Cluster algebra and q-Painlevé equation: higher order generalization and degeneration structure

Takao Suzuki

Kindai University

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Painlevé equation

Problem

Define a new transcendental function as a solution of a differential equation in the complex domain.

The solution should be controlled by the differential equation. Hence we require that the differential equation has no movable branch point (Painlevé property).

Example

The differential equation

$$ny^{n-1}\frac{dy}{dt} = 1 \quad (n \in \mathbb{N}),$$

has a movable branch point. In fact, it has a solution $y = (t-c)^{1/n}$.

All of 1st order meromorphic ordinary differential equations with the Painlevé property were classified by Fuchs and Poincaré in 19th century.

Fact

All differential equations of the form $R(t, y, y') = 0$ with the Painlevé property are reduced to the following 3 types of equations:

- *Solvable by quadratures*
- $y' = a(t)y^2 + b(t)y + c(t)$
- $y' = 4y^3 - g_2 y - g_3 \ (g_2, g_3 \in \mathbb{C})$

At the beginning of 20th century, Painlevé and Gambier tried to classify 2nd order ODEs.

Fact

All differential equations of the form $y'' = R(t, y, y')$ with the Painlevé property are reduced to the following 4 types of equations:

- *Solvable by quadratures*
- *Linear differential equations*
- $y'' = 6y^2 - g_2 \ (g_2 \in \mathbb{C})$
- *Painlevé equations* $P_{\mathrm{I}}, \dots, P_{\mathrm{VI}}$

Discrete Painlevé equation

In 1990's, Grammaticos and his collaborators proposed the singularity confinement as a discrete analogue of the Painlevé property.

Example

Consider a difference equation

$$x_{n+1} + x_{n-1} = \frac{a x_n}{1 - x_n^2}, \quad x_0 = p, \quad x_1 = 1 + \varepsilon.$$

Then we obtain

$$x_2 = -\frac{a}{2\varepsilon} - \frac{a + 4p}{4} + O(\varepsilon), \quad x_3 = -1 + \varepsilon + O(\varepsilon^2), \quad x_4 = -p + O(\varepsilon).$$

Taking a limit $\varepsilon \to 0$, *we can find that a singularity appears at* x_2 *and disappears at* x_4.

That became a trigger for the discovery of various discrete Painlevé equations.

Problem

How many 2nd order discrete Painlevé equations exist?

An answer to this problem was given as follows.

Fact ([Sakai 01])

The 2nd order continuous/discrete Painlevé equations are classified by the geometry of rational surfaces as follows:

	Symmetry/Surface type					
elliptic	E_8/A_0					
multiplicative	E_8/A_0	E_7/A_1	E_6/A_2	D_5/A_3	A_4/A_4	E_3/A_5
	E_2/A_6	$\underset{\lvert\alpha\rvert^2=8}{A_1}/A_7$	A_1/A_7	A_0/A_8		
additive	E_8/A_0	E_7/A_1	E_6/A_2	D_4/D_4	A_3/D_5	$2A_1/D_6$
	A_2/E_6	$\underset{\lvert\alpha\rvert^2=4}{A_1}/D_7$	A_1/E_7	A_0/D_8	A_0/E_8	

Here the symbols E_3 and E_2 stand for $A_2 + A_1$ and $A_1 + \underset{\lvert\alpha\rvert^2=14}{A_1}$ respectively.
Blue-colored types correspond to the continuous Painlevé equations.

The Painlevé VI equation is described as the Hamiltonian system

$$t(t-1)\frac{dq}{dt}=\frac{\partial H_{\rm VI}}{\partial p},\quad t(t-1)\frac{dp}{dt}=-\frac{\partial H_{\rm VI}}{\partial q},$$

$$H_{\rm VI}[\kappa_0,\kappa_1,\kappa_t,\kappa;q,p]=q(q-1)(q-t)p\left(p-\frac{\kappa_0}{q}-\frac{\kappa_1}{q-1}-\frac{\kappa_t-1}{q-t}\right)+\kappa q_i.$$

In 1996, Jimbo and Sakai proposed a q-analogue of the Painlevé VI equation, which is described as

$$\frac{f\overline{f}}{\alpha_3\alpha_4}=\frac{(\overline{g}-t\beta_1)(\overline{g}-t\beta_2)}{(\overline{g}-\beta_3)(\overline{g}-\beta_4)},\quad \frac{g\overline{g}}{\beta_3\beta_4}=\frac{(f-t\alpha_1)(f-t\alpha_2)}{(f-\alpha_3)(f-\alpha_4)},\quad \overline{t}=qt,$$

where $\alpha_1\alpha_2\beta_3\beta_4=q\beta_1\beta_2\alpha_3\alpha_4$.

