Some calculations about earthquake maps in the cross ratio coordinates

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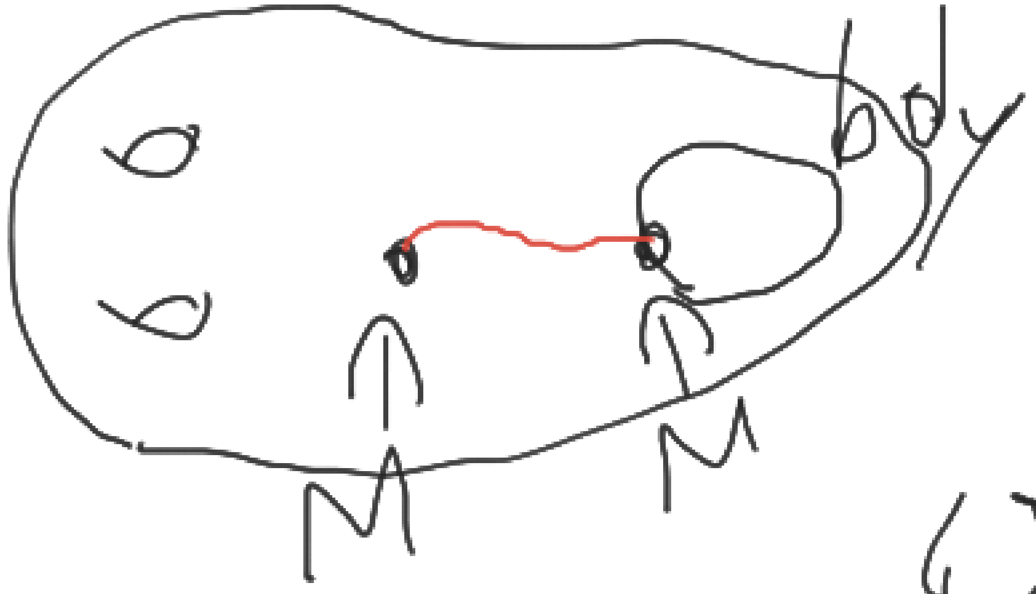
1. The enhanced Teichmüller space
2. The space of measured geodesic laminations with relax signatures
3. The earthquake map
4. The relationship to cluster algebras
5. Calculations

1. The enhanced Teichmüller space.

Σ: oriented cpt. top surf. (with boundaries)

M: $\subset \Sigma$, finite set.



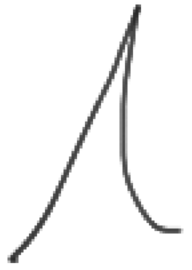
ideal arc connecting points of M

$\triangle$: triangulation from ideal arcs

(Σ, M): marked surf s.t. $\exists \triangle$

$$M_0 := \mathrm{Int}\,\Sigma \cap M, \quad M_\partial := \partial\Sigma \cap M$$

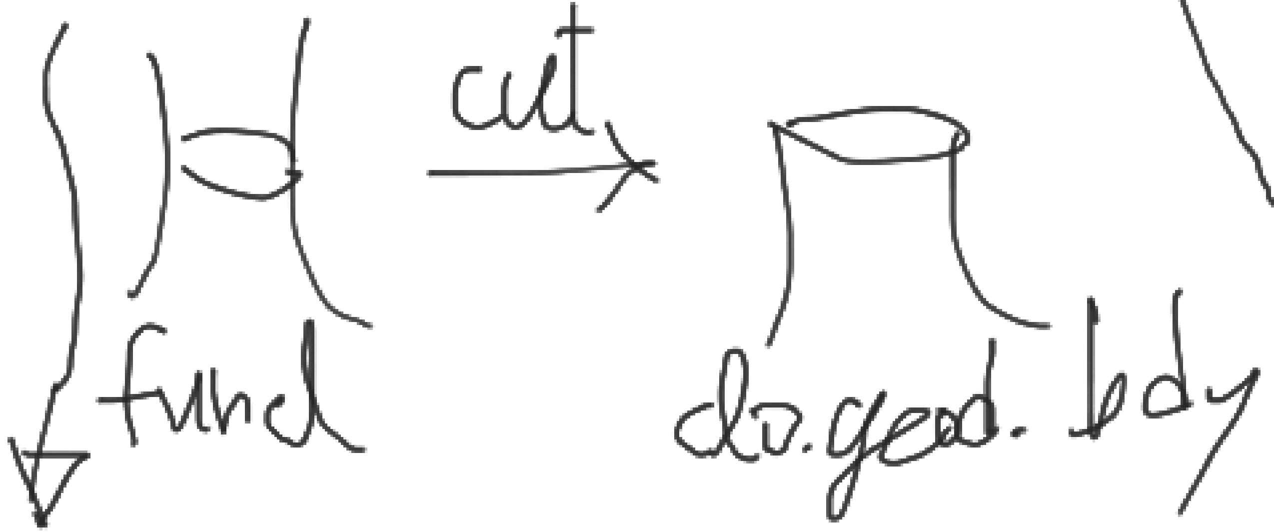
$\mathrm{Hyp}(\Sigma, M) := \{$ hyp metric on Σ

s.t. a point of M_o is  or 

a point of M_∂ is a spike  $\}$

$\Downarrow$

h



Σ^h : hyp. surf.

$f : \mathrm{Int}(\Sigma \setminus M) \to \mathrm{Int}\,\Sigma^h$, ori-pres. homeo is called a signed homeo

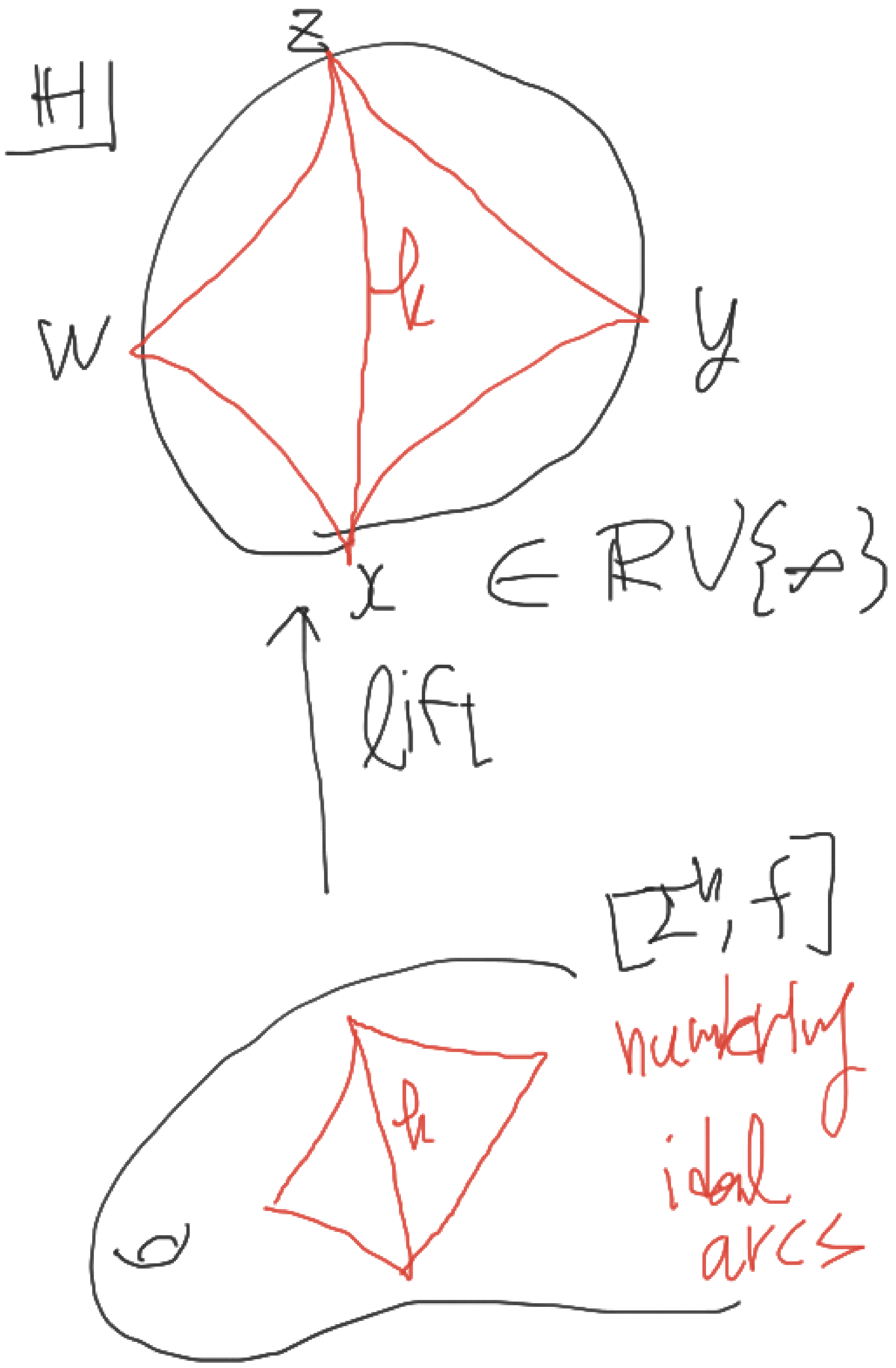
signature of f $\quad (\varepsilon_f)_p := \begin{cases} + \\ 0 \\ - \end{cases}$, $\quad p \in M_0$

