The rigid parts of the elements of the real Grothendieck groups

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Rigid parts of complexes

Let A be a fin. dim. alg. over a field K.
Consider the homotopy category $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ of complexes over the cat. $\operatorname{proj} A$ of fin. gen. proj. A-modules.

- $U \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$: presilting $:\Longleftrightarrow$ $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(U, U[>0]) = 0$.
- Each $U \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ has a unique decomp. $\bigoplus_{i=1}^{m} U_i$ into indec. direct summands up to iso. and reordering.
- For each $U \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, the maximum presilting direct summand of U is well-defined. It can be called the rigid part of U.

Rigid parts of elements in $K_0(\text{proj}\, A)$

Consider the Grothendieck group $K_0(\text{proj}\, A)$.

- Each $\theta \in K_0(\text{proj}\, A)$ admits unique proj. $P_\theta^0, P_\theta^{-1} \in \text{proj}\, A$ such that $\text{add}\, P_\theta^0 \cap \text{add}\, P_\theta^{-1} = \{0\}$ and $\theta = [P_\theta^0] - [P_\theta^{-1}]$.
- Set $\text{Hom}(\theta) := \text{Hom}(P_\theta^{-1}, P_\theta^0)$: the presentation space of θ.
- Each $f \in \text{Hom}(\theta)$ defines a 2-term complex $P_f := (P_\theta^{-1} \xrightarrow{f} P_\theta^0)$.
- [Derksen-Fei] introduced the canonical decomp. $\theta = \bigoplus_{i=1}^m \theta_i$ in $K_0(\text{proj}\, A)$ from the indec. decomp. of P_f for general $f \in \text{Hom}(\theta)$.
- $\theta \in K_0(\text{proj}\, A)$: rigid $:\Longleftrightarrow \ \exists f \in \text{Hom}(\theta)$, P_f: presilting.
- For each $\theta \in K_0(\text{proj}\, A)$, the maximum rigid direct summand of θ is well-defined. It can be called the rigid part of θ.

Problems of canon. decomp.

The definition of rigid parts of $\theta \in K_0(\text{proj } A)$
by using its canon. decomp. $\theta = \bigoplus_{i=1}^{m} \theta_i$ is quite natural.
However, canon. decomp. have the following problem.

Question (cf. [Derksen-Fei])

Let $\theta \in K_0(\text{proj } A)$ and $m \in \mathbb{Z}_{\geq 1}$.
Is the rigid part of $m\theta$ always m times of the rigid part of θ?

Answer [A-Iyama]

No, we found an algebra A and $\theta \in K_0(\text{proj } A)$ such that

- the rigid part of 2θ is nonzero;
- the rigid part of θ is zero, and θ is indec.

To avoid this problem, we have defined the rigid part of $\theta \in K_0(\text{proj } A)$ in a different way (without canon. decomp.).

Rigid parts of elements in $K_0(\text{proj}\, A)_\mathbb{R}$

We use the real Grothendieck group $K_0(\text{proj}\, A)_\mathbb{R} := K_0(\text{proj}\, A) \otimes_\mathbb{Z} \mathbb{R}$.
We define the rigid part of each element $\theta \in K_0(\text{proj}\, A)_\mathbb{R}$.

- For each basic 2-term presilt. complex $U = \bigoplus_{i=1}^m U_i$ (U_i: indec.), $C^\circ(U) := \sum_{i=1}^m \mathbb{R}_{>0}[U_i] \subset K_0(\text{proj}\, A)_\mathbb{R}$: the presilting cone.
- $\theta \in K_0(\text{proj}\, A)_\mathbb{R}$: rigid $:\Longleftrightarrow$ θ is in some presilting cone.