	Symmetry/Surface type					
elliptic	E_8/A_0					
multiplicative	E_8/A_0	E_7/A_1	E_6/A_2	D_5/A_3	A_4/A_4	E_3/A_5
	E_2/A_6	$A_1{}_{\vert\alpha\vert^2=8}/A_7$	A_1/A_7	A_0/A_8		
additive	E_8/A_0	E_7/A_1	E_6/A_2	D_4/D_4	A_3/D_5	$2A_1/D_6$
	A_2/E_6	$A_1{}_{\vert\alpha\vert^2=4}/D_7$	A_1/E_7	A_0/D_8	A_0/E_8	

Cluster mutation

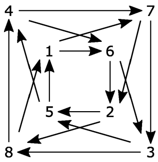
In this talk we use the following notations:

- Q is a quiver without loops and 2-cycles.
- I is a vertex set of Q.
- $\Lambda = (\lambda_{i,j})_{i,j \in I}$ is a skew-symmetric matrix corresponding to Q as follows:
  1. If there are k arrows from i to j, then we set $\lambda_{i,j} = k$ and $\lambda_{j,i} = -k$.
  2. If there is no arrow between i and j, then we set $\lambda_{i,j} = \lambda_{j,i} = 0$.
- y_i is a coefficients corresponding to $i \in I$.
- μ_i is a mutation at $i \in I$, which acts on the coefficients as

$$\mu_i(y_j) = \begin{cases} y_i^{-1} & (j = i) \\ y_j \left(1 + y_i^{-1}\right)^{\lambda_{ij}} & (\lambda_{ij} > 0) \\ y_j (1 + y_i)^{-\lambda_{ij}} & (\lambda_{ij} < 0) \\ y_j & (j \neq i,\ \lambda_{ij} = 0) \end{cases}.$$

- (i, j) is a permutation of $i, j \in I$.

Consider the following quiver Q:



$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact ([Okubo 15])

In the above quiver Q we set

$$T = (5,7)(6,8)\mu_5\mu_6\mu_7\mu_8(1,3)(2,4)\mu_1\mu_2\mu_3\mu_4.$$

Then Q is invariant under the action of T. Moreover, the action T on the coefficients $y_1, \dots, y_8$ provides the q-Painlevé VI equation.

Fact ([Okubo 15], [Bershtein-Gavrylenko-Marshakov 18])

All of the q-Painlevé equations are derived from mutation-periodic quivers.

Generalized (q-)Painlevé VI equation

The Painlevé VI equation is obtained as the isomonodromy deformation of the Fuchsian equation. We propose higher order generalizations from this point of view.

Fact ([Oshima 08])

Irreducible Fuchsian equations with a fixed number of accessory parameters can be reduced to finite types of systems by the Katz's two operations (addition and middle convolution).

Fact ([Haraoka-Filipuk 07])

The isomonodromy deformation equation of the Fuchsian equation is invariant under the Katz's two operations.

Thanks to them, we have a good classification theory of isomonodromy deformation equations of Fuchsian equations.

We list 4 types of representative isomonodromy deformation equations below:

- Garnier system
  - Isomonodromy deformation [Garnier 1912]
- Sasano system
  - Okamoto initial value space and affine Weyl group symmetry [Sasano 07]
  - Similarity reduction of the integrable hierarchy [Fuji-S 08]
  - Isomonodromy deformation [Sakai 10][Fuji-Inoue-Shinomiya-S 13]
- FST system
  - Similarity reduction of the integrable hierarchy [Fuji-S 09][S 13][Tsuda 14]
  - Isomonodromy deformation [Sakai 10]
- Matrix Painlevé system
  - Isomonodromy deformation [Sakai 10][Kawakami 15]