$(\Sigma^{h_1}, f_1) \sim (\Sigma^{h_2}, f_2) \overset{\text{def}}{\iff} f_2 \circ f_1^{-1}$ is homotopic to an isometry

& $\varepsilon_{f_1} = \varepsilon_{f_2}$

the enhanced Teichmüller space

$$\hat{\mathcal{T}}(\Sigma, M) := \{ [\Sigma^h, f] \text{ ; equivalent class} \}$$



$X_k^{\triangle}([\Sigma^h, f]) := [x : y : z : w]$

cross ratio

$\in \mathbb{R}_{>0}$

Thm (Fock–Thurston)

n: the number of ideal arcs

$(X_1^{\triangle}, \dots, X_n^{\triangle}) : \hat{\mathcal{T}}(\Sigma, M) \to (\mathbb{R}_{>0})^n$

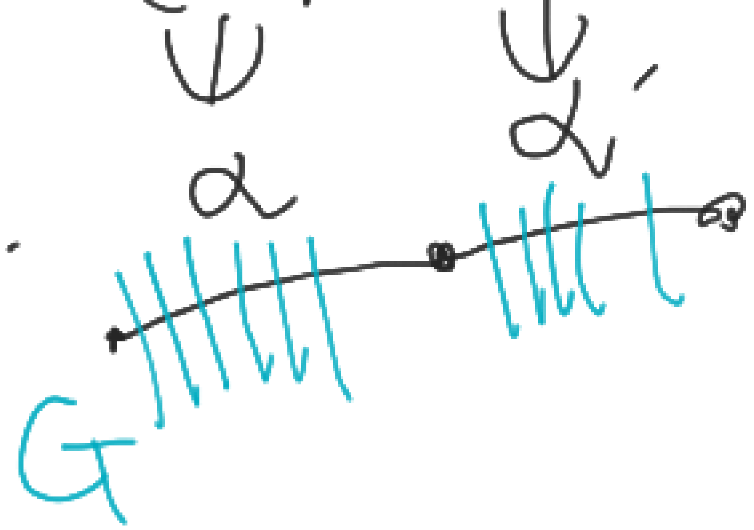
is bijective

We call it the cross ratio coord.

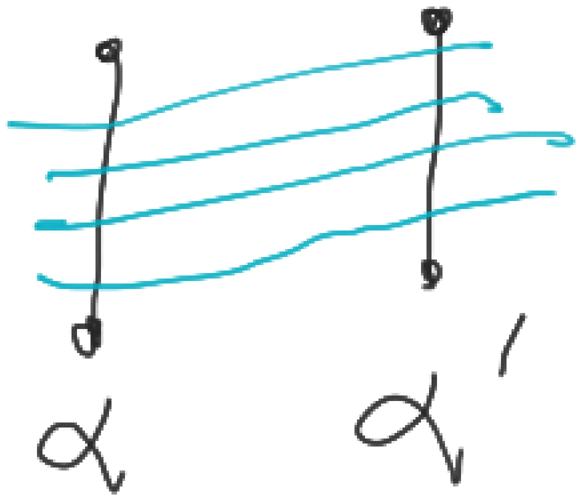
2. For $[\Sigma^n, f] \in \mathcal{T}(\Sigma, M)$, we define a *geodesic lamination* G as a closed subset consisting of a disjoint union of complete simple geodesics.

μ: *measure* along G is $\{\text{cpt arc transverse to } G\} \to \mathbb{R}_{\geq 0}$

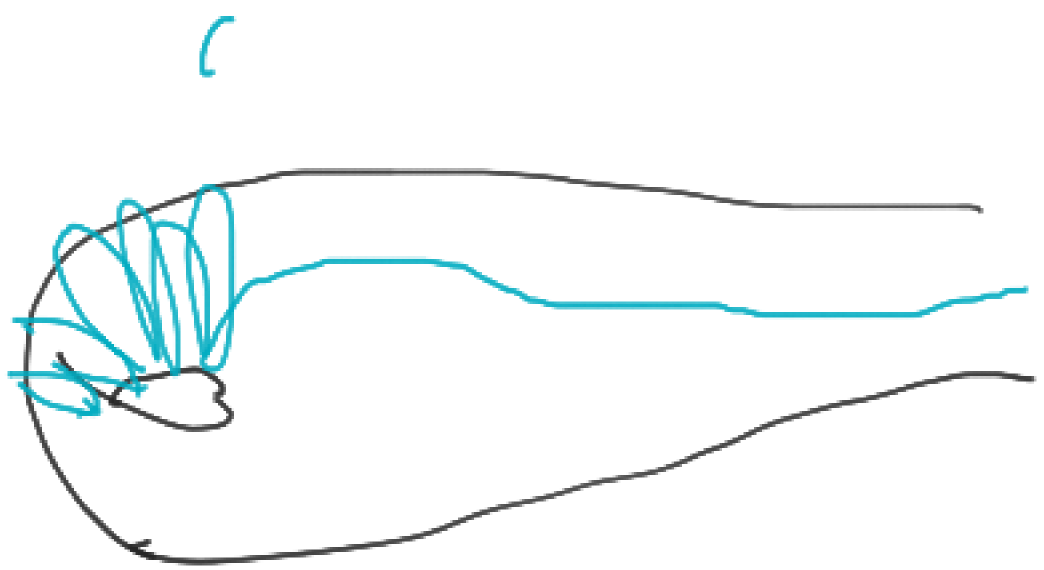
s.t.