Main Theorem [A-Iyama]

The rigid part η of $\theta \in K_0(\text{proj}\, A)_\mathbb{R}$ in our definition satisfies:

(a) η is rigid;

(b) we can take "the weak direct sum" of η and $\theta - \eta$;

(c) η is the maximum element satisfying (a) and (b);

(d) for any $r \in \mathbb{R}_{>0}$, $r\eta$ is the rigid part of $r\theta$;

(e) if $\theta \in K_0(\text{proj}\, A)$, then there exists $l \in \mathbb{Z}_{\geq 1}$ such that $l\eta$ is the rigid part of $l\theta$ defined by canon. decomp.

Weak direct sums in $K_0(\text{proj}\, A)_{\mathbb{R}}$?

Main Theorem [A-Iyama]

The rigid part η of $\theta \in K_0(\text{proj}\, A)_{\mathbb{R}}$ in our definition satisfies:

- **(a)** η is rigid;
- **(b)** we can take "the weak direct sum" of η and $\theta - \eta$;
- **(c)** η is the maximum element satisfying (a) and (b);
- **(d)** for any $r \in \mathbb{R}_{>0}$, $r\eta$ is the rigid part of $r\theta$;
- **(e)** if $\theta \in K_0(\text{proj}\, A)$, then there exists $l \in \mathbb{Z}_{\geq 1}$ such that $l\eta$ is the rigid part of $l\theta$ defined by canon. decomp.

Strategy on (b)

Use numerical torsion pairs by [Baumann-Kamnitzer-Tingley], and define an open neighborhood N_U of $C^\circ(U)$ to define weak direct sums.

Maximum rigid direct summand in $K_0(\text{proj } A)_{\mathbb{R}}$?

Main Theorem [A-Iyama]

The rigid part η of $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$ in our definition satisfies:

- **(a)** η is rigid;
- **(b)** we can take "the weak direct sum" of η and $\theta - \eta$;
- **(c)** η is the maximum element satisfying (a) and (b);
- **(d)** for any $r \in \mathbb{R}_{>0}$, $r\eta$ is the rigid part of $r\theta$;
- **(e)** if $\theta \in K_0(\text{proj } A)$, then there exists $l \in \mathbb{Z}_{\geq 1}$ such that $l\eta$ is the rigid part of $l\theta$ defined by canon. decomp.

Strategy on (c)

Study the boundary of $\overline{N_U}$ as a rational polyhedral cone
to show that there is a unique maximum element η satifying (a)(b).

Setting

Let A be a fin. dim. algebra over a field K.

- proj A: the category of fin. gen. projective A-modules.
- $P_1, P_2, \ldots, P_n$: the non-iso. indec. proj. modules.
- $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$: the homotopy cat. of bounded complexes over proj A.
- mod A: the category of fin. dim. A-modules.
- $S_1, S_2, \ldots, S_n$: the non-iso. simple modules (we may assume there exists a surj. $P_i \longrightarrow S_i$).
- $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$: the derived cat. of bounded complexes over mod A.
- $K_0(\mathcal{C})$: the Grothendieck group of $\mathcal{C}$.
- $K_0(\operatorname{proj} A) = \bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$, $K_0(\operatorname{mod} A) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$.
- $K_0(\mathcal{C})_{\mathbb{R}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.
- $K_0(\operatorname{proj} A)_{\mathbb{R}} = \bigoplus_{i=1}^{n} \mathbb{R}[P_i]$, $K_0(\operatorname{mod} A)_{\mathbb{R}} = \bigoplus_{i=1}^{n} \mathbb{R}[S_i]$.

The Euler form

Proposition

The Euler form is a $\mathbb{Z}$-bilinear form

$$\langle \cdot, \cdot \rangle \colon K_0(\operatorname{proj} A) \times K_0(\operatorname{mod} A) \to \mathbb{Z}$$

such that $\langle [P_i], [S_j] \rangle = \delta_{i,j} \dim_K \operatorname{End}_A(S_j)$.