Their q-analogues are proposed recently (but there is no classification theory):

- q-Garnier system
  - q-Analogue of the isomonodromy deformation [Sakai 05]
  - Pade method [Nagao-Yamada 18]
  - Birational representation of the extended affine Weyl group [Okubo-S 19]
- q-Sasano system
  - Birational representation of the extended affine Weyl group [Masuda 15]
- q-FST system
  - Similarity reduction of the discrete integrable hierarchy [S 15][S 17]
  - Pade method [Nagao-Yamada 18]
  - Birational representation of the extended affine Weyl group [Okubo-S 19]

Weyl group

Let J be an index set and $A = [a_{ij}]_{i,j \in J}$ the generalized Cartan matrix, namely

$$a_{ii} = 2, \quad a_{ij} \in \mathbb{Z}_{\leq 0}, \quad a_{ij} = 0 \;\Leftrightarrow\; a_{ji} = 0 \quad (i \neq j).$$

We define the Weyl group $W(A)$ associated with the generalized Cartan matrix A by the generators s_i $(i \in J)$ and the fundamental relations

$$r_i^2 = 1, \quad (r_i r_j)^{m_{ij}} = 1 \quad (i \neq j).$$

Here m_{ij} is defined by the following table:

$a_{ij}a_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

Example

Let $J = \{1, 2\}$ and

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Then $W(A)$ is the Weyl group of type A_2 with the generators r_1, r_2 and the fundamental relations

$$r_1^2 = r_2^2 = 1, \quad r_1 r_2 r_1 = r_2 r_1 r_2$$

Affine Weyl group of type $A^{(1)}_{N-1}$

Let $J=\mathbb{Z}_N(=\mathbb{Z}/N\mathbb{Z})$. We set

$$A=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

for $N=2$ and

$$A=\begin{bmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & O & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & O & & & 2 & -1 \\ -1 & & & & -1 & 2 \end{bmatrix}$$

for $N\geq 3$. Then $W(A)$ is the affine Weyl group of type $A^{(1)}_{N-1}$ with the generators r_i $(i\in\mathbb{Z}_N)$ and the fundamental relations

$$r_0^2=r_1^2=1,$$

for $N=2$ and

$$r_i^2=1,\quad r_ir_{i+1}r_i=r_{i+1}r_ir_{i+1}\quad (a_{ij}=-1),\quad r_ir_j=r_jr_i\quad (a_{ij}=0)$$

for $N\geq 3$. We denote the affine Weyl group of type $A^{(1)}_{N-1}$ by $W(A^{(1)}_{N-1})$.

Let $\varepsilon_1, \dots, \varepsilon_N, \delta$ be a set of linearly independent vectors. We define vectors called simple roots α_i $(i \in \mathbb{Z}_N)$ by

$$\alpha_0 = \delta - \varepsilon_1 + \varepsilon_N, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i \neq 0).$$

Note that δ is called a null root satisfying

$$\delta = \alpha_0 + \ldots + \alpha_{N-1}.$$

Then a free module $\mathbb{Z}\alpha_0 \oplus \ldots \oplus \mathbb{Z}\alpha_{N-1}$ is called a root lattice, on which $W(A_{N-1}^{(1)})$ can be realized.

Fact

Let r_i $(i \in \mathbb{Z}_N)$ be transformations on the root lattice called simple reflections defined by

$$r_0(\varepsilon_1) = \varepsilon_n + \delta, \quad r_0(\varepsilon_n) = \varepsilon_1 - \delta, \quad r_0(\varepsilon_j) = \varepsilon_j \quad (j \neq 1, n),$$
$$r_i(\varepsilon_i) = \varepsilon_{i+1}, \quad r_i(\varepsilon_{i+1}) = \varepsilon_i, \quad r_i(\varepsilon_j) = \varepsilon_j \quad (i \neq 0,\ j \neq i, i+1).$$

Then they act on the simple roots as

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i,$$

and satisfy the fundamental relations for $W(A_{N-1}^{(1)})$.

