Cluster-additive functions and tropical points

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Frieze patterns

Definition (Frieze pattern)

In mathematics, a frieze or frieze pattern is a two-dimensional design that repeats in one direction. Such patterns occur frequently in architecture and decorative art. — From Wikipedia

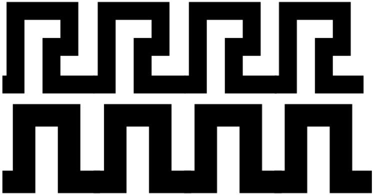


Figure: Examples of frieze patterns

The sequence of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots,\}$ is also a frieze pattern in the sense that we know how to "repeat": $a_{n+1} = a_n + 1$.

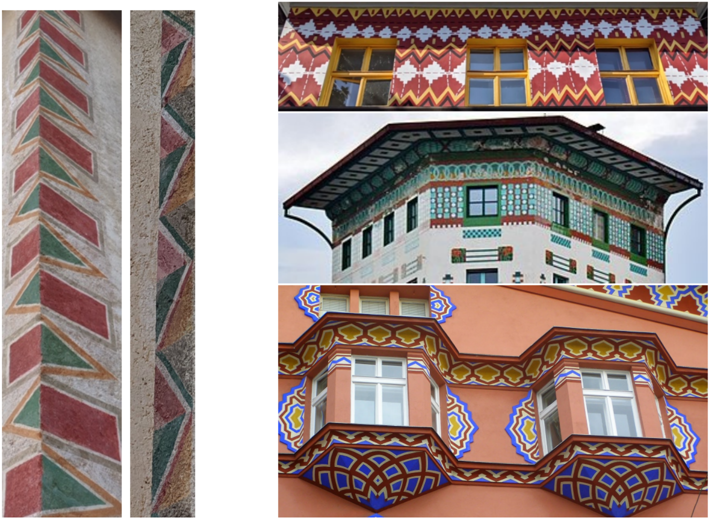
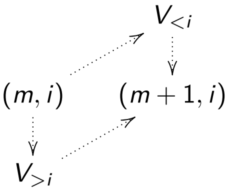


Figure: Frieze patterns in architecture

§1. **Various frieze patterns associated to a Cartan matrix**

- $A = (a_{i,j})$: $r \times r$ symmetrizable Cartan matrix, e.g., $A = \begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}$;
- $[1, r] := \{1, 2, \ldots, r\}$ and $[a]_+ := \max(0, a)$ for $a \in \mathbb{R}$;
- A mesh in the infinite strip $\mathbb{Z} \times [1, r]$ is a diagram of the form



where $V_{>i} = \{(m, j) \mid j > i, a_{j,i} \neq 0\}$ and $V_{<i} = \{(m+1, j) \mid j < i, a_{j,i} \neq 0\}$.

- We are interested in the maps $f : \mathbb{Z} \times [1, r] \to R$ (ring) associated to A satisfying certain "mesh type" relations.

Generic frieze pattern

The generic frieze pattern associated to A is the (unique) map

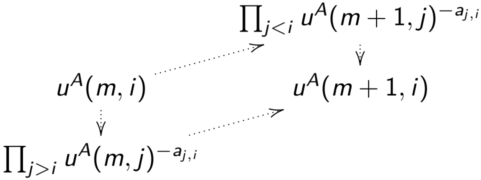
$$u^A : \mathbb{Z} \times [1, r] \to \mathbb{Q}(x_1, \ldots, x_r)$$

such that

(a) $u^A(0, i) = x_i$ for $i = 1, \ldots, r$;

(b) for each mesh starting from (m, i), we have a cluster type relation:

$$u^A(m, i) \cdot u^A(m+1, i) = 1 + \prod_{j>i} u^A(m, j)^{-a_{j,i}} \cdot \prod_{j<i} u^A(m+1, j)^{-a_{j,i}}.$$



Remark

Cluster type relations come from cluster mutations at a source or sink vertex.