These are naturally extended to the real Grothendieck groups:

$$\langle \cdot, \cdot \rangle \colon K_0(\operatorname{proj} A)_{\mathbb{R}} \times K_0(\operatorname{mod} A)_{\mathbb{R}} \to \mathbb{R}.$$

Via the Euler form, each $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ induces the $\mathbb{R}$-linear form

$$\theta := \langle \theta, \cdot \rangle \colon K_0(\operatorname{mod} A)_{\mathbb{R}} \to \mathbb{R}.$$

TF equivalence

Definition [Baumann-Kamnitzer-Tingley]

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.
We define numerical torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\operatorname{mod} A$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

Definition

$\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ are TF equivalent $:\Longleftrightarrow$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'}), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'}).$$

Presilting complexes

Let $U = \bigoplus_{i=1}^{m} U_i$ with U_i: indec.

- U: basic $:\Longleftrightarrow$ $U_i \not\cong U_j$ $(i \neq j)$.
- $|U| := \#\{$isoclasses of indec. direct summands of $U\}$.

Definition [Keller-Vossieck]

Let $U = (U^{-1} \to U^0) \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ be a 2-term complex.

(1) U: presilting $:\Longleftrightarrow$ $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)}(U, U[>0]) = 0$.

(2) U: silting $:\Longleftrightarrow$ U: presilting, $\mathrm{thick}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)} U = \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$.

2-psilt $A := \{$basic 2-term presilting complexes$\}/\cong$.
2-silt $A := \{$basic 2-term silting complexes$\}/\cong$.

Proposition [(1) Aihara, (2) Adachi-Iyama-Reiten]

(1) $\forall U \in$ 2-psilt A, $\exists T \in$ 2-silt A s.t. $U \in \mathrm{add}\, T$.

(2) $U \in$ 2-silt A $\Longleftrightarrow$ $U \in$ 2-psilt A, $|U| = n$.

Presilting cones

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i: indec.

Proposition [Aihara-Iyama]

$[U_1], \ldots, [U_m] \in K_0(\text{proj } A)$ are linearly independent.
If $U \in$ 2-silt A, then they are a $\mathbb{Z}$-basis of $K_0(\text{proj } A)$.

Definition

We define the presilting cones $C(U), C^\circ(U)$ in $K_0(\text{proj } A)_\mathbb{R}$ by

$$C(U) := \sum_{i=1}^{m} \mathbb{R}_{\geq 0}[U_i], \quad C^\circ(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i].$$

$\theta \in K_0(\text{proj } A)_\mathbb{R}$: rigid $:\Longleftrightarrow \theta \in \bigcup_{U \in \text{2-psilt } A} C^\circ(U)$.

$U \neq U' \in$ 2-psilt $A \Rightarrow C^\circ(U) \cap C^\circ(U') = \emptyset$ [Demonet-Iyama-Jasso].

Presilting cones are TF equiv. classes

Theorem (⇒): [Yurikusa, Brüstle-Smith-Treffinger], (⇐): [A]

Let $U \in$ 2-psilt A.
Then, $C^\circ(U)$ is a TF equiv. class such that $\theta \in C^\circ(U) \iff$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = ({}^\perp H^{-1}(\nu U), \operatorname{Sub} H^{-1}(\nu U)),$$
$$(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\operatorname{Fac} H^0(U), H^0(U)^\perp).$$

Definition

For any $U \in$ 2-psilt A, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^\perp H^{-1}(\nu U), \operatorname{Sub} H^{-1}(\nu U)),$$
$$(\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\operatorname{Fac} H^0(U), H^0(U)^\perp).$$

Open neighborhoods of presilting cones

Definition

For any $U \in$ 2-psilt A, we set

$$N_U := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \mathcal{T}_\theta,\ \mathcal{F}_U \subset \mathcal{F}_\theta\}.$$

This is related to τ-tilting reduction by [Jasso].

Lemma

Let $U, V \in$ 2-psilt A.

(1) N_U is a union of TF equiv. classes.

(2) N_U is an open neighborhood of $C^\circ(U)$.

(3) $U \oplus V$: presilting $\iff N_U \cap N_V \neq \emptyset$.
In this case, $N_U \cap N_V = N_{U \oplus V}$.