$\Rightarrow \mu(\alpha + \alpha') = \mu(\alpha) + \mu(\alpha')$



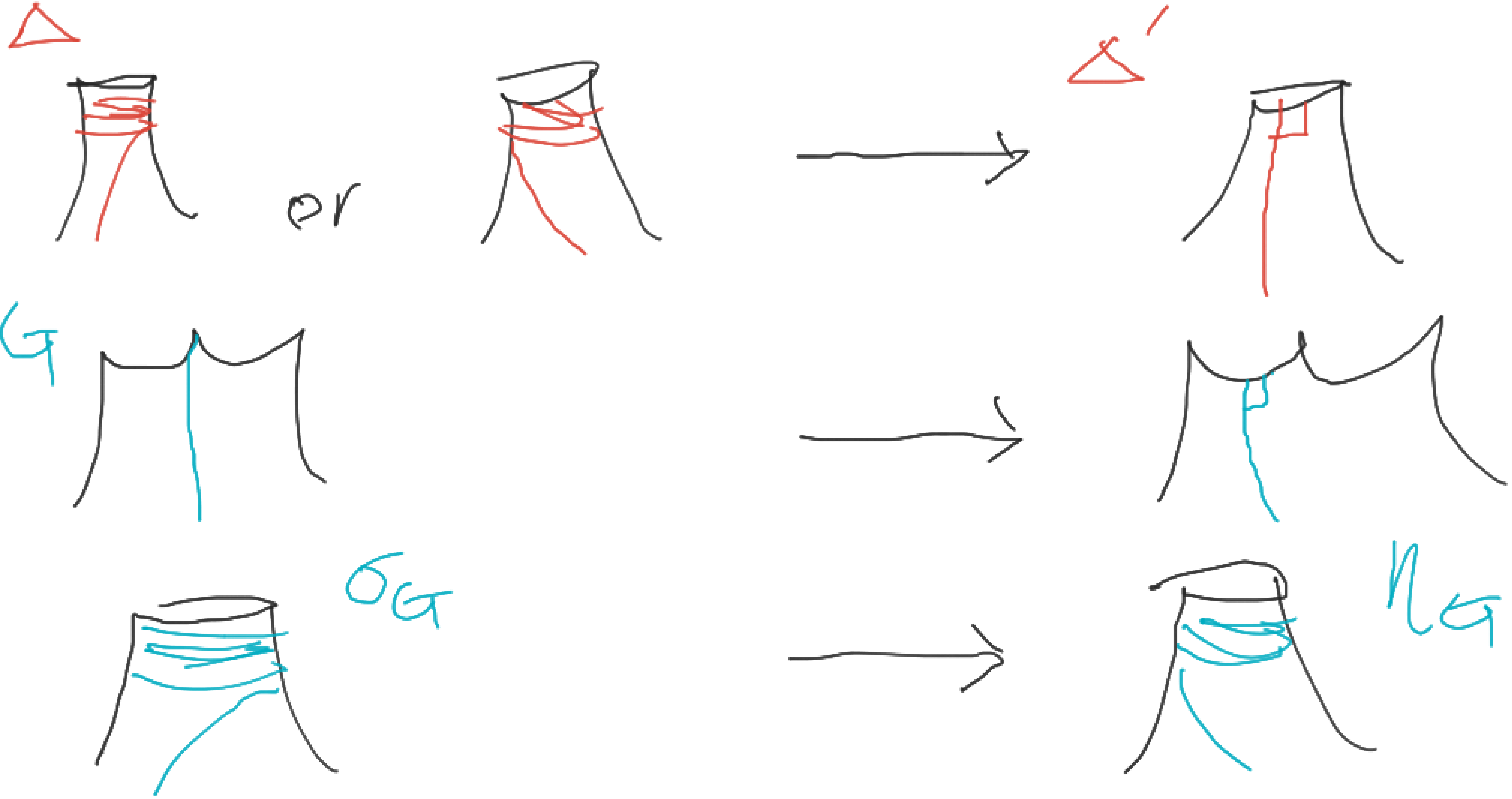
$\Rightarrow \mu(\alpha) = \mu(\alpha')$



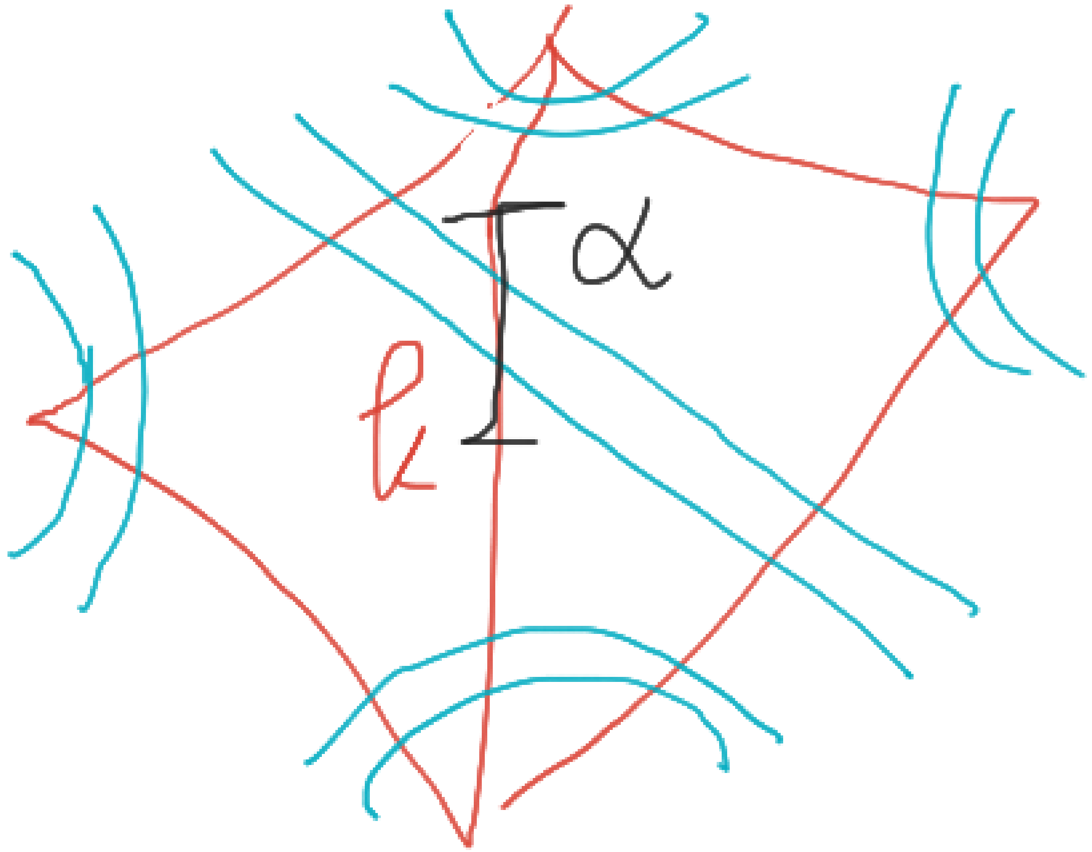
signature of G_f $\quad (\sigma_{G_f})_p := \begin{cases} + & \\ 0 & \text{or} \quad \text{or} \\ - & \end{cases}$ $\quad p \in M_0$

relax signature of G_f $\quad (\eta_{G_f})_p := \begin{cases} (\sigma_{G_f})_p \cdot (\varepsilon_f)_p & p : \text{closed ord. bdy} \\ 0 & \\ + \text{ or } - & \end{cases}$ $\quad p \in M_0$