Let π be a transformation on the root lattice called a Dynkin diagram automorphism defined by

$$\pi(\varepsilon_i) = \varepsilon_{i+1} \quad (i \neq N), \quad \pi(\varepsilon_N) = \varepsilon_1 - \delta.$$

It acts on the simple roots as

$$\pi(\alpha_i) = \alpha_{i+1}.$$

We define transformations called translations T_i $(i \in \mathbb{Z}_n)$ by

$$T_0 = \pi r_{n-1} \dots r_1, \quad T_{n-1} = r_{n-1} \dots r_1 \pi, \quad T_i = r_i \dots r_1 \pi r_{n-1} \dots r_{i+1} \quad (i \neq 0, n-1).$$

They act on the simple roots as

$$T_i(\alpha_i) = \alpha_i + \delta, \quad T_i(\alpha_{i+1}) = \alpha_{i+1} - \delta, \quad T_i(\alpha_j) = \alpha_j \quad (j \neq i, i+1).$$

Fact

The fundamental relations

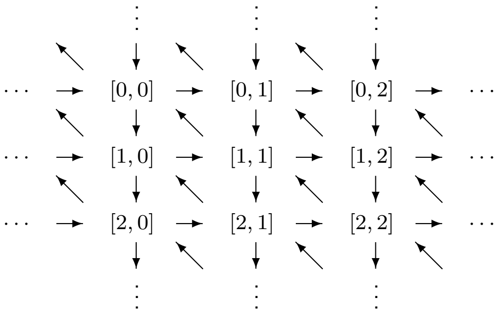
$$\pi^n = 1, \quad \pi r_{i-1} = r_i \pi, \quad T_i T_j = T_j T_i, \quad T_0 \dots T_{n-1} = 1,$$
$$r_i T_{i-1} = T_i r_i, \quad r_i T_i = T_{i-1} r_i, \quad r_i T_j = T_j r_i \quad (j \neq i-1, i), \quad \pi T_{i-1} = T_i \pi,$$

are satisfied. Hence $W(A^{(1)}_{n-1})$ *can be extended by a semi-direct product of groups as*

$$\langle r_0, \dots, r_{n-1}, \pi \rangle = W(A^{(1)}_{n-1}) \rtimes \langle \pi \rangle = \langle T_0, \dots, T_{n-1} \rangle \rtimes \langle r_1, \dots, r_{n-1} \rangle.$$

Extended affine Weyl group of type $A^{(1)}_{mn-1} \times A^{(1)}_{m-1} \times A^{(1)}_{m-1}$

Let $y_{[i,k]}$ $(i \in \mathbb{Z}_{mn}, k \in \mathbb{Z}_m)$ be coefficients corresponding vertices in the following quiver:



We define parameters corresponding to the multiplicative simple roots by

$$a_i = \prod_{l=0}^{m-1} y_{[i,l]} \quad (i \in \mathbb{Z}_{mn}), \quad b_k = \prod_{j=0}^{mn-1} y_{[j,k]}, \quad b'_k = \prod_{j=0}^{mn-1} y_{[j,j+k]} \quad (k \in \mathbb{Z}_m).$$

Assume that

$$\prod_{i=0}^{mn-1} a_i = \prod_{k=0}^{m-1} b_k = \prod_{k=0}^{m-1} b'_k = q.$$

We first define Dynkin diagram automorphisms π, π', ρ by

$$\pi = \prod_{k=0}^{m-1} ([0,k],[1,k+1],\dots,[mn-1,k+mn-1]),$$

$$\pi' = \prod_{k=0}^{m-1} ([0,k],[1,k],\dots,[mn-1,k]),$$

$$\rho = \prod_{i=0}^{\lfloor \frac{mn}{2} \rfloor} \prod_{k=0}^{m-1} ([i,k],[-i,k-i]).$$

Here the cyclic permutation is defined by

$$(N, N-1, \dots, 2, 1) = (N, N-1) \dots (3,2)(2,1).$$

They act on the coefficients as

$$\pi(y_{[i,k]}) = y_{[i+1,k+1]}, \quad \pi'(y_{[i,k]}) = y_{[i+1,k]}, \quad \rho(y_{[i,k]}) = y_{[-i,k-i]}.$$

We next define a simple reflection r_0 by

$$r_0 = \mu_{[0,0]}\mu_{[0,1]}\dots\mu_{[0,m-2]}([0,m-2],[0,m-1])\mu_{[0,m-2]}\dots\mu_{[0,1]}\mu_{[0,0]},$$

