Examples of quiver mutation loops and partition q-series related to affine quivers of type A

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based on the master thesis by Koki Oya (Kansai University)

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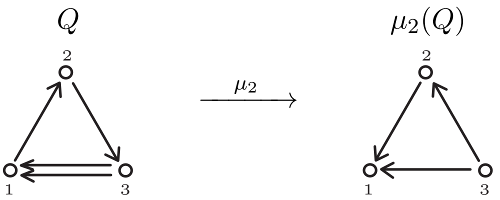
Contents

§1. Introduction

- Quiver mutations (Formin and Zelevinsky, 2001)



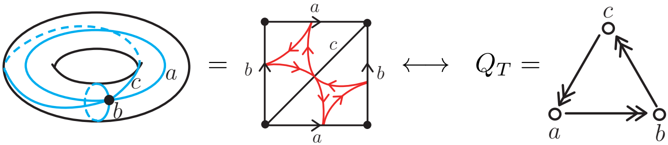
- Partition q-series are defined by using purely combinatorial data (Akishi Kato and Yuji Terashima, 2015):

$$\gamma = \begin{cases} \boldsymbol{m}: \ Q = Q(0) \xrightarrow{\mu_{m_1}} Q(1) \xrightarrow{\mu_{m_2}} \cdots \xrightarrow{\mu_{m_T}} Q(T), \\ \text{(its boundary condition)} = (Q(T) \stackrel{\sim}{\to} Q) \end{cases}$$

$$\longmapsto \quad \text{(partition } q\text{-series)}$$

[KT1] A. Kato and Y. Terashima, CMP **336** (2015), 811–830.
[KT2] A. Kato and Y. Terashima, CMP **338** (2015), 457–481.

- The origin of partition q-series is cluster partition functions of 3-dimensional gauge theories (Yuji Terashima and Masahito Yamazaki, 2014).
- There are two partition q-series denoted by $Z(\gamma)$ and $Z'(\gamma)$; the former contains y-variables and the later does not.
  - $Z'(\gamma)$ are related to some quantum invariant such as the Kashaev invariant.



$$Z'(\gamma) = \sum_{k=0}^{\infty} \frac{1}{(q)_k (q^{-1})_k} = \lim_{N \to \infty} \frac{\langle K \rangle_N}{N^2}$$

for γ induced from $\mu_c \circ \mu_a$ and the boundary condition given by $2\pi/3$-rotation, where $\langle K \rangle_N$ is the Kashaev invariant of the figure-eight knot K.

[TY] Y. Terashima and M. Yamazaki, Prog. Theor, Exp. Phys. (2014), 023B01, (37 pages).

- Kato and Terashima [KT1] show that the "fermionic character formula" conjectured by Kuniba, Nakanishi and Suzuki [KNS] is realized as $Z'(\gamma)$ by taking some special quiver and a mutation sequence.
- $Z(\gamma) \in \widehat{\mathbb{A}}_Q$, which is a non-commutative algebra that the quantum dilogarithm series inhabit.
- Kato and Terashima [KT2] show that $Z(\gamma)$ for reddening sequences are equivalent to combinatorial DT invariants introduced by Keller [K1, K2, K3].
- There are a few examples of computation of partition q-series [KT1, KT2].

[KNS] A. Kuniba, T. Nakanishi and J. Suzuki, Mod. Phys. Lett. A **8** (1993), 1649–1659.

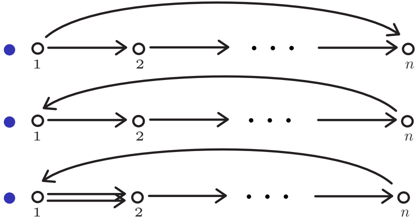
[K1] B. Keller, *On cluster theory and quantum dilogarithm identities*, in: "Representations of algebras and related topics", 85–111, 2011.

[K2] B. Keller, *Cluster algebras and derived categories*, in: "Derived categories in algebraic geometry", 123–183, 2012.

[K3] B. Keller, *Quiver mutation and combinatorial DT-invariants*, DMTCS Proceedings Series Volume AS, Nancy, France, 2013, 9–20, https://www.irif.fr/~chapuy/Archives/fpsac13/pdfAbstracts/dmAS0104.pdf.

The aim of my talk:

to explain main results in Oya's master thesis (2023) in Kansai University: For quivers



systematical examples of special mutation sequences and boundary conditions are given, and their partition q-series with y-variables are computed, though some cases remain conjecture.

It is remarkable that the Cartan matrices of the (affine) Dynkin diagrams of type A appear in the numerators in coefficients of y-variables in these partition q-series.

[O] Koki Oya, *Maximal green sequences and partition series related with affine quivers of type* A, (in Japanese), Kansai University, Master thesis, 2023.

