Categorifications of deformed symmetrizable generalized Cartan matrices

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I. Introduction

It is classical subject to understand categorical concepts in representation theory of quivers in terms of root systems :

Theorem (Gabriel)

Q : Dynkin quiver of type ADE

$$\left\{ \begin{array}{c} \text{indecomposable} \\ \text{rep of } Q \end{array} \right\} \Big/ \simeq \xrightarrow{1:1} \Delta^+(Q)$$

$$M \longmapsto \underline{\dim}\, M$$

$\rightsquigarrow$ Want to consider any orientation simultaneously.

① Gelfand - Ponomarev introduced an algebra, which contains path algebra for any orientation as sub. alg.

$$\underset{\bullet \longrightarrow \bullet \longleftarrow \bullet}{Q} \quad \underset{\text{"double"}}{\rightsquigarrow} \quad \Pi Q \quad \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} / \text{rel.}$$

② Geiss - Leclerc - Schröer introduced quiver algebras ass. w/ symmetrizable GCM C & its symmetrizer.

$$H(C, D, \Omega) \quad \underset{\text{"double"}}{\rightsquigarrow} \quad \Pi(C, D)$$

(When $C = {}^{T}C$, $D = \mathrm{Id}$, they are consistent to ①).

Aim of this talk

- E. Frenkel - Reshetikhin introduced a deformation of CM.

$$C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \quad \rightsquigarrow \quad C(q,t) = \begin{pmatrix} qt^{-1} + q^{-1}t & -(q+q^{-1}) \\ -1 & q^2t^{-1} + q^{-2}t \end{pmatrix}$$

- There are several known definition of deformation of GCM. (=: DGCM) beyond finite types ([Nakajima, Hernandez, Kimura-Pestun]).

We present a novel DGCM which is interpreted by graded structures of $\Pi(C,D)$.

$$\begin{tikzcd}
\Pi(C,D)\text{-modules} \arrow[rr, squiggly, "\text{categorically interpret}"] \arrow[d, squiggly] & & \text{symmetrizable GCM } C. \arrow[d, squiggly] \\
\text{graded } \Pi(C,D)\text{-modules} \arrow[rr, squiggly, "??"] & & C(q,t,\underline{\mu})
\end{tikzcd}$$

① We prove some purely combinatorial properties of $C(q,t,\underline{\mu})$ and relevant concepts (deformation of root system, braid group action e.t.c.).

② For finite types, " q-graded str on $\tilde{\Pi}(C,D)$ " is equivalent to consider certain quiver with potential in cluster theory ([Hernandez-Leclerc]).

⇝ Are there nice characterization of graded str. on $\Pi(C,D)$ for general types?

①②We make a toy model organizing $(q,t,\underline{\mu})$-comb. from several contexts.

II. PPA & DGCM

- $C = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$: symmetrizable GCM

  i.e. $\cdot\ c_{ii} = 2 \quad (\forall i \in I) \qquad (I := \{1, \cdots, n\})$

  $\cdot\ c_{ij} \in \mathbb{Z}_{\leq 0} \ (i \neq j), \quad c_{ij} < 0 \Leftrightarrow c_{ji} < 0$

  $\cdot\ {}^{\exists} D \in \mathrm{Mat}_n(\mathbb{Z}_{>0})$ diagonal s.t. ${}^{T}(DC) = DC$

- We fix a symmetrizer $D = \mathrm{diag}(d_1, \cdots, d_n)$.
- We assume C is connected.