They act on the coefficients as

$$r_0(y_{[0,k]}) = \frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,k+k_2]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,k+k_2+1]}}, \quad r_0(y_{[1,k]}) = y_{[1,k]}\frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,k+k_2]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,k+k_2]}},$$

$$r_0(y_{[-1,k]}) = y_{[-1,k]}\frac{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1} y_{[0,k+k_2+1]}}{\sum_{k_1=0}^{m-1}\prod_{k_2=0}^{k_1-1} y_{[0,k+k_2+1]}}, \quad r_0(y_{[j,k]}) = y_{[j,k]} \quad (j \neq 0, \pm 1).$$

We also define simple reflections $r_1,\dots,r_{mn-1}$ by

$$r_j = \pi^{-1} r_{j-1}\pi \quad (j = 1,\dots,mn-1).$$

In the last, we define a simple reflection s_0 by

$$s_0 = \mu_{[0,0]}\mu_{[1,0]}\dots\mu_{[mn-2,0]}([mn-2,0],[mn-1,0])\mu_{[mn-2,0]}\dots\mu_{[1,0]}\mu_{[0,0]}.$$

They act on the coefficients as

$$s_0(y_{[i,0]}) = \frac{\sum_{i_1=0}^{2n-1}\prod_{i_2=0}^{i_1-1} y_{[i+i_2,0]}}{\sum_{i_1=0}^{2n-1}\prod_{i_2=0}^{i_1} y_{[i+i_2+1,0]}}, \quad s_0(y_{[i,1]}) = y_{[i,0]}y_{[i,1]}\frac{\sum_{i_1=0}^{2n-1}\prod_{i_2=0}^{i_1} y_{[i+i_2+1,0]}}{\sum_{i_1=0}^{2n-1}\prod_{i_2=0}^{i_1-1} y_{[i+i_2,0]}},$$

for $m=2$ and

$$s_0(y_{[i,0]}) = \frac{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1-1} y_{[i+i_2,0]}}{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1} y_{[i+i_2+1,0]}}, \quad s_0(y_{[i,1]}) = y_{[i,1]}\frac{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1} y_{[i+i_2,0]}}{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1-1} y_{[i+i_2,0]}},$$

$$s_0(y_{[i,-1]}) = y_{[i,-1]}\frac{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1} y_{[i+i_2+1,0]}}{\sum_{i_1=0}^{mn-1}\prod_{i_2=0}^{i_1-1} y_{[i+i_2+1,0]}}, \quad s_0(y_{[i,l]}) = y_{[i,l]} \quad (l \neq 0, \pm 1),$$

for $m \geq 3$. We also define simple reflections $s_1,\dots,s_{m-1}$ and $s'_0,\dots,s'_{m-1}$ by

$$s_i = \pi^{-1}s_{i-1}\pi \quad (i=1,\dots,m-1),$$

and

$$s'_i = \rho s_i \rho \quad (i=0,\dots,m-1).$$

The simple reflections and the Dynkin diagram automorphisms act on the parameters as follows:

	r_i	s_k	s'_k	π	π'	ρ
a_j	$a_j a_i^{-a_{ij}}$	a_j	a_j	a_{j+1}	a_{j+1}	a_{-j}
b_l	b_l	$b_l b_k^{-b_{kl}}$	b_l	b_{l+1}	b_l	b'_l
b'_l	b'_l	b'_l	$b'_l(b'_k)^{-b_{kl}}$	b'_l	b'_{l-1}	b_l

Here the matrices $[a_{ij}]_{i,j\in\mathbb{Z}_{mn}}$ and $[b_{kl}]_{k,l\in\mathbb{Z}_m}$ are the generalized Cartan matrices of type $A^{(1)}_{mn-1}$ and $A^{(1)}_{m-1}$ respectively.