(4) $U \in \operatorname{add} V \iff N_V \subset N_U \iff C^\circ(V) \subset N_U$.

The closure $\overline{N_U}$

We focus on the closure $\overline{N_U}$ more today.

Lemma

Let $U, V \in$ 2-psilt A.

(1) $\overline{N_U} = \{\theta \in K_0(\text{proj } A)_{\mathbb{R}} \mid \mathcal{T}_U \subset \overline{\mathcal{T}}_\theta,\ \mathcal{F}_U \subset \overline{\mathcal{F}}_\theta\}$.

(2) $\overline{N_U}$ is a union of TF equiv. classes.

(3) $\overline{N_U}$ is a rational polyhedral cone.

(4) $U \oplus V$: presilting $\iff N_U \cap N_V \neq \emptyset \iff C(V) \subset \overline{N_U}$.

Definition

Let $\eta \in C^\circ(U)$ and $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$.
Then, we say that we can take the weak direct sum of η and θ if $\theta \in \overline{N_U}$.

We will consider properties of $\overline{N_U}$ as a rational polyhedral cone.

Semibricks for U

Definition

(1) $S \in \operatorname{mod} A$: brick $\iff$ $\operatorname{End}_A(S)$: a division ring.

(2) Let $\mathcal{S}$ be a set of bricks.
$\mathcal{S}$: semibrick $\iff$ $\operatorname{Hom}_A(S, S') = 0$ for any $S \neq S' \in \mathcal{S}$.

Theorem [A]

Let $U \in$ 2-psilt A.
Then, there exist semibricks $\mathcal{S}_U, \mathcal{S}'_U$ such that

- $\mathcal{T}_U$ is the smallest torsion class containing $\mathcal{S}_U$.
- $\mathcal{F}_U$ is the smallest torsion-free class containing $\mathcal{S}'_U$.

The explicit forms of $\mathcal{S}_U, \mathcal{S}'_U$

Theorem [A]

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i: indec.
For each $i \in \{1, 2, \ldots, m\}$, we set

$$M_i := H^0(U_i), \qquad X_i := M_i / \sum_{f \in \mathrm{rad}_A(M, M_i)} \mathrm{Im}\, f,$$

$$M'_i := H^{-1}(\nu U_i), \qquad X'_i := \bigcap_{f \in \mathrm{rad}_A(M'_i, M')} \mathrm{Ker}\, f.$$

Then,

$$\mathcal{S}_U = \{X_i \mid i \in \{1, 2, \ldots, m\}\} \setminus \{0\},$$
$$\mathcal{S}'_U = \{X'_i \mid i \in \{1, 2, \ldots, m\}\} \setminus \{0\}.$$

Relationship between N_U and $\mathcal{S}_U, \mathcal{S}'_U$

We have

$$\begin{aligned} N_U &= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid \mathcal{S}_U \subset \mathcal{T}_\theta,\ \mathcal{S}'_U \subset \mathcal{F}_\theta\}, \\ \overline{N_U} &= \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid \mathcal{S}_U \subset \overline{\mathcal{T}}_\theta,\ \mathcal{S}'_U \subset \overline{\mathcal{F}}_\theta\}. \end{aligned}$$

Therefore,

$$\theta \in \partial \overline{N_U} \iff \theta \in \overline{N_U} \text{ and } (\exists i,\, X_i \in \mathcal{W}_\theta \text{ or } X'_i \in \mathcal{W}_\theta),$$

where $\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$: the semistable subcategory [King].
For $\theta \in \overline{N_U}$ and $i \in \{1, 2, \ldots, m\}$, we have unique short exact seq.

$$0 \to \mathsf{t}_\theta X_i \to X_i \to \mathsf{w}_\theta X_i \to 0 \quad (\mathsf{t}_\theta X_i \in \mathcal{T}_\theta,\ \mathsf{w}_\theta X_i \in \mathcal{W}_\theta), \tag{1}$$

$$0 \to \mathsf{w}_\theta X'_i \to X'_i \to \mathsf{f}_\theta X'_i \to 0 \quad (\mathsf{w}_\theta X'_i \in \mathcal{W}_\theta,\ \mathsf{f}_\theta X'_i \in \mathcal{F}_\theta). \tag{2}$$

Facets of $\overline{N_U}$

We set $\mathrm{Facet}\,\overline{N_U} := \{$facets (faces of codim. 1) of $\overline{N_U}\}$.