- $r := \mathrm{lcm}(d_i \mid i \in I)$

  $g_{ij} := \gcd(|c_{ij}|, |c_{ji}|) \ (i \neq j)$

  $f_{ij} := |c_{ij}| / g_{ij} \qquad (i \neq j)$

(e.g.) $\begin{pmatrix} 2 & -2 \\ -4 & 2 \end{pmatrix} \rightsquigarrow D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$

$r = \mathrm{lcm}(2, 1) = 2 \qquad g_{12} = g_{21} = \gcd(2.4) = 2$

$f_{12} = \frac{2}{2} = 1 \qquad f_{21} = \frac{4}{2} = 2.$)

Def

acyclic orientation

$$Q := Q(C, \Omega)$$

$$Q_0 := I = \{1, \cdots, n\}$$

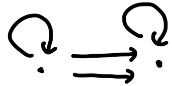
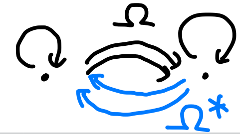
$$Q_1 := \{\alpha_{ij}^{(g)} \;;\; c_{ij} < 0,\ (i,j) \in \Omega,\ 1 \leq g \leq g_{ij}\} \sqcup \{\varepsilon_i \;;\; i \in I\}$$

$$\overset{\bullet}{i} \longleftarrow \overset{\bullet}{j} \qquad\qquad \begin{tikzcd} \underset{i}{\bullet} \arrow[loop above] \end{tikzcd}$$

e.g.) $C = \begin{pmatrix} 2 & -2 \\ -4 & 2 \end{pmatrix} \rightsquigarrow$

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[loop above, "\varepsilon_1"] \arrow[r, shift left, "\alpha_{21}^{(1)}"] \arrow[r, shift right, "\alpha_{21}^{(2)}"'] & \underset{2}{\bullet} \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

take orientation $(2,1)$

Def (GPPA) $\Big($ Q :  $\bar{Q}$:  $\Big)$

Define the PPA $\Pi := \Pi(C, D')$ as a double $\Bbbk\bar{Q}$ w/ rel. P1 ~ P3

$(D' = \ell D \ (\ell \in \mathbb{Z}_{>0}))$

e.g.) $C = \begin{pmatrix} 2 & -2 \\ -4 & 2 \end{pmatrix}$ $D = \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$

$\frac{r}{d_1} = 1$, $\frac{r}{d_2} = 2$. $\Big(\mathrm{diag}\big(\frac{r\ell}{d_i} \mid i \in I\big)$: left symmetrizer of ${}^{T}C$ $\Big)$

P1) $\varepsilon_i^{r\ell/d_i} = 0 \quad (i \in Q_0)$

P2) $\varepsilon_i^{f_{ij}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ji}} \quad (1 \leq g \leq g_{ij},\ C_{ij} < 0)$

P3) $(\forall i \in Q_0)$ $\displaystyle\sum_{\substack{j \in I \\ \text{s.t. } C_{ij} < 0}} \sum_{f=0}^{f_{ij}-1} \mathrm{sgn}(i,j)\, \varepsilon_i^{f} \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ij}-1-f} = 0$

$\left(\mathrm{sgn}(i,j) = \begin{cases} 1 & (i,j) \in \Omega \\ -1 & (i,j) \in \Omega^* \end{cases} \right)$

Rem ① Π does not depend on the choice of orientation up to isom.

② Π is finite dim'l / $\Bbbk$ $\Longleftrightarrow$ C : finite type.

- <u>Conventions of deformation parameters & gradings</u>

$\Gamma :=$ mult. abelian grp

gen. $\{q, t\} \amalg \{\mu_{ij}^{(g)} \mid c_{ij} < 0,\ 1 \leq g \leq g_{ij}\}$

rel. $\mu_{ij}^{(g)} \mu_{ji}^{(g)} = 1$

free abelian of rank $2 + \sum_{(i,j) \in \Omega} g_{ij}$

- We have $\underline{\mu}^{\mathbb{Z}} = \prod_{(i,j) \in \Omega} \prod_{g=1}^{g_{ij}} (\mu_{ij}^{(g)})^{\mathbb{Z}}$ & $\Gamma = q^{\mathbb{Z}} \times t^{\mathbb{Z}} \times \underline{\mu}^{\mathbb{Z}}$

- $\mathbb{Z}[\Gamma] \simeq$ the ring of Laurent poly. w/ $q, t, \mu_{ij}^{(g)}$ $(i,j) \in \Omega$.