Fact ([Masuda-Okubo-Tsuda 18], [S-Okubo 20])

Let

$$G = \langle r_0, \dots, r_{mn-1}\rangle, \quad H = \langle s_0, \dots, s_{m-1}\rangle, \quad H' = \langle s'_0, \dots, s'_{m-1}\rangle,$$
$$\widetilde{G} = \langle G, H, H', \pi, \pi', \rho\rangle.$$

Then we obtain the following properties:

- *G, H and H' are isomorphic to $W(A^{(1)}_{mn-1})$, $W(A^{(1)}_{m-1})$ and $W(A^{(1)}_{m-1})$ respectively.*
- *Any two groups of G, H, H' are mutually commutative.*
- *$\widetilde{G} = \langle G, H, H'\rangle \rtimes \langle \pi, \pi', \rho\rangle$.*

Translations for $m = 2$

Let $m = 2$. We define translations τ_1, τ_2, τ_3 by

$$\tau_1 = s_1' s_1 \pi' \pi^{-1}, \quad \tau_2 = r_i \dots r_{2n-1} \pi r_1 \dots r_{i-1} \pi r_1 \dots r_{2n-1}, \quad \tau_3 = r_1 \dots r_{2n-1} s_1 \pi.$$

They act on the parameters as follows:

$$\tau_1(b_0) = q b_0, \quad \tau_1(b_1) = \frac{b_1}{q}, \quad \tau_1(b_0') = q b_0', \quad \tau_1(b_1') = \frac{b_1'}{q},$$

$$\tau_2(a_0) = \frac{a_0}{q}, \quad \tau_2(a_{n-1}) = q a_{n-1}, \quad \tau_2(a_n) = \frac{a_n}{q}, \quad \tau_2(a_{2n-1}) = q a_{2n-1},$$

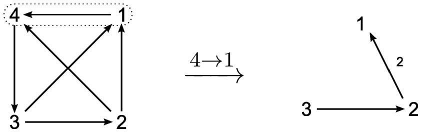
$$\tau_3(a_0) = q a_0, \quad \tau_3(a_1) = \frac{a_1}{q}, \quad \tau_3(b_0) = q b_0, \quad \tau_3(b_1) = \frac{b_1}{q}.$$

Theorem ([Okubo-S 20])

The translations τ_1, τ_2, τ_3 provide higher order q-Painlevé systems as follows:

- τ_1 : *q-FST system*
- τ_2 : *q-Garnier system (of a Sakai's direction)*
- τ_3 : *q-Garnier system (of a Nagao-Yamada's direction)*

We define a confluence of vertices of a quiver $i \to j$ by replacing two vertices i, j and an arrow between them with one vertex j.



At the level of a skew-symmetric matrix, the confluence can be interpreted as follows:

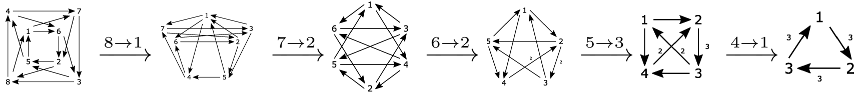
1. Add the i-th row to the j-th row.
2. Add the i-th column to the j-th column.
3. Delete the i-th row and the i-th column.

$$\begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \xrightarrow{4\to 1} \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

There exist the following degeneration structure between the q-Painlevé equations:

$$E_8 \to E_7 \to E_6 \to D_5 \to A_4 \to E_3 \to E_2 \to A_1 \nearrow \underset{|\alpha|^2=8}{A_1} \searrow A_0$$

The degenerations from D_5 to A_0 are derived from confluences of vertices of quivers as follows ([S-Okubo 20]):



Remark

We haven't derived the degeneration $E_8 \to E_7$ from confluences of vertices of quivers yet.

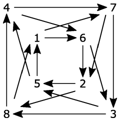


Figure: Quiver Q_{D_5}

The quiver Q_{D_5} is invariant under compositions of mutations and permutations

$$r_0 = (1,4), \quad r_1 = (2,3), \quad r_2 = \mu_1(1,2)\mu_1, \quad r_3 = \mu_5(5,6)\mu_5, \quad r_4 = (5,8),$$
$$r_5 = (6,7), \quad \pi_1 = (1,5,2,6)(4,8,3,7), \quad \pi_2 = (1,2)(3,4)(5,6)(7,8).$$

Fact ([Bershtein-Gavrylenko-Marshakov 18])

A semi-direct product of groups $\langle r_0, \dots, r_5 \rangle \rtimes \langle \pi_1, \pi_2 \rangle$ *is isomorphic to an extended affine Weyl group of type* $D_5^{(1)}$.