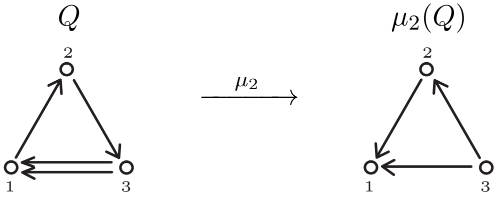
§2. Partition q-series of mutation loops

Throughout this talk, any quiver is assumed to be finite and does not have a loop and a 2-cycle.

Definition 2.1 (Quiver mutations)

Let Q be a quiver and k be its vertex. Then a new quiver $\mu_k(Q)$ is obtained by the following 3 steps:

1. for each path $i \longrightarrow k \longrightarrow j$, add a new arrow $i \longrightarrow j$,
2. reverse orientations of all arrows adjacent to k,
3. remove the arrows in a maximal set of pairwise disjoint 2-cycles.



From now on, we regard the vertex set Q_0 as $\{1,2,\ldots,n\}$:

$$Q_0 = \{1,2,\ldots,n\}.$$

A mutation sequence is a finite sequence $\boldsymbol{m} = (m_1, m_2, \ldots, m_T)$ of vertices of Q.
If the final quiver $\mu_{\boldsymbol{m}}(Q)$ obtained by applying

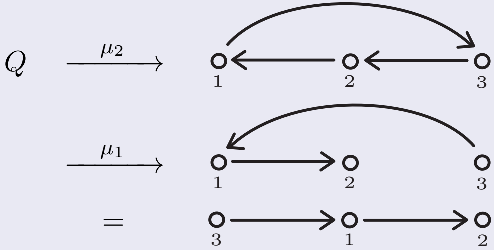
$$\mu_{\boldsymbol{m}} := \mu_{m_T} \circ \cdots \circ \mu_{m_2} \circ \mu_{m_1}$$

to Q, is isomorphic to the initial quiver Q, then we have a bijection $\varphi : \mu_{\boldsymbol{m}}(Q)_0 \longrightarrow Q_0$. The triplet $\gamma = (Q; \boldsymbol{m}, \varphi)$ is called a mutation loop, and φ is called the boundary condition of γ.

Example 2.2

Let $Q =$ 

. Then



Thus we have a mutation loop $(Q; \boldsymbol{m}, \varphi)$, where

$$\boldsymbol{m} = (2, 1),\ \varphi(3) = 1,\ \varphi(1) = 2,\ \varphi(2) = 3.$$

Framed quivers

Given a quiver Q, a new quiver $Q^\sharp$ is constructed by adding a new vertex i' and arrow $i \longrightarrow i'$ for each vertex $i \in Q_0$:

$$Q_0^\sharp = Q_0 \sqcup \{ \, i' \mid i \in Q_0 \, \}, \quad Q_1^\sharp = Q_1 \sqcup \{ \, i \longrightarrow i' \mid i \in Q_0 \, \}.$$

The quiver $Q^\sharp$ is called the framed quiver associated with Q, and i' is called a frozen vertex of $Q^\sharp$.

Similarly, we have a quiver ${}^\sharp Q$ given by

$${}^\sharp Q_0 = Q_0 \sqcup \{ \, i' \mid i \in Q_0 \, \}, \quad {}^\sharp Q_1 = Q_1 \sqcup \{ \, i' \longrightarrow i \mid i \in Q_0 \, \}.$$

The quiver ${}^\sharp Q$ is called the co-framed quiver associated with Q.

Let $\mathrm{Mut}(Q^\sharp)$ be the set of quivers which can be obtained from $Q^\sharp$ by applying mutations at non-frozen vertices.

Definition 2.3 (Keller [K1]; Brüstle-Dupont-Pérotin [BDP])

Let $R \in \mathrm{Mut}(Q^\sharp)$, and i be a non-frozen vertex of $Q^\sharp$.

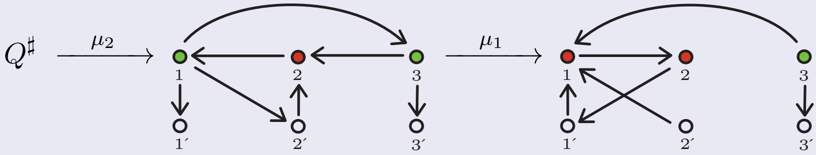
(1) i is green in R if there is no arrow from a frozen vertex to i,

(2) i is red in R if there is no arrow from i to a frozen vertex.

Example 2.4

For the quiver $Q =$ 

,



[BDP] T. Brüstle, G. Dupont and M. Pérotin, IMRN 2014, No. 16, 4547–4586.