Theorem [A-Iyama]

Let $U = \bigoplus_{i=1}^m U_i \in \text{2-psilt}\,A$ with U_i: indec. and $F \in \mathrm{Facet}\,\overline{N_U}$.
Then, there uniquely exist $i \in \{1, 2, \ldots, m\}$ and $L \in \mathrm{mod}\,A \setminus \{0\}$ s.t.

(i) $F = \{\theta \in \overline{N_U} \mid \theta(L) = 0\}$.

(ii) $\forall \theta \in F^\circ$, the short exact sequences (1)(2) are constant, and $(\mathsf{w}_\theta X_i, \mathsf{w}_\theta X_i')$ is $(L, 0)$ or $(0, L)$.

(iii) $\forall j \in \{1, 2, \ldots, m\}$,

$$[U_j](L) = \begin{cases} \dim_K \mathrm{End}_A(X_i) & (i = j,\ (\mathsf{w}_\theta X_i, \mathsf{w}_\theta X_i') = (L, 0)) \\ -\dim_K \mathrm{End}_A(X_i') & (i = j,\ (\mathsf{w}_\theta X_i, \mathsf{w}_\theta X_i') = (0, L)) \\ 0 & (i \neq j) \end{cases}.$$

We write i_F and L_F for i and L above.

Relationship between $\overline{N_U}$ and L_F

For each $F \in \text{Facet}\,\overline{N_U}$, exactly one of the following holds.

(a) $\forall \theta \in F^\circ$, $X_{i_F} \notin \mathcal{T}_\theta$ and $X'_{i_F} \in \mathcal{F}_\theta$.

(b) $\forall \theta \in F^\circ$, $X_{i_F} \in \mathcal{T}_\theta$ and $X'_{i_F} \notin \mathcal{F}_\theta$.

We set $s_F := 1$ if (a) and $s_F := -1$ if (b).

Corollary

For $U \in$ 2-psilt A,

$$\overline{N_U} = \bigcap_{F \in \text{Facet}\,\overline{N_U}} \{\theta \in K_0(\text{proj}\,A)_{\mathbb{R}} \mid s_F \cdot \theta(L_F) \geq 0\}.$$

Corollary

Let $U \in$ 2-psilt A and $F \in \text{Facet}\,\overline{N_U}$.
Then, $C(U/U_{i_F}) \subset F$.

Maximum η such that $\eta \oplus (\theta - \eta)$

Lemma

Let $U \in$ 2-psilt A and $\theta \in \overline{N_U}$.
We set $H := \{\eta \in C(U) \mid \theta - \eta \in \overline{N_U}\}$.
Then, there exist $a_1, a_2, \ldots, a_m \in \mathbb{R}_{\geq 0}$ such that

$$H = \left\{ \sum_{i=1}^{m} x_i[U_i] \mid x_i \in [0, a_i] \, (i \in \{1, 2, \ldots, m\}) \right\}.$$

Moreover, if $\theta \in K_0(\text{proj } A)$, then $a_1, a_2, \ldots, a_m \in \mathbb{Z}_{\geq 0}$.

η_U and η'_U are piecewise-projections

Definition

We define $\eta_U \colon \overline{N_U} \to C(U) \subset \overline{N_U}$ and $\eta'_U \colon \overline{N_U} \to \overline{N_U}$ by

$$\eta_U(\theta) := \sum_{i=1}^{m} a_i[U_i], \quad \eta'_U(\theta) := \theta - \eta_U(\theta).$$

Lemma

Let $U \in$ 2-psilt A.