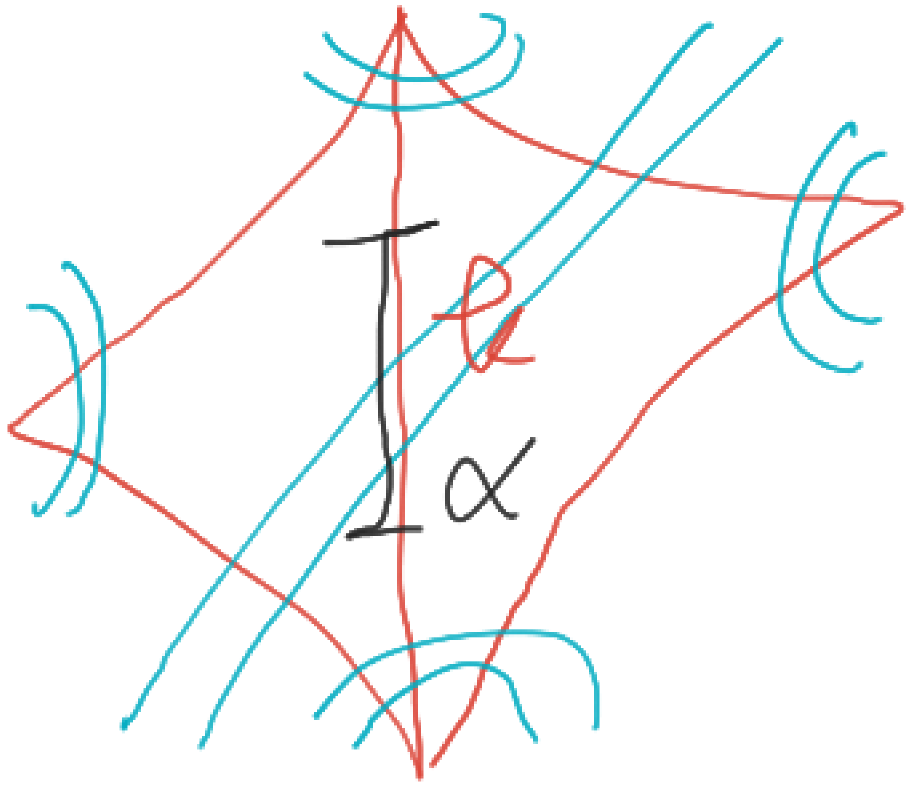
$$\widehat{ML}([\Sigma^h, f]) := \{(G_f, \mu, \eta)\}$$



Δ
or
Δ'
G
σG
nG



$s_{\ell_k} := +$



$s_{\ell_k} := -$

$\triangle$

G

$$x^{\triangle}_{k}(G,\mu,\eta) := s_{\ell_k}\,\mu(\alpha) \in \mathbb{R}^{\mathrm{trop}}$$

$$\begin{pmatrix} +\,,\ \times \\ \text{in } \mathbb{R} \end{pmatrix} \to \begin{pmatrix} \min,\ + \\ \text{in } \mathbb{R}^{\mathrm{trop}} \end{pmatrix}$$

<u>Thm</u> (Fock-Goncharov, Benedetti-Bonsante)

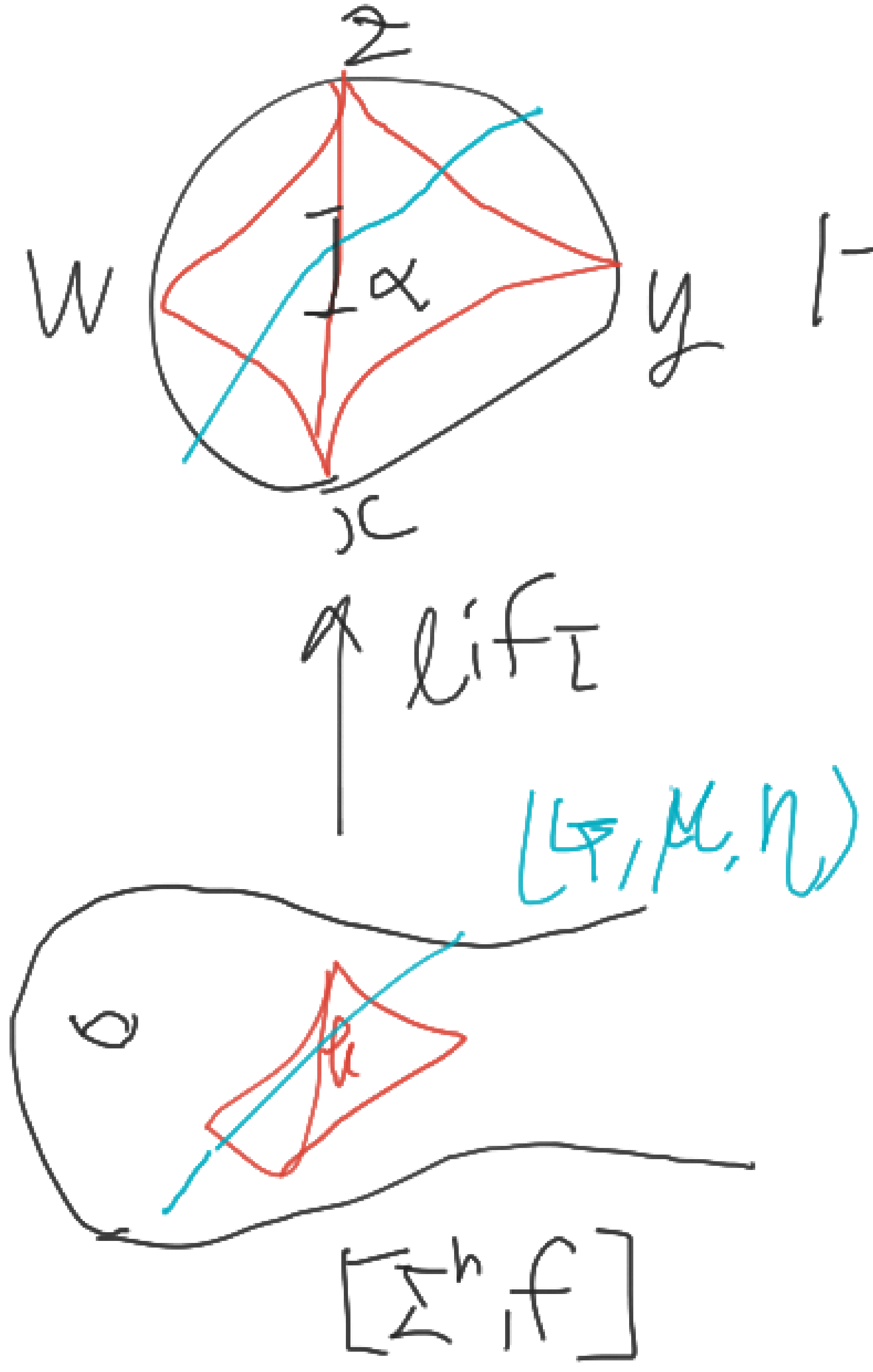
$(x^{\triangle}_1, \dots, x^{\triangle}_n) : \widehat{\mathcal{ML}}([\Sigma^n, F]) \to (\mathbb{R}^{\mathrm{trop}})^n$ is bijective