$\boldsymbol{c}$-vectors

Let Q be a quiver with n vertices, and $R \in \mathrm{Mut}(Q^\sharp)$. For each $i, j \in Q_0$ set

$$c_{i,j}(R) := \sharp\{\ \alpha \in R_1 \mid \alpha : i \longrightarrow j'\ \} - \sharp\{\ \alpha \in R_1 \mid \alpha : j' \longrightarrow i\ \},$$

and define the row vector $\boldsymbol{c}_i(R)$ by

$$\boldsymbol{c}_i(R) := (c_{i,1}(R), \ldots, c_{i,n}(R)).$$

For $i \in Q_0$,

$$i \text{ is green in } R \iff \text{all entries of } \boldsymbol{c}_i(R) \text{ are non-negative},$$
$$i \text{ is red in } R \iff \text{all entries of } \boldsymbol{c}_i(R) \text{ are non-positive}.$$

Theorem 2.5 (sign coherence theorem; Derksen-Weyman-Zelevinsky, Nagao, Brüstle-Dupont-Pérotin [BDP])

For all $R \in \mathrm{Mut}(Q^\sharp)$, any vertex in Q is green or red in R.

Let $\boldsymbol{m} = (m_1, m_2, \dots, m_T)$ be a mutation sequence of Q, and we set $Q(0) := Q$ and

$$Q(t) := (\mu_{m_t} \circ \cdots \circ \mu_{m_2} \circ \mu_{m_1})(Q)$$

for $t = 1, \dots, T$. We introduce the sign ε_t of the mumation μ_{m_t} by

$$\varepsilon_t = \begin{cases} 1 & (\text{if } m_t \text{ is green in } Q(t-1)), \\ -1 & (\text{if } m_t \text{ is red in } Q(t-1)). \end{cases}$$

s- and k-variables for $\gamma = (Q, m = (m_1, \ldots, m_T), \varphi)$

(i) For a vertex $i \in Q_0$, introduce an s-variable s_i.

(ii) In order of $t = 1, \ldots, T$, add a new s-variable s'_{m_t} for m_t.

(iii) For a vertex $i \in Q_0$, identify s_i with the last added s-variable for i in the final quiver under φ.

(iv) For each $t = 1, \ldots, T$, define $k_t, k_t^\vee$ by

$$k_t = \begin{cases} s_{m_t} + s'_{m_t} - \sum_{i \to m_t} s_i & (\text{if } \varepsilon_t = 1), \\ \sum_{m_t \to j} s_j - (s_{m_t} + s'_{m_t}) & (\text{if } \varepsilon_t = -1), \end{cases} \tag{2.1}$$

$$k_t^\vee = \begin{cases} s_{m_t} + s'_{m_t} - \sum_{m_t \to j} s_j & (\text{if } \varepsilon_t = 1), \\ \sum_{i \to m_t} s_i - (s_{m_t} + s'_{m_t}) & (\text{if } \varepsilon_t = -1), \end{cases} \tag{2.2}$$

where s_{m_t}, s'_{m_t} are the last added s-variables for m_t in $Q(t-1), Q(t)$, respectively.

If the linear equations (2.1) running over $t = 1, \ldots, T$ can be solved with respect to s-variables, then γ is called non-degenerate.

Assume that γ is non-degenerate. Then all $k_t^\vee$ are expressed by $\mathbb{Q}$-linear combinations of $k_1, \ldots, k_T$.

Let q be an indeterminate. For each t, a weight function $W_{m_t} : \mathbf{N}^T \longrightarrow \mathbb{Q}(q^{\frac{1}{2}})$ and a row vector $\boldsymbol{\alpha}_t \in \mathbf{N}^n$ are defined by

$$W_{m_t}(k_1, \ldots, k_T) = \frac{q^{\frac{\varepsilon_t}{2} k_t k_t^\vee}}{(q^{\varepsilon_t})_{k_t}} \quad ((k_1, \ldots, k_T) \in \mathbf{N}^T),$$

$$\boldsymbol{\alpha}_t = \varepsilon_t \boldsymbol{c}_{m_t}(Q(t-1)^\sharp),$$

where

$$(q^{\varepsilon_t})_{k_t} = \prod_{i=1}^{k_t} (1 - q^{\varepsilon_t i}), \tag{2.3}$$

which is called a $\boldsymbol{q}$-Pochhammer symbol, and $Q(t-1)^\sharp = (\mu_{m_{t-1}} \circ \cdots \circ \mu_{m_2} \circ \mu_{m_1})(Q^\sharp)$.

Partition q-series

A partition q-series is defined as an element in the formal quantum affine space $\widehat{\mathbb{A}}_Q$ that the quantum dilogarithm series inhabit.