- For finite types, $\forall (i,j) \in I \times I$, $\mu_{i,j} := \mu_{i,i_1} \cdot \mu_{i_1 i_2} \cdot \cdots \cdot \mu_{i_k, j}$ $(i \sim i_1 \sim i_2 \sim \cdots \sim i_k \sim j)$ (well-defined by rel.)

We define DGCM $C(q,t,\underline{\mu})$ of C as

$$C_{ij}(q,t,\underline{\mu}) := \begin{cases} q^{d_i}t^{-1} + q^{-d_i}t & (i=j) \\ -[f_{ij}]_{q^{d_i}} \sum_{g=1}^{g_{ij}} \mu_{ij}^{(g)} & (i\neq j) \end{cases}$$

<u>Rem</u> ① "$TC(q,t,\underline{\mu}) \neq (TC)(q,t,\underline{\mu})$"

② Our DGCM recovers

- Frenkel-Reshetikhin (all finite types after $\underline{\mu} \to 1$)
- Kimura-Pestun ($\forall\, i\sim j,\ f_{ij}=1$ or $f_{ji}=1$ $(*)$

  $\left.\begin{array}{l}\text{all finite and affine types} \\ \text{all symmetric types}\end{array}\right)$ satisfy $(*)$)

e.g. 1)

$$C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \rightsquigarrow C(q,t,\underline{\mu}) = \begin{pmatrix} qt^{-1} + q^{-1}t & -\mu_{12}(q+q^{-1}) \\ -\mu_{21} & q^2t^{-1} + q^{-2}t \end{pmatrix}$$

$$\det C(q,t,\underline{\mu}) = q^3t^{-2} + q^{-3}t^2 = q^3t^{-2}(1+q^{-6}t^4)$$

$$\widetilde{C}(q,t,\underline{\mu}) = \frac{q^{-3}t^2}{1+q^{-6}t^4} \boxed{\begin{pmatrix} q^2t^{-1} + q^{-2}t & \mu_{21}(q+q^{-1}) \\ \mu_{12} & qt^{-1} + q^{-1}t \end{pmatrix}} \ (*)$$

$$\in \mathrm{Mat}_I \mathbb{Z}[P_0]((t))$$

$$\widetilde{C}_{ij}(q,t) := \left[\widetilde{C}_{ij}(q,t)\right]_{\underline{\mu}=1} = \sum_{u,v} \tilde{c}_{ij}(u,v)\, q^u t^v \qquad (\tilde{c}_{ij}(u,v) \in \mathbb{Z})$$

- $\tilde{c}_{ij}(-u-6, v+4) = -\tilde{c}_{ij}(u,-v)$
- $(*)$ are invariant under $(q,t) \longleftrightarrow (q^{-1}, t^{-1})$ with non-negative coefficient.

e.g. 2)

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \text{(not invertible)}$$

$$C(q,t,\underline{\mu}) = \begin{pmatrix} qt^{-1} + q^{-1}t & -(\mu_{12}^{(1)} + \mu_{12}^{(2)}) \\ -(\mu_{21}^{(1)} + \mu_{21}^{(2)}) & qt^{-1} + q^{-1}t \end{pmatrix}$$

$$\det(C(q,t,\underline{\mu})) = q^2t^{-2} - (\mu_{12}^{(1)}\mu_{21}^{(2)} + \mu_{21}^{(1)}\mu_{12}^{(2)}) + q^{-2}t^2$$
$$\in (\mathbb{Z}[\Gamma_0]((t)))^{\times}$$

$\rightsquigarrow$ $C(q,t,\underline{\mu})$ is invertible in $\mathrm{Mat}_n(\mathbb{Z}[\Gamma_0]((t)))$.