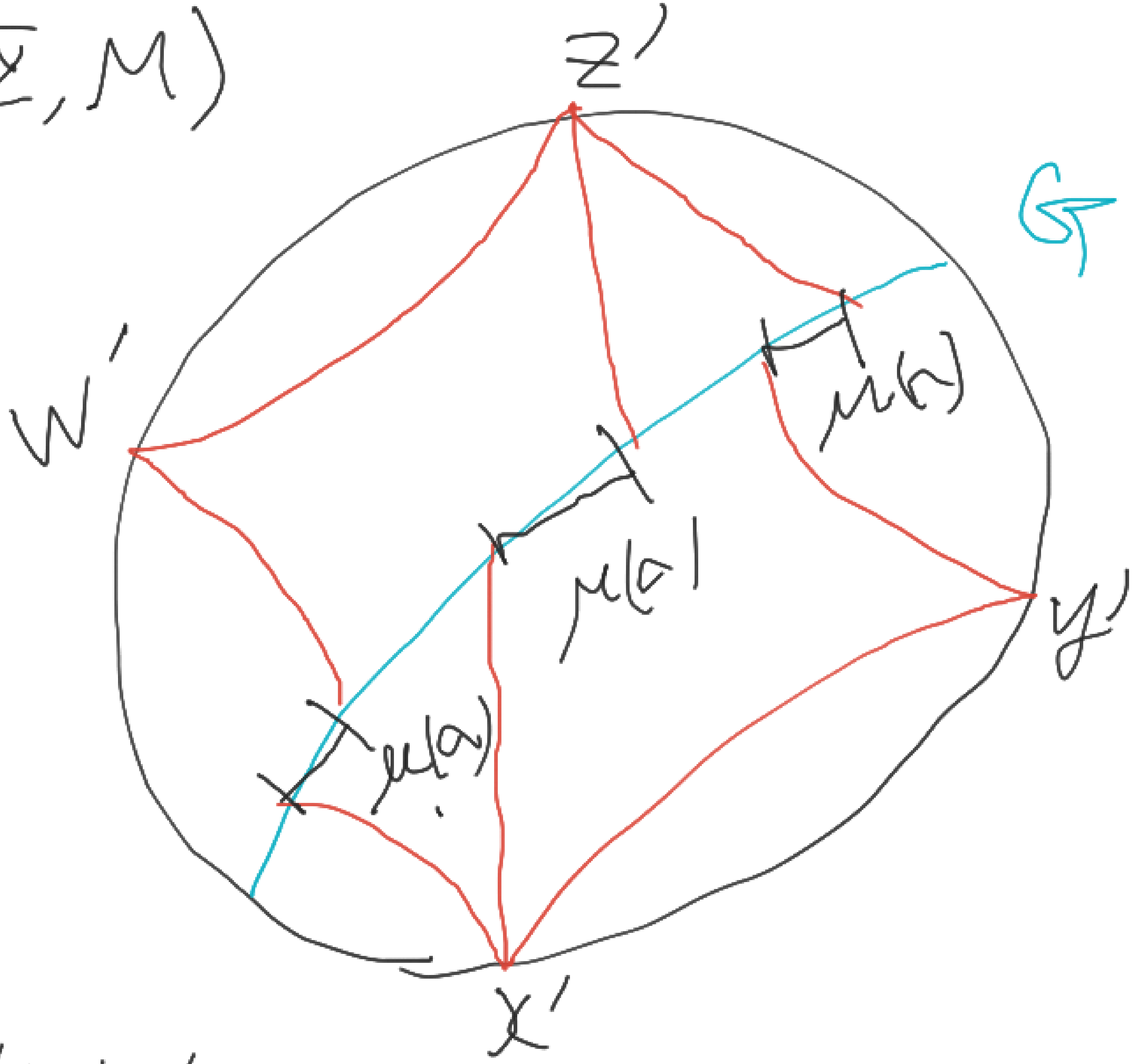
<u>shear coord.</u>

Thm (B-B)

We consider the trivial bdl $\pi : \widehat{\mathcal{ML}}(\Sigma, M) \to \widehat{\mathcal{T}}(\Sigma, M)$ whose fiber at $[\Sigma^h, f]$ is $\widehat{\mathcal{ML}}([\Sigma^h, f])$

3. $E : \widehat{\mathcal{ML}}(\Sigma, M) \to \mathcal{T}(\Sigma, M)$





s.t. $X^i_\Delta(E([\Sigma^h, f], (G, \mu, \eta))) := [x'; y'; z'; w']$

<u>earthquake map</u>

Thm (earthquake theorem) (Thurston, Bonsante-Krasnov-Schlenker)

(i) If we fix $[\Sigma^h, f]$,

$$E([\Sigma^h, f], \cdot) : \hat{\mathcal{ML}}([\Sigma^h, f]) \to \hat{\mathcal{T}}(\Sigma, M)$$

is homeo

(ii) $$E(\phi(\cdot), \phi(\cdot)) = \phi(E(\cdot, \cdot))$$

$\forall \phi$; element of the mapping class group

4.

Def $\varepsilon \in \mathbb{Z}^{n\times n}$ skew-symmetrizable,

$(X_1, \dots, X_n)$, $(x_1, \dots, x_n)$ = indeterminates

For $k = 1, \dots, n$, $\mu_k : (\varepsilon, (X_i)_i, (x_i)_i) \longmapsto$

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} + \dfrac{|\varepsilon_{ik}|\,\varepsilon_{kj} + \varepsilon_{ik}\,|\varepsilon_{kj}|}{2} & \text{otherwise,} \end{cases}$$

$$X'_i = \begin{cases} X_k^{-1} & \text{if } i = k, \\ X_i\left(1 + X_k^{-\operatorname{sgn}(\varepsilon_{ik})}\right)^{-\varepsilon_{ik}} & \text{if } i \neq k. \end{cases}$$

$$x'_i = \begin{cases} -x_k & \text{if } i = k, \\ x_i - \varepsilon_{ik}\min\{0, -\operatorname{sgn}(\varepsilon_{ik})x_k\} & \text{if } i \neq k, \end{cases}$$

mutation

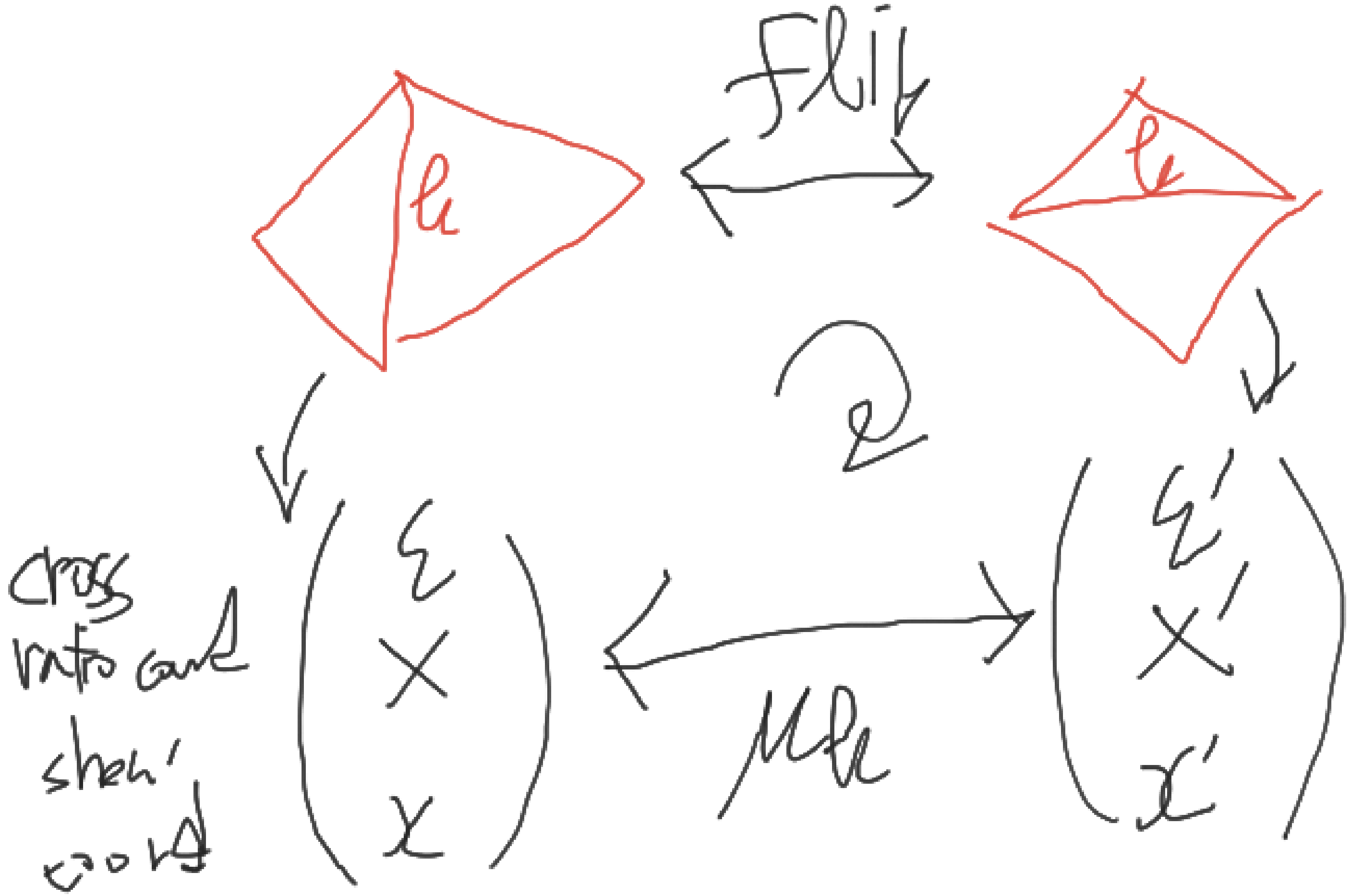
mutation class $\mathcal{S}_\varepsilon := \{ (\varepsilon', X', x')$ generated by performing mutations $\}$