Examples of generic frieze pattern

- Take $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Then the generic frieze pattern u^A is given

$$u^A = \begin{matrix} \cdots & {\color{red}x_1} & \frac{1+x_2}{x_1} & \frac{1+x_1}{x_2} & x_2 & \cdots \\ \cdots & {\color{red}x_2} & \frac{1+x_1+x_2}{x_1x_2} & x_1 & \frac{1+x_2}{x_1} & \cdots \end{matrix}$$

- Take $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Then

$$u^A = \begin{matrix} \cdots & {\color{red}x_1} & \frac{1+x_2}{x_1} & \frac{x_1^3+(1+x_2)^2}{x_1^2x_2} & \frac{1+x_1^3+x_2}{x_1x_2} & x_1 & \cdots \\ \cdots & {\color{red}x_2} & \frac{x_1^3+(1+x_2)^3}{x_1^3x_2} & \frac{x_1^6+3x_1^3x_2+2x_1^3+(1+x_2)^3}{x_1^3x_2^2} & \frac{1+x_1^3}{x_2} & x_2 & \cdots \end{matrix}$$

- In the finite type case, the generic frieze patterns are periodic.

Tropical frieze pattern

A tropical frieze pattern associated to A is a map $f : \mathbb{Z} \times [1, r] \to \mathbb{Z}$ such that for each mesh starting from (m, i), we have a tropical type relation:

$$f(m,i) + f(m+1,i) = \Big[\sum_{j>i}(-a_{j,i}) \cdot f(m,j) + \sum_{j<i}(-a_{j,i}) \cdot f(m+1,j)\Big]_+.$$

We can compare it with cluster type relation:

$$u^A(m,i) \cdot u^A(m+1,i) = 1 + \prod_{j>i} u^A(m,j)^{-a_{j,i}} \cdot \prod_{j<i} u^A(m+1,j)^{-a_{j,i}}.$$

Remark

(i) *Tropical frieze patterns were introduced and studied in [Guo' 13];*

(ii) *Tropical type relations are obtained from the cluster type relations by tropicalization over* $(\mathbb{Z}, +, \max)$*;*

(iii) *Such relations are closely related with the final-seed mutation rule at sink or source vertices for d-vectors in cluster algebras;*

Additive function

An additive function or additive frieze pattern associated to A is a map $f : \mathbb{Z} \times [1, r] \to \mathbb{Z}$ such that for each mesh starting from (m, i), we have an additive relation:

$$f(m,i) + f(m+1,i) = \sum_{j>i}(-a_{j,i}) \cdot f(m,j) + \sum_{j<i}(-a_{j,i}) \cdot f(m+1,j).$$

We can compare it with tropical type relation:

$$f(m,i) + f(m+1,i) = \left[\sum_{j>i}(-a_{j,i}) \cdot f(m,j) + \sum_{j<i}(-a_{j,i}) \cdot f(m+1,j)\right]_+.$$

Remark

$\mathbb{Z}^r$-valued additive functions naturally appear in representation theory by taking Grothendieck group.

Cluster-additive function

A cluster-additive function associated to A is a map $f : \mathbb{Z} \times [1, r] \to \mathbb{Z}$ such that for each mesh starting from (m, i), we have a cluster-additive relation:

$$f(m,i) + f(m+1,i) = \sum_{j>i}(-a_{j,i}) \cdot \left[f(m,j)\right]_+ + \sum_{j<i}(-a_{j,i}) \cdot \left[f(m+1,j)\right]_+.$$

We can compare it with tropical type relation:

$$f(m,i) + f(m+1,i) \quad = \quad \left[\sum_{j>i}(-a_{j,i}) \cdot f(m,j) + \sum_{j<i}(-a_{j,i}) \cdot f(m+1,j)\right]_+.$$

Aim (i): Try to understand the cluster-additive relations introduced by [Ringel'12]. Where such relations come from?

Answers: They come from

(a) sequence of initial-seed mutations for g-vectors;

(b) sequence of mutations for coefficients row;

(c) initial-seed mutation rule for d-vectors at sink or source vertices.