Definition 2.6 (Kato-Terashima [KT2])

Let $\gamma = (Q, \boldsymbol{m}, \varphi)$ be a non-degenerate mutation loop, and $\boldsymbol{m} = (m_1, \dots, m_T)$. The element

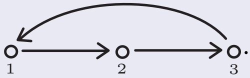
$$Z(\gamma) = \sum_{\boldsymbol{k}=(k_1,\dots,k_T)\in\mathbf{N}^T} \Big(\prod_{t=1}^{T} W_{m_t}(\boldsymbol{k})\Big) y^{\sum_{t=1}^{T} k_t \boldsymbol{\alpha}_t} \in \widehat{\mathbb{A}}_Q \tag{2.4}$$

is called the partition q-series associated with γ.

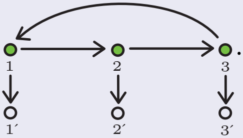
The definition of $\widehat{\mathbb{A}}_Q$ would be later explained.

Example 2.7 ($A_2^{(1)}$-quiver [KT2])

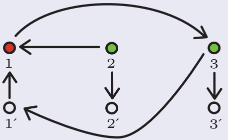
Let us consider the quiver

$Q =$ 

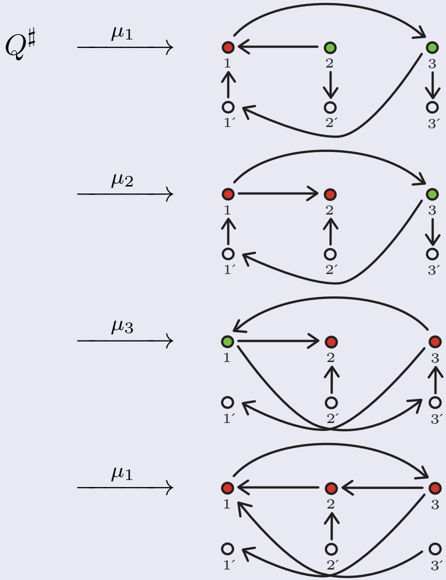
Its framed quiver is

$Q^\sharp =$ 

Then

$Q^\sharp \xrightarrow{\mu_1}$ 

Example 2.7 (continued)



Example 2.7 (continued)

By setting $\boldsymbol{m} = (1, 2, 3, 1)$ and $\varphi = (1\ 3)$, we have a mutation loop $\gamma = (Q, \boldsymbol{m}, \varphi)$.

Introduce initial s-variables s_1, s_2, s_3, and s-variables s'_1, s'_2, s'_3, s''_1 corresponding to the vertices in $\boldsymbol{m}$. Under the boundary condition $\varphi = (1\ 3)$, we identify

$$s_1 = s'_3, \quad s_2 = s'_2, \quad s_3 = s''_1.$$

Define k-variables $k_1, \ldots, k_4$ and $k_1^\vee, \ldots, k_4^\vee$ as follows:

$$\begin{aligned}
k_1 &= s_1 + s'_1 - s_3, & k_1^\vee &= s_1 + s'_1 - s_2,\\
k_2 &= s_2 + s'_2 = 2s_2, & k_2^\vee &= 2s_2 - s'_1,\\
k_3 &= s_3 + s'_3 - s'_1 = s_3 + s_1 - s'_1, & k_3^\vee &= s_3 + s_1,\\
k_4 &= s'_1 + s''_1 - s'_3 = s'_1 + s_3 - s_1, & k_4^\vee &= s'_1 + s_3 - s_2
\end{aligned}$$

Example 2.7 (continued)

These equations can be solved as

$$s_1 = \frac{k_1 + k_3}{2}, \quad s_1' = \frac{k_1 + k_4}{2}, \quad s_2 = \frac{k_2}{2}, \quad s_3 = \frac{k_3 + k_4}{2}.$$

Thus γ is non-degenerate, and

$$k_1^\vee = \frac{2k_1 - k_2 + k_3 + k_4}{2}, \qquad k_2^\vee = \frac{-k_1 + 2k_2 - k_4}{2},$$
$$k_3^\vee = \frac{k_1 + 2k_3 + k_4}{2}, \qquad k_4^\vee = \frac{k_1 - k_2 + k_3 + 2k_4}{2}.$$

On the other hand,

$$\begin{aligned}
\boldsymbol{\alpha}_1 &= \boldsymbol{c}_1(Q^\sharp) = (1,0,0),\\
\boldsymbol{\alpha}_2 &= \boldsymbol{c}_2(Q(1)^\sharp) = (0,1,0),\\
\boldsymbol{\alpha}_3 &= \boldsymbol{c}_3(Q(2)^\sharp) = (1,0,1),\\
\boldsymbol{\alpha}_4 &= \boldsymbol{c}_4(Q(3)^\sharp) = (0,0,1).
\end{aligned}$$

Example 2.7 (continued)