(Γ - graded vector space)

$V = \bigoplus_{g \in \Gamma} V_g$: locally fin. (i.e. $\forall g \in \Gamma$, $\dim_{\mathbb{k}} V_g < \infty$)

$\rightsquigarrow$ • $\dim_\Gamma V = \sum_{g \in \Gamma} \dim_{\mathbb{k}}(V_g)\, g \in \mathbb{Z}[\![\Gamma]\!]$.

• For $x \in \Gamma$, $xV := \bigoplus_{g \in \Gamma} V_{x^{-1}g}$.

More generally, for $a = \sum_{g \in \Gamma} a_g g \in \mathbb{Z}_{\geq 0}[\![\Gamma]\!]$

$$V^{\oplus a} := \bigoplus_{g \in \Gamma} (gV)^{\oplus a_g}$$

($V^{\oplus a}$: locally fin $\Rightarrow \dim_\Gamma V^{\oplus a} = a \cdot \dim_\Gamma V$.)

Def (Γ-grading on Π)

$$\deg(\alpha_{ij}^{(g)}) = q^{-d_i f_{ij}} t \mu_{ij}^{(g)} \qquad \deg(\varepsilon_i) = q^{2d_i}$$

$$\left(\begin{array}{l} \Gamma = t^{\mathbb{Z}} \times \Gamma_0 \\ \rightsquigarrow \text{For } V = \bigoplus_{g \in \Gamma} V_g \ , \ V_n := \bigoplus_{g \in \Gamma_0} V_{t^n g} \quad (n \in \mathbb{Z}) \\ \qquad (V_{\geq n} := \bigoplus_{m \geq n} V_m \ , \ V_{>n} := \bigoplus_{m > n} V_m). \end{array} \right)$$

- Π satisfies $\Pi = \Pi_{\geq 0}$ & $\dim_{\mathbb{k}} \Pi_n < \infty$ $(\forall n \in \mathbb{Z}_{\geq 0})$

Def
- $\Pi\text{-mod}_{\Gamma}^{\geq n}$:= the category of Γ-graded Π-modules M s.t. $M = M_{\geq n}$ & $\dim_{\mathbb{k}} M_m < \infty$ $(\forall m \geq n)$
- $\Pi\text{-mod}_{\Gamma}^{+} := \bigcup_{n \in \mathbb{Z}} \Pi\text{-mod}_{\Gamma}^{\geq n}$

(They are abelian categories)

Def We have a filtration of subgrps $K(\Pi\text{-mod}_{\mathcal{T}}^{\geq n})$

$$\hat{K}(\Pi\text{-mod}_{\mathcal{T}}^{+}) := \varprojlim_{n} K(\Pi\text{-mod}_{\mathcal{T}}^{+}) / K(\Pi\text{-mod}_{\mathcal{T}}^{\geq n})$$

$\rightsquigarrow$ • $\hat{K}(\Pi\text{-mod}_{\mathcal{T}}^{+})$ has a natural $\mathbb{Z}[\Gamma_0]((t))$-mod. str.

$$\left(\begin{array}{l} a[M] = [M^{\oplus a_+}] - [M^{\oplus a_-}] \\ a_+, a_- \in \mathbb{Z}_{\geq 0}[\Gamma_0]((t)) \text{ s.t. } a = a_+ - a_- \end{array} \right.$$

• $\hat{K}(\Pi\text{-mod}_{\mathcal{T}}^{+})$ has a free $\mathbb{Z}[\Gamma_0]((t))$-basis

$\{[S_j]\}_{j\in I}$

simple module

<u>Rem</u> • Π can be considered as a Jacobian algebra of the quiver Q with the algebraic potential

$$W = \sum_{\substack{i,j\in I \\ i\sim j}} \sum_{g=1}^{g_{ij}} \mathrm{sgn}_\Omega(i,j)\, \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ij}}$$

with relation $(P1)$: $\varepsilon_i^{r\ell/d_i} = 0 \quad (i \in Q_0)$.