§2. **Ringel's results and conjectures**

Ringel's bijection and cluster-hammock functions

Theorem (Ringel's bijection, [Ringel'12])

For each given $m \in \mathbb{Z}$, there is a natural bijection between the cluster-additive functions associated to A and the points in $\mathbb{Z}^r$, given by the restriction to m-th slice

$$f \mapsto (f(m,1), f(m,2), \ldots, f(m,r))^{\mathrm{T}} \in \mathbb{Z}^r.$$

- For each vertex (m,i), we have a unique cluster-additive function $h^{(m,i)}$ such that the restriction of $h^{(m,i)}$ to the m-th slice is $-\mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th column of the identity matrix I_r.
- We call $h^{(m,i)}$ the cluster-hammock function at vertex (m,i).
- Cluster-hammock functions look like:

$$\begin{matrix} \ldots & * & 0 & * & \ldots \\ \ldots & * & -1 & * & \ldots \\ \ldots & * & 0 & * & \ldots \end{matrix}$$

“Cone structure" on cluster-additive functions

- Let $f, g : \mathbb{Z} \times [1, r] \to \mathbb{Z}$ be two cluster-additive functions associated to A.
- f and g are said to be compatible (or sign-coherent) if $f(m, i) \cdot g(m, i) \geq 0$ for any $(m, i) \in \mathbb{Z} \times [1, r]$.

Theorem ([Ringel'12])

Let $f_1, \ldots, f_m$ be cluster-additive functions associated to A. Then the sum $f_1 + \ldots + f_m$ is cluster-additive if and only if any two of them are compatible.

- As a set, cluster-additive functions are in bijection with the points in $\mathbb{Z}^r$. However, the set of cluster-additive functions has no natural linear structure;
- Ringel's theorem tells us that the set of cluster-additive functions has some “cone structure".
- Aim (ii): try to understand this cone structure from the cluster theory.
- One result is that: under certain bijection, every g-vector cone can be identified with a subcone of the cone structure of cluster-additive functions.

Ringel's first conjecture

- Take $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. The following are two examples of cluster-additive functions associated to A.

$$f_1 = \begin{matrix} \cdots & 5 & 6 & 1 & -1 & 5 & \cdots \\ \cdots & 11 & 7 & -4 & 4 & 11 & \cdots \end{matrix}$$

$$f_2 = \begin{matrix} \cdots & 7 & 6 & -1 & 1 & 7 & \cdots \\ \cdots & 13 & 5 & -5 & 8 & 13 & \cdots \end{matrix}$$

- For A above, the generic frieze pattern $u^A : \mathbb{Z} \times [1, r] \to \mathbb{Q}(x_1, \ldots, x_r)$ is

$$u^A = \begin{matrix} \cdots & x_1 & \frac{1+x_2}{x_1} & \frac{x_1^3+(1+x_2)^2}{x_1^2 x_2} & \frac{1+x_1^3+x_2}{x_1 x_2} & x_1 & \cdots \\ \cdots & x_2 & \frac{x_1^3+(1+x_2)^3}{x_1^3 x_2} & \frac{x_1^6+3x_1^3x_2+2x_1^3+(1+x_2)^3}{x_1^3 x_2^2} & \frac{1+x_1^3}{x_2} & x_2 & \cdots \end{matrix}$$

- Ringel's first conjecture: For a finite type Cartan matrix A, if $u^A(m, i) = u^A(n, j)$, then $f(m, i) = f(n, j)$ for any cluster-additive function f associated to A.

Ringel's second conjecture

- Keep the example before. $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ and f_1, f_2 are cluster-additive functions associated to A.

$$f_1 = \begin{matrix} \dots & 5 & 6 & 1 & -1_{(3,1)} & 5 & \dots \\ \dots & 11 & 7 & -4_{(2,2)} & 4 & 11 & \dots \end{matrix}$$

$$f_2 = \begin{matrix} \dots & 7 & 6 & -1_{(2,1)} & 1 & 7 & \dots \\ \dots & 13 & 5 & -5_{(2,2)} & 8 & 13 & \dots \end{matrix}$$

- Easy to check $f_1 = 4h^{(2,2)} + h^{(3,1)}$ and $f_2 = 5h^{(2,2)} + h^{(2,1)}$, where $h^{(m,i)}$ is the cluster-hammock function at vertex (m, i);
- Ringel's second conjecture: For a finite type Cartan matrix A, any cluster-additive function f associated to A is a non-negative linear combination of cluster-hammock functions associated to A.
- [Ringel'12] proved his two conjectures in type A and [Guo' 13] proved Ringel's conjectures for type A, D, E.
- Both Ringel and Guo studied the cluster-additive functions from the perspectives of representation theory.