It follows that

$$Z(\gamma) = \sum_{(k_1,k_2,k_3,k_4)\in\mathbf{N}^4} \frac{q^{\frac{1}{2}(k_1^2-k_1k_2+k_1k_3+k_1k_4+k_2^2-k_2k_4+k_3^2+k_3k_4+k_4^2)}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}(q)_{k_4}} \times y^{(k_1+k_3,k_2,k_3+k_4)}.$$

The quadratic form of the numerator of the coefficients of y is given by

$$\frac{1}{4}(k_1\ k_2\ k_3\ k_4)\begin{pmatrix} 2 & -1 & 1 & 1 \\ -1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix}.$$

The non commutative algebra $\widehat{\mathbb{A}}_Q$

Let Q be a quiver. For $i, j \in Q_0$ we set

$$b_{ij}(Q) = \sharp\{ \alpha \in Q_1 \mid \alpha : i \longrightarrow j \} - \sharp\{ \alpha \in Q_1 \mid \alpha : j \longrightarrow i \}.$$

A skew-symmetric bilinear form $\langle \ , \ \rangle : \mathbf{Z}^n \times \mathbf{Z}^n \longrightarrow \mathbb{Z}$ is defined by

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = b_{ij}(Q) \qquad (i, j \in Q_0), \tag{2.5}$$

where "$\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$" is the standard basis of

$$\mathbf{Z}^n = \{ (a_1, \ldots, a_n) \mid a_i \in \mathbb{Z} \ (i = 1, \ldots, n) \}.$$

Consider the commutative algebra $R = \mathbb{Q}(q^{\frac{1}{2}})$. Introducing formal symbols $y^{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathbf{N}^n = \{ (a_1, \ldots, a_n) \in \mathbf{Z}^n \mid a_i \geq 0 \ (i = 1, \ldots, n) \}$, we define the free R-module

$$\mathbb{A}_Q = \bigoplus_{\boldsymbol{\alpha} \in \mathbf{N}^n} R y^{\boldsymbol{\alpha}}.$$

The R-module $\mathbb{A}_Q$ is an associative algebra with the product

$$y^{\boldsymbol{\alpha}} y^{\boldsymbol{\beta}} = q^{\frac{1}{2}\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle} y^{\boldsymbol{\alpha}+\boldsymbol{\beta}} \tag{2.6}$$

and the identity element $1_{\mathbb{A}_Q} := y^{\mathbf{0}}$. Since the product of $\mathbb{A}_Q$ is compatible with the grading by $y^{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha} \in \mathbf{N}^n$) it induces an R-algebra structure on

$$\widehat{\mathbb{A}}_Q := \prod_{\boldsymbol{\alpha} \in \mathbf{N}^n} R y^{\alpha}.$$

Remark 2.8

Setting $y_i := y^{\boldsymbol{e}_i}$ $(i = 1, \ldots, n)$,

we have

$$y^{\boldsymbol{\alpha}} = q^{-\frac{1}{2} \sum\limits_{1 \leq i < j \leq n} b_{ij}(Q) \alpha_i \alpha_j} y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n},$$

$$y_i y_j = q^{b_{ij}(Q)} y_j y_i$$

for all $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $i, j \in \{1, \ldots, n\}$.

A mutation sequence $\boldsymbol{m}$ of Q is said to be reddening if all vertices in $\mu_{\boldsymbol{m}}(Q)$ are red.

Theorem 2.9 (Keller [K3])

For two reddening sequences $\boldsymbol{m}, \boldsymbol{m}'$ of Q, there is an isomorphism $\mu_{\boldsymbol{m}}(Q^\sharp) \longrightarrow \mu_{\boldsymbol{m}'}(Q^\sharp)$ whose restriction to the frozen vertices is identity.

Theorem 2.10 (Brüstle-Dupont-Pérotin [BDP])

For a reddening sequence $\boldsymbol{m}$ of Q, there is an isomorphism $\mu_{\boldsymbol{m}}(Q^\sharp) \longrightarrow {}^\sharp Q$ whose restriction to the frozen vertices is identity.

By Theorem 2.10 any reddening sequence $\boldsymbol{m}$ of Q gives rise to a canonical boundary condition $\varphi : \mu_{\boldsymbol{m}}(Q) \longrightarrow Q$. The mutation loop $(Q; \boldsymbol{m}, \varphi)$ is said to be a reddening mutation loop corresponding to $\boldsymbol{m}$.