• Any G-grading (G: free abelian) s.t. W is homogeneous "factors through" $\Gamma' := \langle \mathrm{Im}(\deg : Q_1 \longrightarrow \Gamma) \rangle \subset \Gamma$

$$H := \langle [a] \mid a \in Q_1 \rangle / (W \text{ is homogeneous})$$

$$\begin{tikzcd}
H/\mathrm{tors} \arrow[rr, "\deg", "\sim"'] \arrow[dr] & & \Gamma' \arrow[dl] \\
& G &
\end{tikzcd}$$

<u>Aim</u> Explain DGCM via Γ-graded Euler-Poincaré pairing:

$$\langle M, N \rangle_\Gamma := \sum_{k=0}^{\infty} (-1)^k \dim_\Gamma \operatorname{tor}_k^\Pi (M^\phi, N).$$

where
- $(-)^\phi : \Pi\text{-mod}_\Gamma^+ \longrightarrow \Pi^{op}\text{-mod}_\Gamma^+$ induced from an involution $\phi : \Pi \longrightarrow \Pi^{op}$
  $$\phi(e_i) = e_i, \quad \phi(\alpha_{ij}^{(g)}) = \alpha_{ji}^{(g)} \quad \phi(\varepsilon_i) = \varepsilon_i$$
- tor_k^Π : k-th left derived functor of $M \mapsto M \otimes_\Pi N$

※ This pairing is well-defined as an element of $\mathbb{Z}[\![\Gamma]\!]$ by

$$\forall \gamma \in \Gamma \quad \operatorname{tor}_k^\Pi (M^\phi, N)_\gamma = 0 \qquad \text{for } k \gg 0.$$

Lem (Geiss-Leclerc-Schröer, Fujita-M).

$$q^{-2d_i}t^2 P_i \xrightarrow{\varphi^{(i)}} \bigoplus_{j\sim i} P_j^{\oplus(-q^{d_i}t\,C_{ij}(q,t,\underline{\mu})^{\phi})} \longrightarrow P_i \longrightarrow E_i \longrightarrow 0.$$

exact

(E_i: maximal self-extension of S_i)

$\mu_0(\varpi_i) = -\varpi_{i^*}$

$\phi(q)=q$ $\phi(t)=t$, $\phi(\mu_{ij}) = \mu_{ji}$

① C: infinite type $\Rightarrow$ $\mathrm{Ker}\,\varphi^{(i)} = 0$

② C: finite type $\Rightarrow$ $\mathrm{Ker}\,\varphi^{(i)} \simeq q^{-rh^\vee}t^h \mu_{i^*i} E_{i^*}$

$$\rightsquigarrow \quad \langle E_i, S_j\rangle_\Gamma = \begin{cases} \dfrac{q^{-d_i}t\left(C_{ij}(q,t,\underline{\mu}) - q^{-rh^\vee}t^h\mu_{ii^*}C_{i^*j}(q,t,\underline{\mu})\right)}{1-q^{-2rh^\vee}t^{2h}} & \text{(finite type)} \\ q^{-d_i}t\,C_{ij}(q,t,\underline{\mu}) & \text{(infinite type)} \end{cases}$$

$$\rightsquigarrow \quad \left(\langle E_i, S_j \rangle_\Gamma \right)_{i,j\in I} = \begin{cases} \dfrac{q^{-D} t \left(\mathrm{id} - q^{-rh^\vee} t^h \mu_{ii^*} \nu \right)}{1-(q^{-rh^\vee} t^h)^2} C(q,t,\underline{\mu}) \\ \qquad (\nu := (\delta_{ii^*})_{i,j\in I}) \\ q^{-D} t\, C(q,t,\underline{\mu}) \end{cases} \quad (\star)$$