§3. **Cluster algebras part**

Seed pattern

Fix an initial seed $\Sigma = (\mathbf{u}, \mathbf{p}, \widetilde{B})$, where

- $\widetilde{B} = \begin{pmatrix} B \\ P \end{pmatrix}$ is an $(r+s) \times r$ integer matrix with B skew-symmetrisable.
- $\mathbf{u} = (u_1, \dots, u_r)$ unfrozen part and $\mathbf{p} = (p_1, \dots, p_s)$ frozen part.

Denote by $\mathbb{T}_r$ the r-regular tree rooted at a vertex t_0.

We can form a seed pattern $\mathcal{S} := \{\Sigma_t = (\mathbf{u}_t, \mathbf{p}, \widetilde{B}_t) \mid t \in \mathbb{T}_r\}$ on $\mathbb{T}_r$ such that

- $\Sigma_{t_0} = \Sigma$;
- $\Sigma_{t'} = \mu_k(\Sigma_t)$ whenever $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_r$.

Tropical points

Definition (Tropical point)

Given a seed pattern $\mathcal{S} = \{\Sigma_t \mid t \in \mathbb{T}_r\}$. A tropical point

$$[\mathbf{g}] = \{\mathbf{g}_t \in \mathbb{Z}^{r+s} \mid t \in T_r\}$$

associated to $\mathcal{S}$ is an assignment of a column vector $\mathbf{g}_t \in \mathbb{Z}^{r+s}$ to each vertex $t \in \mathbb{T}_r$ such that we have an relation of matrix mutation

$$[\widetilde{B}_{t'}, \mathbf{g}_{t'}] = \mu_k[\widetilde{B}_t, \mathbf{g}_t]$$

for any edge $t \overset{k}{\text{———}} t'$ in $\mathbb{T}_r$.

Remark

[Nakanishi-Zelevinsky' 2012] proved that the initial-seed mutation rule for (extended) g-vectors of cluster variables in $\mathcal{S}$ is the same with the transformation from $\mathbf{g}_t$ to $\mathbf{g}_{t'}$ given above.

Acyclic skew-symmetrizable matrix B_A

- Let $A = (a_{i,j})$ be a Cartan matrix and denote by

$$B_A = \begin{bmatrix} 0 & -a_{1,2} & -a_{1,3} & \dots & -a_{1,r} \\ a_{2,1} & 0 & -a_{2,3} & \dots & -a_{2,r} \\ a_{3,1} & a_{3,2} & 0 & \dots & -a_{3,r} \\ \dots & \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & a_{r,3} & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \geq 0 \\ \leq 0 & 0 \end{bmatrix}$$

  the acyclic skew-symmetrizable matrix associated to A.
- For the transposed Cartan matrix A^{T}, we have $B_{A^{\mathrm{T}}} = -B_A^{\mathrm{T}}$.
- Given a skew-symmetrizable matrix B. We say k is a source vertex of B if the k-th column of B is non-positive.
- Easy to see $B_A = \mu_r \dots \mu_2\mu_1(B_A)$ and $\mu_r \dots \mu_2\mu_1(B_A)$ is a sequence of mutations at source vertices.

Acyclic belt

- Keep A and B_A as before. Let $\{\Sigma_t^{(B_A)} \mid t \in \mathbb{T}_r\}$ be the seed pattern with trivial coefficients associated to B_A.
- The r-regular tree $\mathbb{T}_r$ rooted at t_0 has a unique 2-regular sub-tree $\mathbb{T}_r^{\dagger}$ with the following form

$$\cdots \overset{r-1}{—} t(-2,r) \overset{r}{—} t(-1,1) \overset{1}{—} \cdots \overset{r-1}{—} t(-1,r) \overset{r}{—} {\color{red}t(0,1)} \overset{1}{—} t(0,2) \overset{2}{—} \cdots \overset{r-1}{—} t(0,r) \overset{r}{—} t(1,1) \overset{1}{—} \cdots,$$

where each $t(m,i)$ denotes a vertex of $\mathbb{T}_r$ and ${\color{red}t(0,1) = t_0}$.

