^2 + (t - 3) = txyz. \tag{2}$$

Relation of q-Deformations and t-Deformations

Theorem([K], arXiv2008.12913v3)
(i)For t-deformation $f_w(t)$ and q-deformation of Markov number associated with christoffel ab-word w,
$\Rightarrow$

$$f_w([3]_q/q) = qh_w(q). \tag{3}$$

(ii) There exists one to one correpondence between the set of q-deformation of a Markov triple and t-deformation of the Markov triple.
(iii) For the value q such taht $A_qB_q = B_qA_q$
namely $q = -1$ or $q = e^{\pm\frac{2}{3}\pi\sqrt{-1}}$
$\Rightarrow$
$x^2 + y^2 + z^2 - 4 = xyz$ (Zagier type).

Relation of q-Deformations and t-Deformation

Example

(i) $f_{a^2b}(q^{-1}[3]_q) = q^{-3}[3]_q^3 - q^{-2}[3]_q^2 - 2q^{-1}[3]_q + 1$
$= q^{-3}\{q^6 + 2q^5 + 2q^4 + 3q^3 + 2q^2 + 2q + 1\}$
$= qh_{a^2b}(q)$

(ii) $qh_{a^2bab}(q) =$
$q^{-6}(q^{12}+4q^{11}+9q^{10}+16q^9+23q^8+29q^7+30q^6+29q^5+23q^4+16q^3+9q^2+4q+1)$
$= (q^{-1}[3]_q)^6 - 2(q^{-1}[3]_q)^5 - 2(q^{-1}[3]_q)^4 + 4(q^{-1}[3]_q)^3 + (q^{-1}[3]_q)^2 - (q^{-1}[3]_q) - 1$
$= t^6 - 2t^5 - 2t^4 + 4t^3 + t^2 - t - 1$
$= f_{a^2bab}(t)$.

Future Problems

Classify and charaterize prehomogeneous vector spaces coming from F-polynomials associated to quivers of type A.

Thanks

Thank you for your attention!

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