Quantum dilogarithm series $\mathbb{E}(y;q)$

The quantum dilogarithm series $\mathbb{E}(y;q)$ is a formal power series in $\mathbb{Q}(q^{\frac{1}{2}})[[y]]$ defined by

$$\mathbb{E}(y;q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k}{2}}}{(q)_k} y^k = \sum_{k=0}^{\infty} \frac{q^{-\frac{k^2}{2}}}{(q^{-1})_k} y^k. \tag{2.7}$$

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $\varepsilon \in \{\pm 1\}$ we set

$$\mathbb{E}(y^{\boldsymbol{\alpha}}; q^{\varepsilon}) = \sum_{k=0}^{\infty} \frac{q^{-\varepsilon \frac{k^2}{2}}}{(q^{-\varepsilon})_k} y^{k\boldsymbol{\alpha}}. \tag{2.8}$$

Furthermore, for a mutation sequence $\boldsymbol{m} = (m_1, \ldots, m_T)$ of Q, we define $\mathbb{E}(Q;\boldsymbol{m}) \in \widehat{\mathbb{A}}_Q$ by

$$\mathbb{E}(Q;\boldsymbol{m}) := \mathbb{E}(y^{\boldsymbol{\alpha}_1}; q^{\varepsilon_1}) \mathbb{E}(y^{\boldsymbol{\alpha}_2}; q^{\varepsilon_2}) \cdots \mathbb{E}(y^{\boldsymbol{\alpha}_T}; q^{\varepsilon_T}). \tag{2.9}$$

Theorem 2.11 (Keller [K2]; Nagao)

Let $\boldsymbol{m}, \boldsymbol{m}'$ be two mutation sequences of Q. If there is an isomorphism $\mu_{\boldsymbol{m}}(Q^\sharp) \longrightarrow \mu_{\boldsymbol{m}'}(Q^\sharp)$ whose restriction to the frozen vertices is identity, then $\mathbb{E}(Q; \boldsymbol{m}) = \mathbb{E}(Q; \boldsymbol{m}')$.

By the above theorem, $\mathbb{E}(Q; \boldsymbol{m}) \in \widehat{\mathbb{A}}_Q$ does not depend on the choice of reddening sequences $\boldsymbol{m}$. The power series $\mathbb{E}(Q; \boldsymbol{m})$ is called the combinatorial DT-invariant of Q, which is introduced by Keller.

Theorem 2.12 (Kato-Terashima [KT2])

Let $\gamma = (Q, \boldsymbol{m}, \varphi)$ be a reddening mutation loop. Then

$$Z(\gamma) = \overline{\mathbb{E}(Q; \boldsymbol{m})}, \tag{2.10}$$

where $\overline{}$ in the RHS is the anti-automorphism on $\widehat{\mathbb{A}}_Q$ over $\mathbb{Q}$ determined by $\overline{y^\alpha} = y^\alpha$ $(\alpha \in \mathbf{N}^n)$, $\overline{q} = q^{-1}$, where $n = \sharp Q_0$.

Maximal green sequences

The mutation sequence $\boldsymbol{m} = (m_1, m_2, \ldots, m_T)$ of Q is said to be green if $\varepsilon_t = 1$, that is, m_t is green in $Q(t-1)$ for all t. If all vertices in $Q(T)$ are red, then $\boldsymbol{m}$ is called a maximal green sequence. A maximal green sequence is reddening, however the converse is not true.

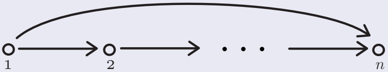
§3. Partition q-series related to affine quivers of type A

Consider a special A-quiver $\overrightarrow{A}$ such as

There are two affinizations of $\overrightarrow{A}$ according to whether it has a cycle or not.

Theorem 3.1 (Oya [O])

For the affine A-quiver

$Q =$ 

the mutation sequence $\boldsymbol{m} = (1, 2, \ldots, n-1, n)$ is maximal green. The induced boundary condition is id, and the partition q-series of $\gamma = (Q, \boldsymbol{m}, \mathrm{id})$ is given by

Theorem 3.1 (Oya [O] (Continued))

$$Z(\gamma) = \sum_{(k_1,\ldots,k_n)\in\mathbf{N}^n} \frac{q^{\frac{1}{2}\left(\sum_{i=1}^{n} k_i^2 - \sum_{i=1}^{n-1} k_i k_{i+1} - k_1 k_n\right)}}{(q)_{k_1}\cdots(q)_{k_n}} y^{(k_1,\ldots,k_n)}. \quad (3.1)$$

Remark 3.2

(1) The exponent of the numerator of the coefficient of $y^{(k_1,\ldots,k_n)}$ in (3.1) is expressed as

$$\frac{1}{4}\boldsymbol{k}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \ddots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}\boldsymbol{k}^{\mathrm{T}},$$

which coincides with the Cartan matrix of type affine A_{n-1}.