- The acyclic belt associated to B_A is the collection of seeds on the 2-regular sub-tree $\mathbb{T}_r^{\dagger}$

$$\begin{aligned}
\{\Sigma_t^{(B_A)} \mid t \in \mathbb{T}_r^{\dagger}\} &= \{\Sigma_{t(m,i)}^{(B_A)} \mid (m,i) \in \mathbb{Z} \times [1,r]\} \\
&= \begin{matrix} \cdots & \Sigma_{t(m,1)}^{(B_A)} & \Sigma_{t(m+1,1)}^{(B_A)} & \cdots \\ \cdots & \vdots & \vdots & \cdots \\ \cdots & \Sigma_{t(m,r)}^{(B_A)} & \Sigma_{t(m+1,r)}^{(B_A)} & \cdots \end{matrix}
\end{aligned}$$

Generic frieze pattern and acyclic belt

Cluster variables on the acyclic belt are exactly the variables appearing in the generic frieze pattern. More precisely,

Proposition

Given $u^A : \mathbb{Z} \times [1, r] \to \mathbb{Q}(u_1, \ldots, u_r)$ *and* $\{\Sigma_{t(m,i)}^{(B_A)} \mid (m, i) \in \mathbb{Z} \times [1, r]\}$*. Then*

(i) $u^A(m, i)$ *is the* i*-th cluster variable of seed* $\Sigma_{t(m,i)}^{(B_A)}$;

(ii) *The cluster* $\mathbf{u}_{t(m,i)}$ *of* $\Sigma_{t(m,i)}^{(B_A)}$ *is given by*

$$\begin{array}{cc} & u^A(m+1, 1) \\ & \vdots \\ & u^A(m+1, i-1) \\ u^A(m, i) & \\ u^A(m, i+1) & \\ \vdots & \\ u^A(m, r) & \end{array}$$

§4. **Our results**

Cluster-additive functions and tropical points

- Keep A and B_A as before. Recall $B_{A^{\mathrm{T}}} = -B_A^{\mathrm{T}}$.
- We will use cluster algebra $\mathcal{A}(-B_A^{\mathrm{T}})$ to study cluster-additive functions associated to A.
- For a tropical point $[\mathbf{g}] = \{\mathbf{g}_t \in \mathbb{Z}^r \mid t \in \mathbb{T}_r\}$ associated to $-B_A^{\mathrm{T}}$, we denote by $g(m,1), \ldots, g(m,r)$ the diagonal of the matrix $[\mathbf{g}_{t(m,1)}, \ldots, \mathbf{g}_{t(m,r)}]$.

Theorem (C.-de St. Germain-Lu)

(i) *Given tropical point* $[\mathbf{g}] = \{\mathbf{g}_t \in \mathbb{Z}^r \mid t \in \mathbb{T}_r\}$ *associated to* $-B_A^{\mathrm{T}}$. *The map* $T_{[\mathbf{g}]} : (m,i) \mapsto -g(m,i)$ *is a cluster-additive function associated to* A. *Thus*

$$(T_{[\mathbf{g}]}(m,1), \ldots, T_{[\mathbf{g}]}(m,r)) = -(g(m,1), \ldots, g(m,r)).$$

(ii) *The map* $T : [\mathbf{g}] \mapsto T_{[\mathbf{g}]}$ *induces a bijection between the tropical points associated to* $-B_A^{\mathrm{T}}$ *and the cluster-additive functions associated to* A.