(1) $\eta_U \colon \overline{N_U} \to C(U) \subset \overline{N_U}$ and $\eta'_U \colon \overline{N_U} \to \overline{N_U}$ are piecewise-linear.

(2) $\eta_U \circ \eta_U = \eta_U, \eta_U \circ \eta'_U = 0, \eta'_U \circ \eta_U = 0, \eta'_U \circ \eta'_U = \eta'_U$.

(3) If $U = V \oplus W$, then $\eta_U = \eta_V + \eta_W, \eta_W = \eta_W \circ \eta'_V$ on $\overline{N_U}$.

Decompositions of $\theta \in \overline{N_U}$

The image of η_U is $C(U)$, and the image of η'_U is

$$J_U := \{\theta \in \overline{N_U} \mid \eta_U(\theta) = 0\} = \overline{N_U} \setminus \left(\bigcup_{V \in \text{2-psilt}\, A,\ 0 \neq V \in \text{add}\, U} N_V \right).$$

Proposition

For any $U \in$ 2-psilt A, there exists a bijection

$$(\eta_U, \eta'_U) \colon \overline{N_U} \to C(U) \times J_U; \quad \theta \mapsto (\eta_U(\theta), \eta'_U(\theta)).$$

In other words, for any $\theta \in \overline{N_U}$,
there uniquely exist $\eta \in C(U)$ and $\eta' \in J_U$ such that $\theta = \eta + \eta'$,
and actually, $\eta = \eta_U(\theta)$ and $\eta' = \eta'_U(\theta)$.

A stratification in $K_0(\mathrm{proj}\, A)_{\mathbb{R}}$

$\text{2-psilt}_U\, A := \{V \in \text{2-psilt}\, A \mid U \in \mathrm{add}\, V\}$.

Definition

For $U \in \text{2-psilt}\, A$, we set

$$R_U := N_U \setminus \bigcup_{V \in (\text{2-psilt}_U\, A) \setminus \{U\}} N_V.$$

- If $\theta \in R_U$, then U is the max. $V \in \text{2-psilt}\, A$ such that $\theta \in N_V$.
- $K_0(\mathrm{proj}\, A)_{\mathbb{R}} = \bigsqcup_{U \in \text{2-psilt}\, A} R_U$.

Remark

The family $\{R_U\}_{U \in \text{2-psilt}\, A}$ is a stratification in $K_0(\mathrm{proj}\, A)_{\mathbb{R}}$.

The rigid parts of elements in $K_0(\operatorname{proj} A)_{\mathbb{R}}$

Definition

For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, take the unique $U \in$ 2-psilt A such that $\theta \in R_U$.
Then, we call $\eta_U(\theta)$ as the rigid part of θ.

Corollary

For any $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ and $r \in \mathbb{R}_{>0}$,
the rigid part of $r\theta$ is r times of the rigid part of θ.

The rigid part of θ is maximum in the following sense.

Theorem [A-Iyama]

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i: indec. and $\theta \in R_U$.
Assume that $V \in$ 2-psilt A and $\eta \in C^\circ(V)$ satisfies $\theta - \eta \in \overline{N_V}$.

(1) $V \in \operatorname{add} U$.

(2) Write $\eta_U(\theta) = \sum_{i=1}^{m} a_i[U_i]$ and $\eta = \sum_{i=1}^{m} b_i[U_i]$.
Then, for each i, we have $a_i \geq b_i$.

The propeties of rigid parts

Theorem [A-Iyama]

Let $U \in \text{2-psilt}\, A$.
Then, there exists a bijection

$$(\eta_U, \eta'_U)\colon R_U \to C^\circ(U) \times (\overline{N_U} \cap R_0); \quad \theta \mapsto (\eta_U(\theta), \eta'_U(\theta)).$$

We have

$$K_0(\text{proj}\, A)_\mathbb{R} = \bigsqcup_{U \in \text{2-psilt}\, A} R_U = \bigsqcup_{U \in \text{2-psilt}\, A} (C^\circ(U) \times (\overline{N_U} \cap R_0)).$$

In particular, for each $\theta \in K_0(\text{proj}\, A)_\mathbb{R}$,
there uniquely exist $U \in \text{2-psilt}\, A$ and $\eta \in C^\circ(U)$, $\eta' \in \overline{N_U} \cap R_0$
satisfying $\theta = \eta + \eta'$.