Remark 3.2 (Continued)

(2) (Kato and Terashima [KT1])
Let Q be an alternative Dynkin quiver of type A, D or E, and γ be a special mutation loop of Q. Then the partition q-series $Z'(\gamma)$ without y variables is given by

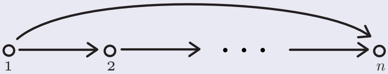
$$Z'(\gamma) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbf{N}^n} \frac{q^{\boldsymbol{k}C^{-1}\boldsymbol{k}^{\mathrm{T}}}}{(q)_{k_1}\cdots(q)_{k_n}},$$

where C is the Cartan matrix of the underlying Dynkin diagram of Q.

Combining Theorems 3.1 and 2.12 we have:

Corollary 3.3

The combinatorial DT invariant associated with the affine A-quiver

$Q =$ 

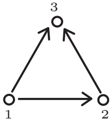
is given by

$$\mathbb{E}(Q,\boldsymbol{m}) = \sum_{(k_1,\dots,k_n)\in\mathbf{N}^n} \frac{q^{-\frac{1}{2}\left(\sum_{i=1}^{n} k_i^2 - \sum_{i=1}^{n-1} k_i k_{i+1} - k_1 k_n\right)}}{(q^{-1})_{k_1}\cdots(q^{-1})_{k_n}} y^{(k_1,\dots,k_n)},$$

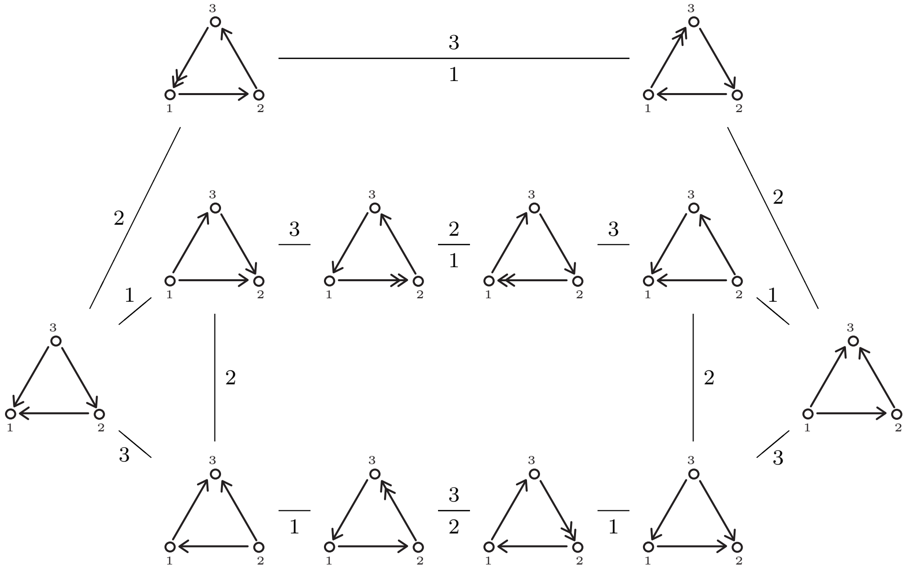
where $\boldsymbol{m} = (1,2,\dots,n-1,n)$.

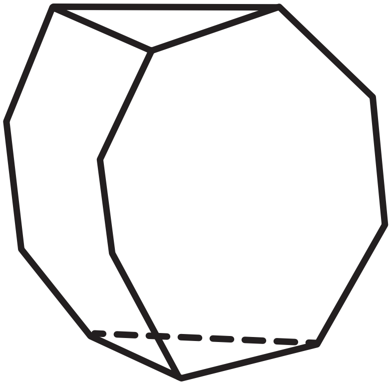
Some associahedron

Consider the quiver in Theorem 3.1 in the case $n=3$:

$Q =$ 

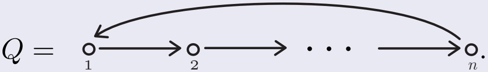
Then we have the following polyhedron:





Theorem 3.4 (Oya [O])

Let $n \geq 3$, and consider the affine A-quiver



Then $\boldsymbol{m} = (1, 2, \ldots, n-1, n, n-2, n-3, \ldots, 1)$ is a maximal green sequence of Q, and the induced boundary condition is given by $\varphi = (n \; n-2 \; n-3 \; \cdots \; 3 \; 2 \; 1)$.