d-compatibility degree

- For any $0 \neq b \in \mathcal{A}(\widetilde{B})$ and any seed Σ_t, we have the Laurent expansion

$$b = \frac{N(u_{1;t}, \ldots, u_{r;t})}{u_{1;t}^{d_1} \cdots u_{r;t}^{d_r}},$$

  where $N \in \mathbb{Z}[\mathbf{p}][\mathbf{u}_t]$ is coprime with each $u_{i;t} \in \mathbf{u}_t$;
- The integer vector $(d_1, \ldots, d_r)^{\mathrm{T}}$ is called the d-vector of b w.r.t. seed Σ_t;
- Each d_k only depends on b and $u_{k;t}$, not depend on the choice of cluster containing $u_{k;t}$, c.f., [Cao-Li' 20].
- Call d_k the d-compatibility degree of b w.r.t. $u_{k;t}$ and denote it by

$$(u_{k;t} \mid\mid b)_d := d_k.$$

- Next result is using d-compatibility degree give an realization of all the cluster-addition functions.

Cluster-additive functions and d-compatibility degree

- Let $u^{A^{\mathrm{T}}} : \mathbb{Z} \times [1, r] \to \mathbb{Q}(u_1, \ldots, u_r)$ be the generic frieze pattern associated to the transposed Cartan matrix A^{T};
- Each $u^{A^{\mathrm{T}}}(m, i)$ is actually a cluster variable of $\mathcal{A}(B_{A^{\mathrm{T}}}) = \mathcal{A}(-B_A^{\mathrm{T}})$;

Theorem (C.-de St. Germain-Lu)

Let $[\mathbf{g}]$ be a tropical point associated to $-B_A^{\mathrm{T}}$ and $T_{[\mathbf{g}]}$ the corresponding cluster-additive function associated to A. Then there is an element $u_{[\mathbf{g}]}$ in $\mathcal{A}(-B_A^{\mathrm{T}})$ such that

$$T_{[\mathbf{g}]}(m, i) = (u^{A^{\mathrm{T}}}(m, i) \mid\mid u_{[\mathbf{g}]})_d.$$

Corollary

Ringel's first conjecture is true for any Cartan matrix A (even infinite type).

Proof: $u^A(m, i) = u^A(n, j) \Longrightarrow u^{A^{\mathrm{T}}}(m, i) = u^{A^{\mathrm{T}}}(n, j) \Longrightarrow T_{[\mathbf{g}]}(m, i) = T_{[\mathbf{g}]}(n, j).$

Sign-coherent set of tropical points

- Let B be an $r \times r$ skew-symmetrisable matrix and $U = \{[\mathbf{g}^{(1)}], \ldots, [\mathbf{g}^{(q)}]\}$ be a finite (multiple) set of tropical points associated to B;
- Write $[\mathbf{g}^{(i)}] = \{\mathbf{g}_t^{(i)} \in \mathbb{Z}^r \mid t \in \mathbb{T}_r\}$ for $i = 1, \ldots, q$ and take the formal sum

$$\sum_{[\mathbf{g}] \in U} [\mathbf{g}] := \{\sum_{i=1}^{q} \mathbf{g}_t^{(i)} \mid t \in \mathbb{T}_r\}.$$

- In general, $\sum_{[\mathbf{g}] \in U} [\mathbf{g}]$ does not form a tropical point associated to B.
- We say that $U = \{[\mathbf{g}^{(1)}], \ldots, [\mathbf{g}^{(q)}]\}$ is sign-coherent, if each row of the matrix $[\mathbf{g}_t^{(1)}, \ldots, \mathbf{g}_t^{(q)}]$ is either non-negative, or non-positive for each $t \in \mathbb{T}_r$.
- Easy to check: If U is sign-coherent, then the sum $\sum_{[\mathbf{g}] \in U} [\mathbf{g}]$ is still a tropical point associated to B.

Sign-coherent decomposition of tropical points

Definition (Sign-coherent decomposition)

Let $[\mathbf{g}] = [\mathbf{g}^{(1)}] + \ldots + [\mathbf{g}^{(q)}]$ be a decomposition of tropical points associated to B. We say this decomposition is sign-coherent, if

$$U = \{[\mathbf{g}^{(1)}], \ldots, [\mathbf{g}^{(q)}]\}$$

forms a sign-coherent set of tropical points. In this case, we denote by $[\mathbf{g}] = [\mathbf{g}^{(1)}] \uplus \ldots \uplus [\mathbf{g}^{(q)}]$.