Relationship between canon. decomp.

By canon. decomp. $\theta = \bigoplus_{i=1}^{m} \theta_i$ in $K_0(\operatorname{proj} A)$ of [Derksen-Fei],
the max. rigid direct summand θ_{ri} of θ is defined independently.
We set $\theta_{\mathrm{nr}} := \theta - \theta_{\mathrm{ri}}$.

Theorem [A-Iyama]

Let K be alg. closed and $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.

(1) $\exists l \in \mathbb{Z}_{\geq 1}$, the rigid part of $l\theta$ (in our sense) is $(l\theta)_{\mathrm{ri}}$.

(2) l in (1) satisfies $\forall m \in \mathbb{Z}_{\geq 1}$, $(ml\theta)_{\mathrm{ri}} = m \cdot (l\theta)_{\mathrm{ri}}$, $(ml\theta)_{\mathrm{nr}} = m \cdot (l\theta)_{\mathrm{nr}}$.

Future problem

We found an algebra A and $\theta \in K_0(\operatorname{proj} A)$ such that
we cannot let $l = 1$ in Theorem.
When can we let $l = 1$?

Example of rigid parts (1)

Let A be the algebra

$$K\left(1 \overset{\alpha_1}{\underset{\alpha_3}{\overset{\xrightarrow{\alpha_2}}{\longrightarrow}}} 2 \overset{\beta_1}{\underset{\beta_3}{\overset{\xrightarrow{\beta_2}}{\longrightarrow}}} 3 \right) / \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \\ \alpha_1\beta_1 + \alpha_2\beta_3, \alpha_2\beta_2 + \alpha_3\beta_1, \alpha_3\beta_3 + \alpha_1\beta_2 \end{matrix} \right\rangle$$

Consider the element $\theta := [P_1] + [P_2] - [P_3] \in K_0(\text{proj}\, A)$.

- $\theta \in R_{P_1}$ and $\eta_{P_1}(\theta) = [P_1]$: the rigid part of θ (in our sense).
- The canon. decomp. of θ is θ itself, and $\theta_{\text{ri}} = 0$.
- We can take the weak direct sum of $[P_1]$ and $[P_2] - [P_3]$, but their direct sum in the sense of [Derksen-Fei] is NOT allowed.

We found this example by using results of [Fei].

Example of rigid parts (2)

Let A be the algebra

$$K\left(1 \overset{\alpha_1}{\underset{\alpha_3}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\alpha_2}}}}} 2 \overset{\beta_1}{\underset{\beta_3}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\beta_2}}}}} 3 \right) \Big/ \left\langle \begin{matrix} \alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \\ \alpha_1\beta_1 + \alpha_2\beta_3, \alpha_2\beta_2 + \alpha_3\beta_1, \alpha_3\beta_3 + \alpha_1\beta_2 \end{matrix} \right\rangle$$

Consider the element $\theta := [P_1] + [P_2] - [P_3] \in K_0(\operatorname{proj} A)$.

- $2\theta \in R_{P_1}$ and $\eta_{P_1}(2\theta) = 2[P_1]$: the rigid part of θ (in our sense).
- The canon. decomp. of 2θ is $[P_1] \oplus [P_1] \oplus (2[P_2] - 2[P_3])$ and $\theta_{\mathrm{ri}} = 2[P_1]$.
- We can take the weak direct sum of $[P_1]$ and $2[P_2] - 2[P_3]$, and their direct sum in the sense of [Derksen-Fei] is also allowed.
- For any $m \in \mathbb{Z}_{\geq 1}$, $\eta_U(2m\theta) = (2m\theta)_{\mathrm{ri}} = 2m[P_1]$.

Thank you for your attention.

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