Since $\langle P_i, S_j \rangle_\Gamma = \delta_{ij}$, we introduce a module $\overline{P}_i$
s.t. $\langle \overline{P}_i, E_j \rangle_\Gamma = \delta_{ij}$. $\left(\overline{P}_i := \left(\Pi / \Pi e_i \right) e_i \right)$

$$\rightsquigarrow \quad \mathrm{Id} = \left(\langle P_i, S_j \rangle_\Gamma \right)_{i,j\in I} = \left(\dim_\Gamma e_i \overline{P}_k \right)_{i,k\in I} \cdot \left(\langle E_l, S_j \rangle_\Gamma \right)_{l,j\in I}$$

By $(\star)$, we have a formula of $\widetilde{C}(q,t,\underline{\mu})$.

Cor In $\mathbb{Z}[\Gamma_0]((t))$,

① $$\tilde{C}_{ij}(q,t,\underline{\mu}) = \frac{q^{-d_i} t}{1-q^{-2rh^\vee} t^h} \left(\dim_{\mathbb{P}}(e_i \bar{P}_j) - q^{-rh^\vee} t^h \mu_{ii^*} \dim_{\mathbb{P}}(e_{i^*} \bar{P}_j) \right)$$

(finite type)

② $$\tilde{C}_{ij}(q,t,\underline{\mu}) = q^{-d_j} t \dim_{\mathbb{P}}(e_i \bar{P}_j)$$ (infinite type)

Thm (Fujita-M: positivity of infinite type)

C: infinite type $\Rightarrow \tilde{C}_{ij}(q,t,\underline{\mu}) \in \mathbb{Z}_{\geq 0}[\Gamma_0][[t]]$

($\because$ dimension of vector sp is non-negative.)

For finite types, (for simplicity $\underline{u} \to 1$)

<u>Cor</u> (Kashiwara - Oh ($q \to 1$), Fujita - M)

The coefficients $\{\tilde{c}_{ij}(u,v)\}_{u,v \in \mathbb{Z}}$ satisfy

① $\tilde{c}_{ij}(u,v) = -\tilde{c}_{ij^*}(u - rh^\vee, v+h)$ $(\forall u \leq 0, \forall v \geq 0)$

(periodicity of proj. resol.)

②. $\tilde{c}_{ij}(u,v) \geq 0$ $(-rh^\vee \leq u \leq 0, 0 \leq v \leq h)$

$\left(\dim_{q,t} e_i \bar{P}_j = q^{-d_i} t \sum_{u=0}^{rh^\vee} \sum_{v=0}^{h} \tilde{c}_{ij}(-u,v) q^{-u} t^{v} \right)$

③. $\tilde{c}_{ij}(-rh^\vee - u, h - v) = \tilde{c}_{ij^*}(u,v)$ $(-rh^\vee \leq \forall u \leq 0, 0 \leq \forall v \leq h)$

($\Bbbk$-duality $\mathbb{D}(\bar{P}_j^\phi) \simeq q^{2d_j - rh^\vee} t^{h-2} \bar{P}_{j^*}$)

Braid group action on deformed root lattice

$$Q_P := \mathbb{Q}(P) \otimes_{\mathbb{Z}} Q = \bigoplus_{i \in I} \mathbb{Q}(P)\alpha_i .$$

$$(\alpha_i, \alpha_j)_P := [d_i]_q C_{ij}(q, t, \underline{\mu})$$

(This satisfy $(ax, by)_P = a^{\phi} b (x, y)_P$, $(x, y)_P = (y, x)_P^{\phi}$
$(\forall x, y \in Q_P, a, b \in \mathbb{Q}(P))$)