$\mathcal{S}_\varepsilon$ is of finite type $\overset{\text{def}}{\iff} \# \mathcal{S}_\varepsilon < \infty$

$\mathcal{S}_\varepsilon$ is of finite mut. type $\overset{\text{def}}{\iff} \# \{ \varepsilon' \mid (\varepsilon', X', x') \text{ in } \mathcal{S}_\varepsilon \} < \infty$

Def $\mathcal{X}$-manifold $\mathcal{X}_\Sigma(\mathbb{R}_{>0}) := \mathbb{R}_{>0}{}^n$ with $(x_i)_i$ in Σ_4 as local coordinates

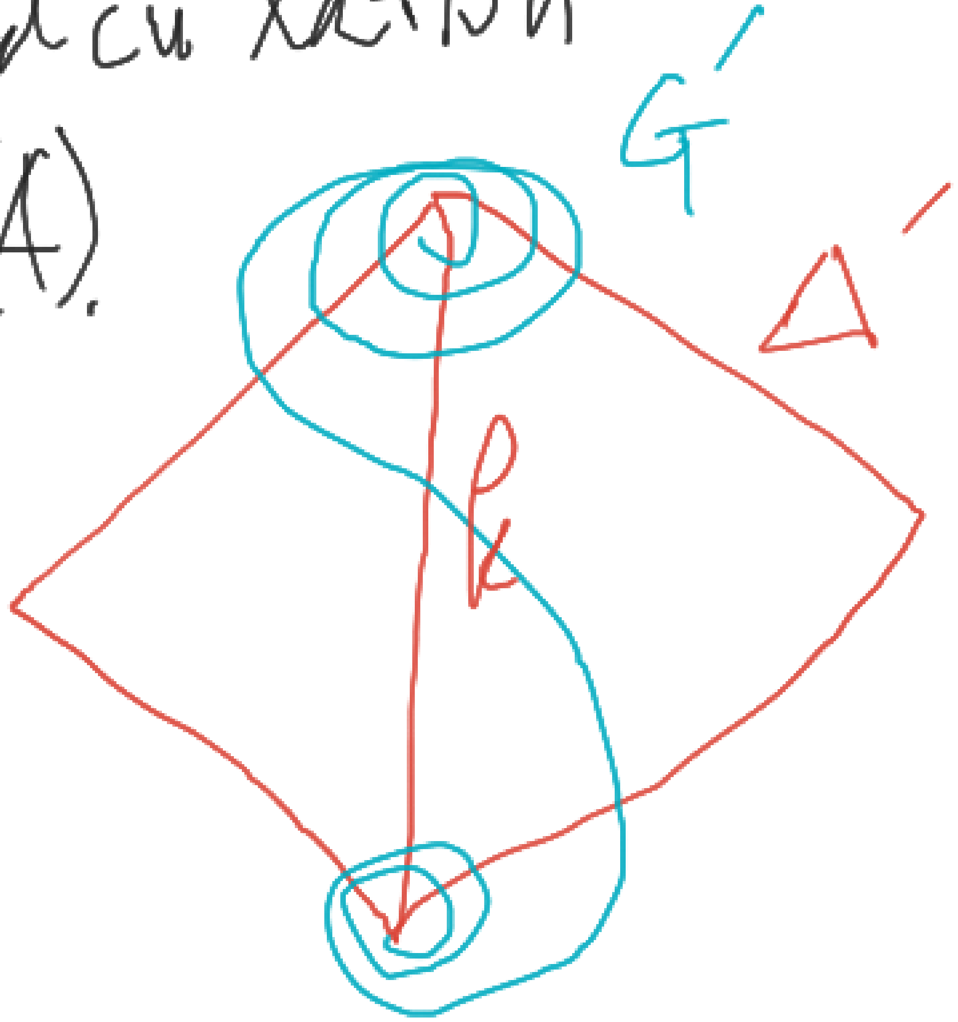
trop. $\mathcal{X}$-manifold $\mathcal{X}_\Sigma(\mathbb{R}^{\mathrm{trop}}) := (\mathbb{R}^{\mathrm{trop}})^n$ with $(x_i)_i$



5. Calculation

Prop (A).

(i)



G_T; good lam. along an ideal arc

$$\hat{X}^{\Delta}_{\ell}\left(E\left([\Sigma^{n}, f], (G_T, \mu, \eta)\right)\right)$$

$$= e^{\chi^{\hat{\Delta}}_{\ell}(G, \mu, \eta)} \cdot \hat{X}^{\Delta}_{\ell}([\Sigma^{n}, f])$$

$$\hat{X}^{\Delta}_{j}\left(\quad '' \quad \right)$$

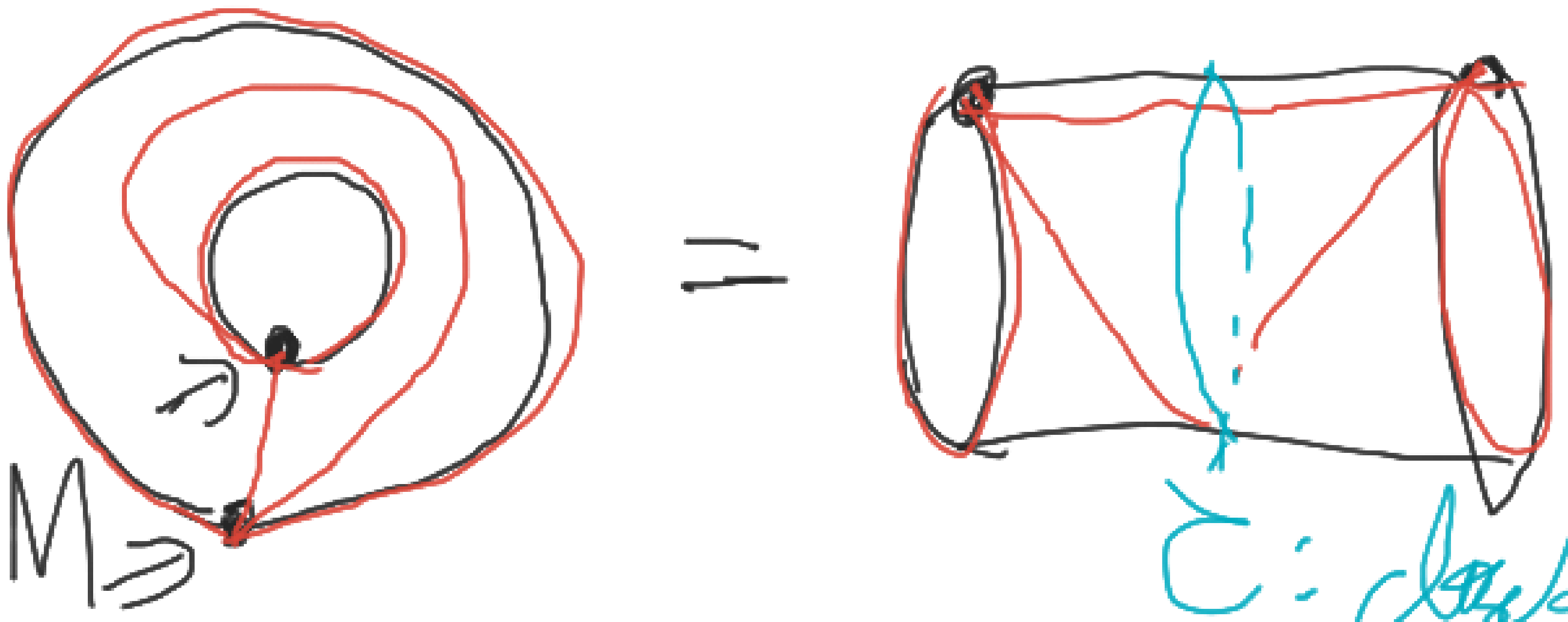
$$= \hat{X}^{\Delta}_{j}([\Sigma^{n}, f])$$