Remark ([Sakai 01], [Tsuda-Masuda 06])

A translation $\pi_1 r_2 r_1 r_0 r_2 r_3 r_5 r_4 r_3$ *provides the* q*-Painlevé VI equation.*

The parameters corresponding to the multiplicative simple roots are given by

$$\alpha_0 = \frac{y_4}{y_1}, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_1 y_2, \quad \alpha_3 = y_5 y_6, \quad \alpha_4 = \frac{y_8}{y_5}, \quad \alpha_5 = \frac{y_7}{y_6}.$$

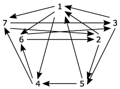


Figure: Quiver Q_{A_4}

The quiver Q_{A_4} is invariant under compositions of mutations and permutations

$$r_0 = \mu_1\mu_2(2,6)\mu_2\mu_1, \quad r_1 = (2,3), \quad r_2 = \mu_2(2,4)\mu_2, \quad r_3 = \mu_5(5,6)\mu_5,$$
$$r_4 = (6,7), \quad \pi_1 = (1,5,3,7,4)(2,6)\,\mu_2.$$

Fact ([Bershtein-Gavrylenko-Marshakov 18])

A semi-direct product of groups $\langle r_0, \dots, r_4\rangle \rtimes \langle \pi_1 \rangle$ is isomorphic to an extended affine Weyl group of type $A_4^{(1)}$.

The parameters corresponding to the simple roots are given by

$$\alpha_0 = y_1\,y_2\,y_6, \quad \alpha_1 = \frac{y_3}{y_2}, \quad \alpha_2 = y_2\,y_4, \quad \alpha_3 = y_5\,y_6, \quad \alpha_4 = \frac{y_7}{y_6}.$$

In the confluence $8 \to 1$ a degeneration of the coefficients is given by a replacement

$$y_1 \to y_1/\varepsilon, \quad y_8 \to \varepsilon,$$

and taking a limit $\varepsilon \to 0$. This limiting procedure induces a degeneration of the simple reflections as follows:

Q_{D_5}	$r_2r_4r_3r_4r_2 = \mu_1\mu_8\mu_2(2,6)\mu_2\mu_8\mu_1$	r_1	$r_0r_2r_0$	r_3	r_5
Q_{A_4}	r_0	r_1	r_2	r_3	r_4

For example, the action $r_2r_4r_3r_4r_2(y_1y_8)$ in Q_{D_5} is reduced to the one $r_0(y_1)$ in Q_{A_4} as follows:

$$\begin{aligned} r_2\,r_4\,r_3\,r_4\,r_2(y_1\,y_8) &= \frac{(1+y_1+y_1\,y_6+y_1\,y_6\,y_2)(1+y_8+y_8\,y_1+y_8\,y_1\,y_6)}{y_1\,y_6\,(1+y_2+y_2\,y_8+y_2\,y_8\,y_1)(1+y_6+y_6\,y_2+y_6\,y_2\,y_8)} \\ &\xrightarrow[y_8\to\varepsilon]{y_1\to y_1/\varepsilon} \frac{(\varepsilon+y_1+y_1\,y_6+y_1\,y_6\,y_2)(1+\varepsilon+y_1+y_1\,y_6)}{y_1\,y_6\,(1+y_2+y_2\,\varepsilon+y_2\,y_1)(1+y_6+y_6\,y_2+y_6\,y_2\,\varepsilon)} \\ &\xrightarrow{\varepsilon\to 0} \frac{1+y_1+y_1\,y_6}{y_6\,(1+y_2+y_2\,y_1)} \\ &= r_0(y_1). \end{aligned}$$

Remark

We haven't clarified degenerations of the Dynkin diagram automorphisms π_1, π_2 yet.

Toward a classification theory of the continuous/discrete Painlevé systems, we have to consider the following problems.

Problem

Establish a q-analogue of the Katz-Oshima classification theory.

Problem

Formulate higher order elliptic difference Painlevé systems as the master equations.

It is important to consider the solution.

Problem

Give a particular solution in terms of the hypergeometric function.

Problem

Describe the general solution explicitly in the form of the asymptotic expansion.

For those purpose, we expect that the theory of the cluster algebra is a powerful tool.

Thank you for your attention.