- $\{\varpi_i^\vee\}$: coweights (dual basis of $\{\alpha_i\}_{i \in I}$ w.r.t. $(\ ,\ )_P$)
- $\alpha_i^\vee := q^{-d_i} t [d_i]_q^{-1} \alpha_i$ $\quad ((\alpha_i^\vee, \alpha_j)_P = q^{-d_i} t C_{ij}(q, t, \underline{\mu}))$
- $\tilde{\varpi}_i := [d_i]_q \varpi_i^\vee$

$\hat{K}(\Pi\text{-mod}_{\mathbb{P}}^{+})$ vs $Q_{\mathbb{P}}$

$$\mathbb{F} := \overline{\mathbb{Q}(\mathbb{P}_0)((t))} \quad \text{and} \quad Q_{\mathbb{F}} := Q_{\mathbb{P}} \otimes_{\mathbb{Q}(\mathbb{P})} \mathbb{F}.$$

<u>Lem</u> Take $\kappa_\ell \in \mathbb{F}$, s.t. $\kappa_\ell^2 = q^{r\ell}[r\ell]_q t^{-1}$

$\exists$ $\mathbb{F}$-linear isom $\chi_\ell : \hat{K}(\Pi\text{-mod}_{\mathbb{P}}^{+})_{\mathbb{F}} \longrightarrow Q_{\mathbb{F}}$

- $\alpha_i = \kappa_\ell \cdot \chi_\ell([S_i])$ $\qquad$ $\alpha_i^\vee = \kappa_\ell^{-1} \chi_\ell([E_i])$

  $\overline{\varpi}_i^\vee = \kappa_\ell^{-1} \Psi \chi_\ell([P_i])$ $\qquad$ $\overline{\varpi}_i = q^{-d_i} t \kappa_\ell \Psi \chi_\ell([\overline{P}_i])$

where $\underset{\mathbb{F}\text{-lin. auto}}{\Psi} := \begin{cases} \dfrac{\mathrm{id} - q^{-rh^\vee} t^h \nu}{1 - q^{-2rh^\vee} t^{2h}} & (\text{finite}) \\ \mathrm{id}. & (\text{infinite}) \end{cases}$

$(\nu(\alpha_i) = \mu_{i^*i} \alpha_{i^*})$

- $\langle x, y \rangle_{\mathbb{P}} = (\Psi \chi_\ell(x), \chi_\ell(y))_{\mathbb{P}}$ $\quad (\forall x, y \in \hat{K}(\Pi\text{-mod}_{\mathbb{P}}^{+})_{\mathbb{F}})$

q $\qquad\qquad$ q,t

<u>Def</u> (Chari, Bouwknegt-Pilch, finite types)

$\mathbb{Q}(\Gamma)$-linear auto T_i of Q_Γ

$$T_i x := x - (\alpha_i^\vee, x)_\Gamma \alpha_i \qquad (x \in Q_\Gamma)$$

($(q, t, \underline{\mu})$-analogue of simple refl)

- We will see T_i $(i \in I)$ define braid grp action by an analogue of [Amiot-Iyama-Reiten-Todolov].

<u>Fact</u> $J_i := \Pi(1-e_i)\Pi$

C : infinite type $\Rightarrow$ J_i has proj. dim at most 1.

$\rightsquigarrow$ We have $J_i \overset{L}{\otimes} M \in D^b(\Pi\text{-mod}_\Gamma^+)$ $\quad (\forall M \in \Pi\text{-mod}_\Gamma^+)$

By $[J_i e_i] = [P_j] - \delta_{ij}[E_i]$

$$[J_i \overset{L}{\otimes}_\pi M] = [M] - \langle E_i, M\rangle_P [S_i]$$

$$\Leftrightarrow \chi_\ell[J_i \overset{L}{\otimes}_\pi M] = \chi_\ell[M] - (\underline{I}\alpha_i^\vee, \chi_\ell[M])_P \alpha_i.$$

$\therefore$ C: infinite type $\Rightarrow \chi_\ell[J_i \overset{L}{\otimes} M] = T_i \chi_\ell[M]$.