$(j \neq \ell)$

Fock-Goncharov fan C^{+}_{Δ}

Prop (ii)

$\Sigma_{1,1}$



$L(f_2) :=$ the length of c

$t \in \mathbb{R}_{>0}$

c : closed geod bdry

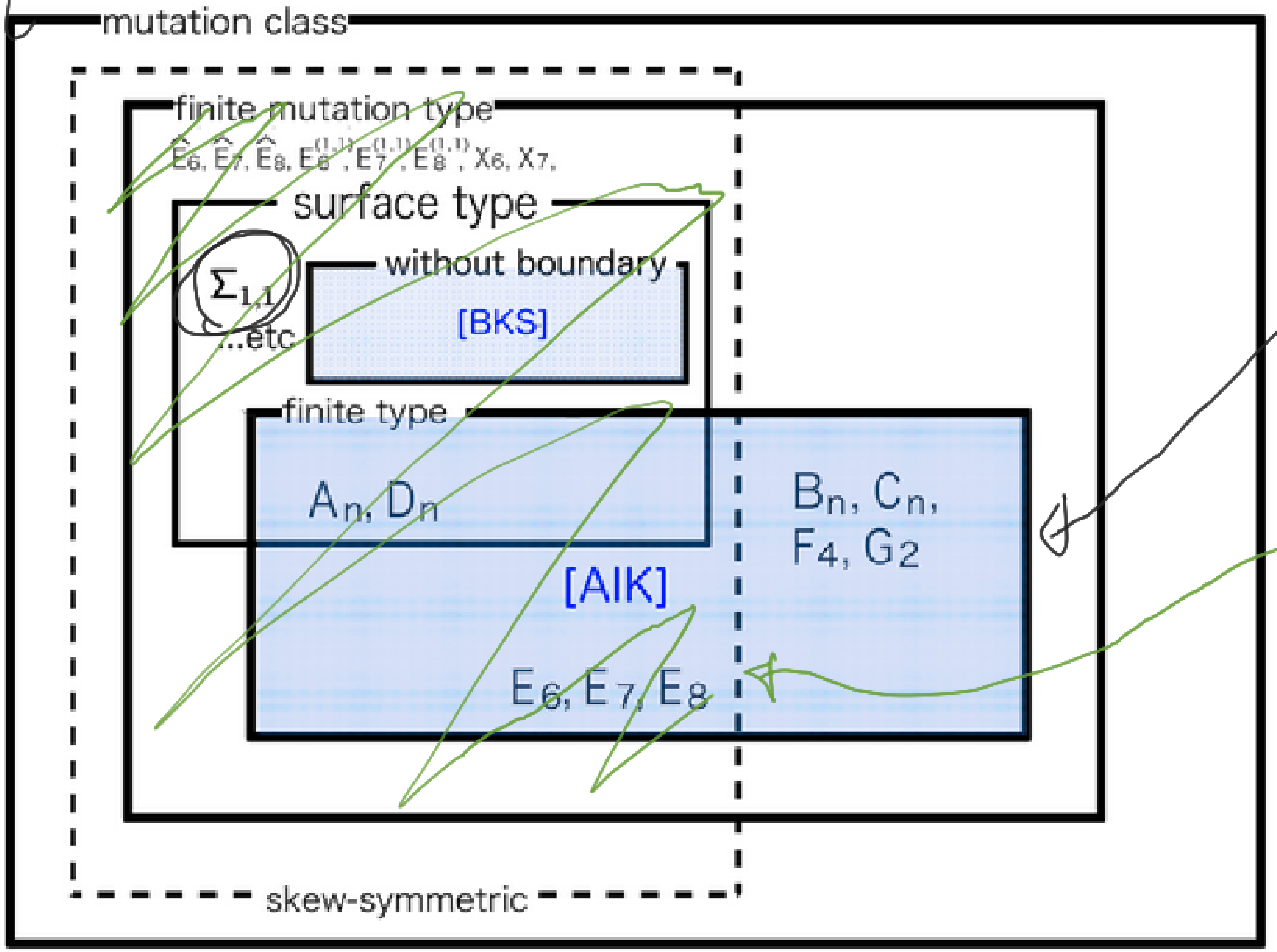
earthquake map along c = the Fenchel–Nielsen twist along c

$$
\begin{aligned}
X_1(E_{tC}(g)) =& X_1 \cdot \frac{2\left(X_1 X_2 \cosh(L(f_2)) - 2\sqrt{X_1 X_2}\cosh\left(\frac{L(f_2)}{2}\right) + X_1 + 1\right) e^{t\,L(f_2)}}{\left(\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - X_1 - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - X_1 - 1\right)\right)^2},\\
X_2(E_{tC}(g)) =& X_2 \cdot \frac{\left(\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - 1\right)\right)^2}{2\left(X_1 X_2 \cosh(L(f_2)) - 2\sqrt{X_1 X_2}\cosh\left(\frac{L(f_2)}{2}\right) + X_1 + 1\right) e^{t\,L(f_2)}},\\
X_3(E_{tC}(g)) =& X_3 \cdot \frac{\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - X_1 - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - X_1 - 1\right)}{\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - 1\right)},\\
X_4(E_{tC}(g)) =& X_4 \cdot \frac{\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - X_1 - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - X_1 - 1\right)}{\left(\sqrt{X_1 X_2}\, e^{-\frac{L(f_2)}{2}} - 1\right) e^{t\,L(f_2)} - \left(\sqrt{X_1 X_2}\, e^{\frac{L(f_2)}{2}} - 1\right)}.
\end{aligned}
$$