Conjecture 3.5 (Oya [O])

For the mutation loop $\gamma = (Q, \boldsymbol{m}, \varphi)$ given in Theorem 3.5,

$$Z(\gamma) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbf{N}^n} \frac{q^{\frac{1}{4}\boldsymbol{k}A\boldsymbol{k}^{\mathrm{T}}}}{(q)_{k_1}\cdots(q)_{k_n}} y^{\sum_{t=1}^{n} k_t \boldsymbol{\alpha}_t}, \tag{3.2}$$

where

$$\boldsymbol{\alpha}_t = \begin{cases} \boldsymbol{e}_t & (1 \leq t \leq n-1), \\ \sum_{i=1}^{2n-2-t} \boldsymbol{e}_i + \boldsymbol{e}_n & (n \leq t \leq 2n-2) \end{cases}$$

and A is given as follows:

Conjecture 3.5 (Oya [O] (Continued))

$$A = \left(\begin{array}{c|c} B & C \\ \hline C^{\mathrm{T}} & D \end{array}\right)$$

where

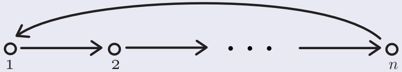
$$B = \begin{pmatrix} 2 & -1 & & & \mathrm{O} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ \mathrm{O} & & & -1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} \mathrm{O} & & & 1 & 1 \\ & & 1 & 1 & \\ & \iddots & \iddots & & \\ 1 & 1 & & & \mathrm{O} \\ 0 & -1 & \cdots & -1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 1 & & & 1 \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ 1 & & & 1 & 2 \end{pmatrix}.$$

It is confirmed by Oya that Conjecture 3.5 holds for $n = 3, 4, 5, 6$. So, as a corollary we have:

Corollary 3.6

Let $n = 3, 4, 5, 6$. The combinatorial DT invariant associated with the affine A-quiver

$Q =$ 

is given by

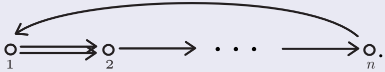
$$\mathbb{E}(Q, \boldsymbol{m}) = \sum_{\boldsymbol{k}=(k_1,\dots,k_n)\in\mathbf{N}^n} \frac{q^{-\frac{1}{4}\boldsymbol{k}A\boldsymbol{k}^{\mathrm{T}}}}{(q^{-1})_{k_1}\cdots(q^{-1})_{k_n}} y^{\sum_{t=1}^{n} k_t \boldsymbol{\alpha}_t},$$

where $\boldsymbol{m} = (1, 2, \dots, n-1, n, n-2, n-3, \dots, 1)$, $\boldsymbol{\alpha}_t$ and A are the same given in Conjecture 3.5.

For the quiver obtained by adding one arrow $1 \longrightarrow 2$ to the affine A-quiver with cycle, we have:

Theorem 3.7 (Oya [O])

Let $n \geq 3$, and consider the quiver

$Q =$ 

Then $\boldsymbol{m} = (1, n, n-1, \ldots, 3, 2, 4, 5, \ldots, n, 1)$ is a maximal green sequence of Q, and the induced boundary condition is $\varphi = (n \ n-1 \ \cdots \ 4 \ 3 \ 1)$.

Conjecture 3.8 (Oya [O])

For the mutation loop $\gamma = (Q, \boldsymbol{m}, \varphi)$ given in Theorem 3.7

$$Z(\gamma) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbf{N}^n} \frac{q^{\frac{1}{4}\boldsymbol{k}A\boldsymbol{k}^{\mathrm{T}}}}{(q)_{k_1}\cdots(q)_{k_n}} y^{\sum_{t=1}^{n} k_t \boldsymbol{\alpha}_t}, \tag{3.3}$$

where A is the following square matrix of size $(2n-2)$:

Conjecture 3.8 (Oya [O] (Continued))

$$A = \left(\begin{array}{c|c} D & C' \\ \hline C'^{\mathrm{T}} & B \end{array}\right),$$

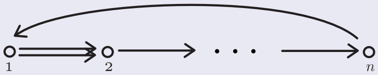
where B, D are the same given in Conjecture 3.5, and

$$C' = \begin{pmatrix} -2 & \text{O} & & & & 1 \\ -2 & & & & 1 & 1 \\ -2 & & & 1 & 1 & \\ \vdots & & \iddots & \iddots & & \\ -2 & & 1 & 1 & & \\ -2 & 1 & 1 & & & \\ -1 & 1 & & & \text{O} & \end{pmatrix}.$$

It is confirmed by Oya that Conjecture 3.8 holds for $n = 3, 4, 5$. So, we have:

Corollary 3.9

Let $n = 3, 4, 5$. The combinatorial DT invariant associated with the quiver

$Q =$ 

is given by

$$\mathbb{E}(Q, \boldsymbol{m}) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_n)\in\mathbf{N}^n} \frac{q^{-\frac{1}{4}\boldsymbol{k}A\boldsymbol{k}^{\mathrm{T}}}}{(q^{-1})_{k_1}\cdots(q^{-1})_{k_n}} y^{\sum_{t=1}^{n} k_t \boldsymbol{\alpha}_t},$$

where $\boldsymbol{m} = (1, n, n-1, \ldots, 3, 2, 4, 5, \ldots, n, 1)$, $\boldsymbol{\alpha}_t$ and A are the same given in Conjecture 3.8.

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