<u>Thm</u> (Chari ($t, \underline{u} \to 1$), Fujita-M)

$\{T_i\}_{i\in I}$ defines an action of the braid grp.

ass. to $(W, \{s_i\}_{i\in I})$

$$\begin{cases} T_i T_j = T_j T_i & (c_{ij} = 0) \\ T_i T_j T_i = T_j T_i T_j & (c_{ij} c_{ji} = 1) \\ (T_i T_j)^k = (T_j T_i)^k & (c_{ij} c_{ji} = k \text{ w/ } k \in \{2,3\}) \end{cases}$$

(sketch) Infinite types

e.g.) $c_{ij} \cdot c_{ji} = 1$

- $J_i \overset{L}{\otimes}_\Pi J_j \overset{L}{\otimes}_\Pi J_i \simeq J_i \otimes J_j \otimes_\Pi J_i \simeq J_i J_j J_i$
- $J_i J_j J_i = J_j J_i J_j$ ([Buan-Iyama-Reiten-Scott, Fu-Geng])

$\rightsquigarrow$ $J_i \overset{L}{\otimes}_\Pi J_j \overset{L}{\otimes}_\Pi J_i \simeq J_j \overset{L}{\otimes}_\Pi J_i \overset{L}{\otimes}_\Pi J_j$

$\rightsquigarrow$ $T_i T_j T_i = T_j T_i T_j$

(By restricting untwisted affine type, we can prove for finite types.)

· <u>Numerical formula of $\tilde{C}(q,t,\underline{\mu})$ via braid grp action</u>

We have a nice filtration

$$\Pi = F_0 \supset F_1 \supset \cdots$$

$$F_k = J_{i_1} \cdot \cdots \cdot J_{i_k} \quad (\text{s.t. this satisfies } \bigcap_k F_k = 0)$$

$$F_{k-1} e_i / F_k e_i \simeq \begin{cases} J_{i_1} \cdots J_{i_{k-1}} \otimes_\Pi E_i & (i_k = i) \\ 0 & (\text{other}) \end{cases} \qquad (\forall k \geq 1)$$

$\therefore$ In $\hat{K}(\Pi\text{-mod}^{+}_{P})$, we have

$$[P_i] = \sum_{k=1}^{\infty} [F_{k-1} e_i / F_k e_i] = \sum_{k: i_k = i} [J_{i_1} \cdots J_{i_{k-1}} \otimes_\Pi E_i]$$

$$\xrightarrow[x_\ell]{} \quad \varpi_i = q^{-d_i} t \sum_{k: i_k = i} T_{i_1} \cdots T_{i_{k-1}} \alpha_i .$$

We take $(i_1, i_2, \cdots)$ be a seq in I

s.t.

$$\left.\begin{array}{lll} \bullet \text{ (finite type)} & (i_1, \cdots, i_\ell) : \text{red. word of } w_0 \in W & \\ & i_{k+\ell} = i_k^* \quad (\forall k) & \\ \bullet \text{ (infinite type)} & \text{subseq } (i_1, \cdots, i_k) \text{ is reduced } (\forall k) & \\ & |\{k \mid i_k = i\}| = \infty \quad (\forall i \in I) & \end{array}\right\} (*)$$

By the rel. $\alpha_i = \sum_{j \in I} C_{ji}(q, t, \underline{\mu}) \varpi_j$, we have

Thm (Hernandez-Leclerc (ADE & red. expression adapted to the quiver $q \to 1$, $\underline{\mu} \to 1$)
Fujita-M (general with $(*)$) finite

$$\widetilde{C}_{ij}(q, t, \underline{\mu}) = q^{-d_j} t \sum_{\substack{k \in \mathbb{Z}_{>0} \\ i_k = j}} (\varpi_i^\vee, T_{i_1} \cdots T_{i_{k-1}} \alpha_j)\, t^k$$

$(\forall i, j \in I)$

Thank you very much !!