$t=1 \rightsquigarrow E_{tC} = E_C$ is Dehn twist along C

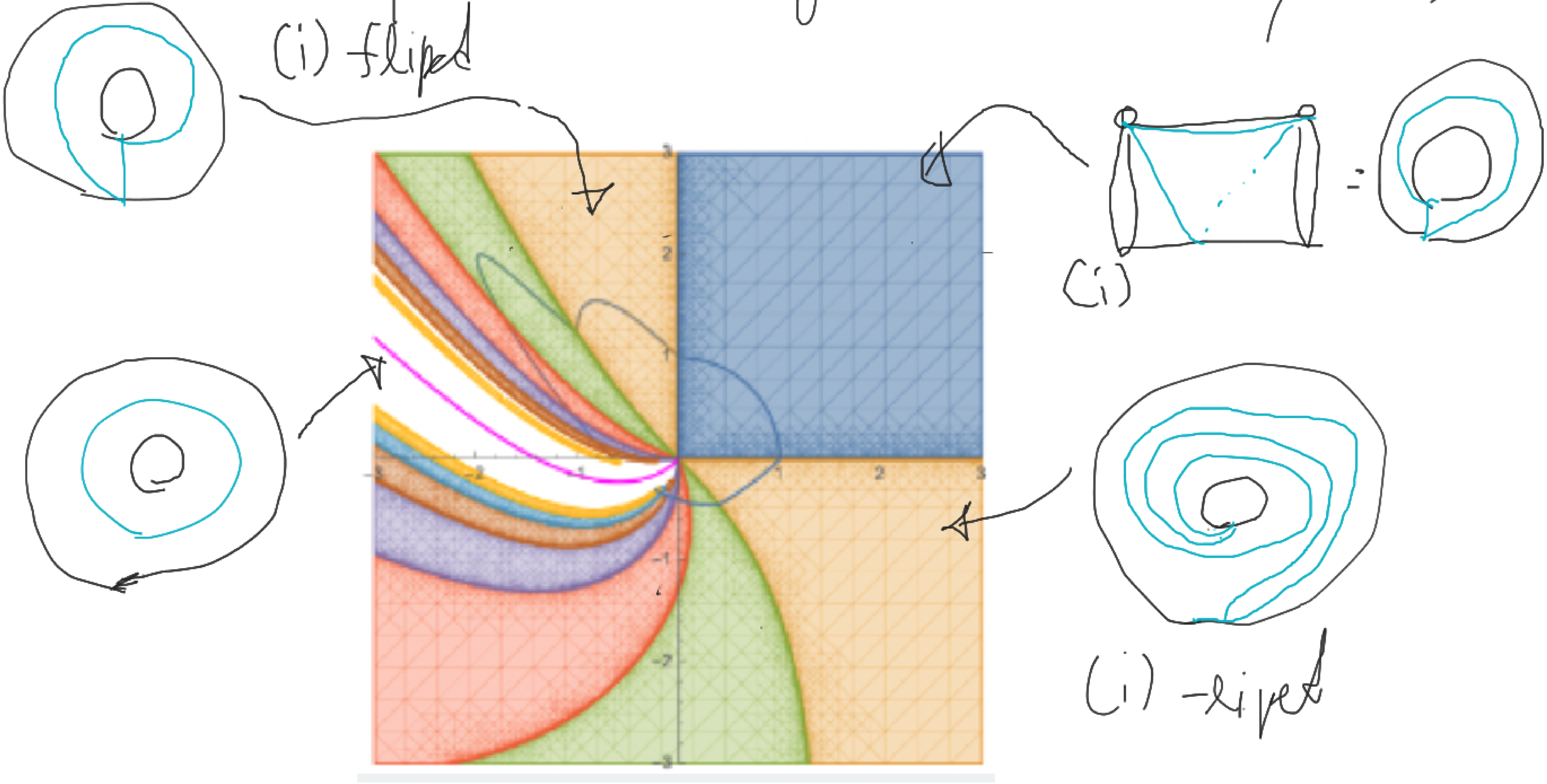
We defined an earthquake map (cluster earthquake map) and proved the earthquake theorem for mut. class of finite type.

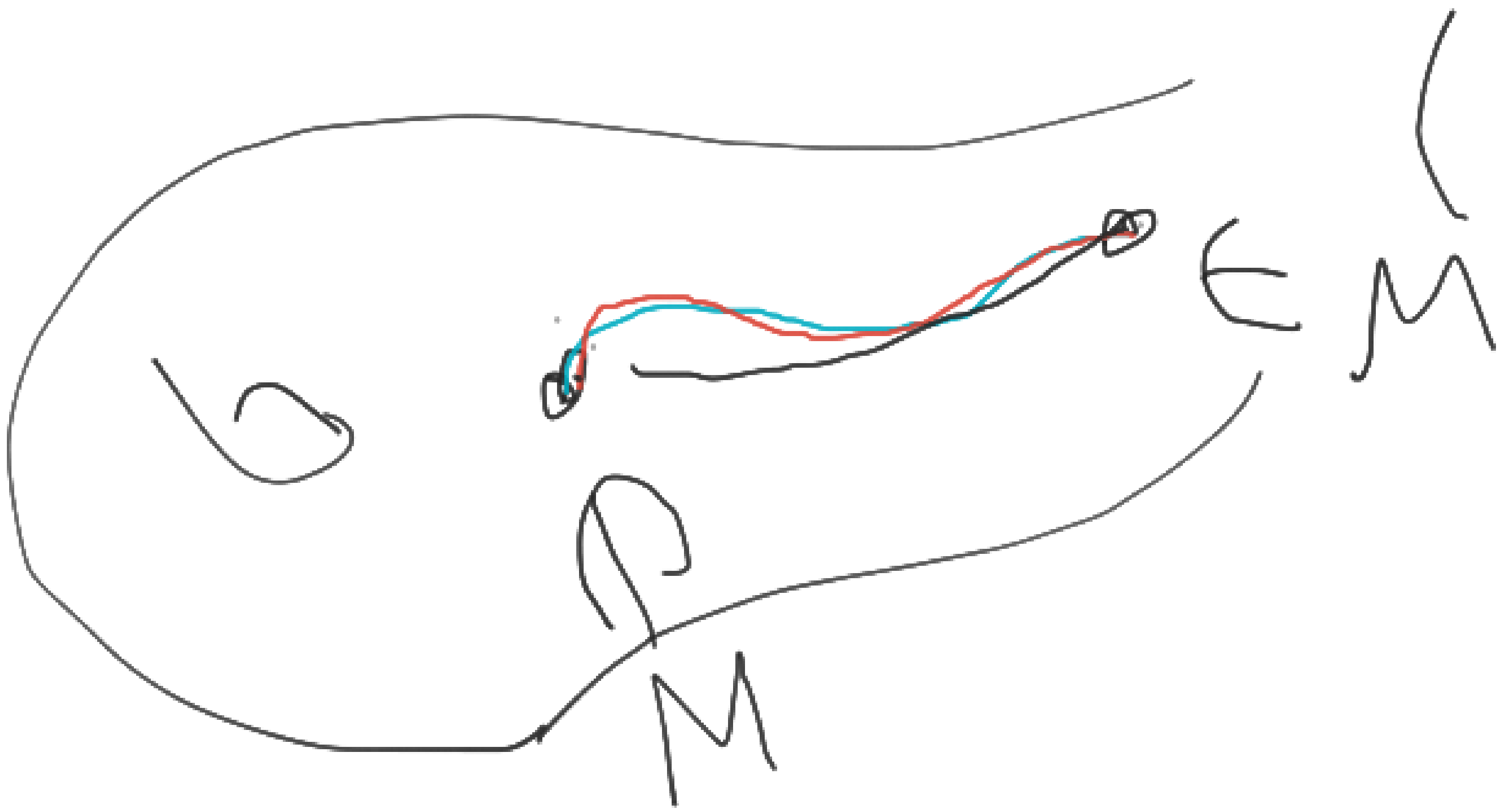
(A-Ishibashi-Kano)



the support of FG fan is dense in $\mathcal{X}_{\mathrm{uf}}(\mathbb{R}^{\mathrm{trop}})$
(Yurikusa)

We want to prove the earthquake theorem. by (ii)





We can take initial tri Δ by mutations and transform Δ along γ