Each sign-coherent decomposition of tropical points induces a decomposition of cluster-additive functions. More precisely,

Proposition

If $[\mathbf{g}] = [\mathbf{g}^{(1)}] \uplus \ldots \uplus [\mathbf{g}^{(q)}]$, *then* $T_{[\mathbf{g}]} = T_{[\mathbf{g}^{(1)}]} + \ldots + T_{[\mathbf{g}^{(q)}]}$.

Cluster monomials and tropical points

- A monomial in cluster variables from the same cluster is called a cluster monomial.
- Let $w \in \mathbb{T}_r$ and $\mathbf{a} \in \mathbb{Z}^r$. Denote by $[(\mathbf{a}, w)] := \{\mathbf{g}_t : t \in \mathbb{T}_r\}$ the unique tropical point associated to B determined by $\mathbf{g}_w = \mathbf{a}$.
- Let $b = \mathbf{u}_w^{\mathbf{a}} \in \mathcal{A}(B)$ be a cluster monomial in seed Σ_w. Then b corresponds to tropical point associated to B via

$$b \mapsto [\mathbf{g}(b)] := [(\mathbf{a}, w)],$$

which is well-defined, i.e., it does not depend on the choice of w and $\mathbf{a}$.

Cluster monomials and cluster-additive functions

Recall that $u^{A^{\mathrm{T}}} : \mathbb{Z} \times [1, r] \to \mathbb{Q}(u_1, \ldots, u_r)$ denote the generic frieze pattern associated to the transposed Cartan matrix A^{T}.

Proposition

Let $b = \mathbf{u}_t^{\mathbf{a}} = u_{1;t}^{a_1} \cdots u_{r;t}^{a_r}$ *a cluster monomial of* $\mathcal{A}(-B_A^{\mathrm{T}})$. *Then*

(i) $T_{[\mathbf{g}(b)]}(m, i) = (u^{A^{\mathrm{T}}}(m, i) \mid\mid b)_d$ *for any* $(m, i) \in \mathbb{Z} \times [1, r]$;

(ii) $T_{[\mathbf{g}(b)]} = a_1 T_{[\mathbf{g}(u_{1;t})]} + \ldots + a_r T_{[\mathbf{g}(u_{r;t})]}$.

(iii) *Suppose that* $u_{k;t}$ *appears in the generic frieze pattern, say,* $u_{k;t} = u^{A^{\mathrm{T}}}(n, j)$. *Then* $T_{[\mathbf{g}(u_{k;t})]} = h^{(n,j)}$ *is the cluster-hammock function at vertex* (n, j).

Corollary

Ringel's second conjecture is true.

Proof: Key reasons: In the finite type case, (i) each cluster-additive function can be given by a cluster monomial of $\mathcal{A}(-B_A^{\mathrm{T}})$ and (ii) each cluster variable of $\mathcal{A}(-B_A^{\mathrm{T}})$ corresponds to a cluster-hammock function.

Duality of d-compatibility degree on the acyclic belt

We mainly use cluster algebra $\mathcal{A}(-B_A^{\mathrm{T}})$ to study the cluster-additive functions associated to A.

Actually, cluster algebra $\mathcal{A}(B_A)$ can be also used to study "some" nice cluster-additive functions associated to A, for example, cluster-hammock functions.

Theorem (C.-de St. Germain-Lu)

Let $h^{(n,j)}$ be the cluster-hammock function associated to A. Then

$$(u^{A^{\mathrm{T}}}(m,i) \mid\mid u^{A^{\mathrm{T}}}(n,j))_d = h^{(n,j)}(m,i) = (u^{A}(n,j) \mid\mid u^{A}(m,i))_d.$$

Namely, for any two cluster variables x, z on the acyclic belt of $\mathcal{A}(B_A)$, we have

$$(x \mid\mid z)_d = (z^{\vee} \mid\mid x^{\vee})_d,$$

where $x^{\vee}$ and $z^{\vee}$ are the corresponding cluster variables in $\mathcal{A}(-B_A^{\mathrm{T}})$.

In general, the above duality does not hold for arbitrary cluster variables x and z.

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Thank you!