Cluster transformations, the tetrahedron equation and three-dimensional gauge theories

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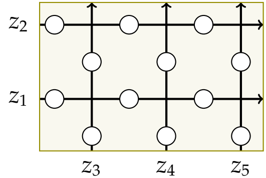
Based on joint work with Xiao-yue Sun [arXiv:2211.10702]

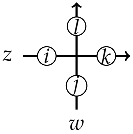
Lattice spin model in classical statistical mechanics:



Spectral parameters $z_1, z_2, \ldots, z_{m+n} \in \mathbb{C}$ assigned to lines

Spin variables $\bigcirc \in \{1, 2, \ldots, N\}$ on edges interact at vertices.

Energy of a spin configuration is a sum of local energies:



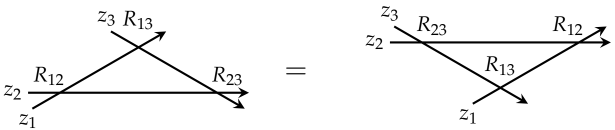
$$\rightsquigarrow \; R(z,w)_{ij}^{kl}\,, \qquad E = -k_B T \sum_{\text{vertices}} \log R$$

R-matrix $R(z,w) = (R(z,w)_{ij}^{kl}) \in \operatorname{End}(V^{\otimes 2})$, $\dim V = N$

For special models, R satisfies the Yang–Baxter equation (YBE)

$$R_{23}(z_2 - z_3)R_{13}(z_2 - z_3)R_{12}(z_1 - z_2) = R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) \in \mathrm{End}(V^{\otimes 3})\,.$$

Equality between two configurations of 3 lines in $\mathbb{R}^2$:



YBE with spectral parameters implies integrability:

- 2D classical lattice model ↔ (1+1)D quantum spin chain
- Commuting conserved charges acting on the Hilbert space, generating the center of the Yangian.

In the past 15 years, YBE has appeared in many supersymmetric quantum field theories (SUSY QFTs):

- 2D $\mathcal{N} = (2,2)$ SUSY gauge theories [Nekrasov–Shatashvili]
- 4D $\mathcal{N} = 2$ SUSY gauge theories [Nekrasov–Shatashvili]
- 3D $\mathcal{N} = 4$ SUSY gauge theories

  [Bullimore–Dimofte–Gaiotto, Braverman–Finkelberg–Nakajima]
- 4D $\mathcal{N} = 1$ SUSY gauge theories

  [Gaiotto–Rastelli–Razamat, Gadde–Gukov, Maruyoshi–Y]
- 4D Chern–Simons theory (= Ω-deformed 6D MSYM)

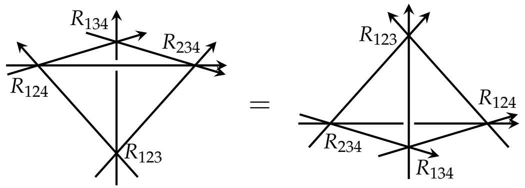
  [Costello, Costello–Witten–Yamazaki]

All of these have realization in string theory and are related by dualities. [Costello–Y]

3D analog of YBE: tetrahedron equation (TE) [Zamolodchikov '80]

$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234}$$

Equality between two configurations of 4 planes in $\mathbb{R}^3$:



Relatively long history. (Before BPZ on 2D CFT!)

Far less developed than YBE. (Only one book [Kuniba '22] on TE!)

But it could be as rich. (Recent progress from the viewpoint of quantized coordinate rings [Kapranov–Voevodsky, Kuniba–Okado, . . .])

Reminder: we live in 3D space!

A well-known solution of TE [Kapranov–Voevodsky, Bazhanov–Sergeev] and its super version [Sergeev, Yoneyama] are conjectured to arise from a brane system in M-theory [Y '22].

Today: my work with Xiao-yue Sun [2211.10702], where we

- constructed solutions of TE using trivial cluster transformations; and
- expressed them as partition functions of 3D $\mathcal{N} = 2$ SUSY gauge theories on S^3.

First time for TE to appear in gauge theory.[1]

Should be related to 3-manifolds ("3D-3D correspondence").

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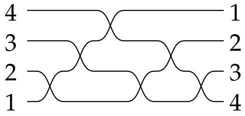
[1] RLLL relations had been found to arise from gauge theories and cluster algebras [Yamazaki '16, Gavrylenko–Semenyakin–Zenkevich '20].

Symmetric group S_n: group of permutations on $\{1, 2, \ldots, n\}$, generated by the adjacent transpositions $\{s_a\}_{a=1}^{n-1}$ satisfying

$$\begin{aligned} & s_a^2 = 1\,, \\ & s_a s_b = s_b s_a \quad \text{for } |a-b| \geq 2 \quad \text{(far commutativity)}\,, \\ & s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1} \qquad \text{(braid relation)}\,. \end{aligned}$$

An expression $s_{a_1} s_{a_2} \cdots s_{a_k}$ can be represented by a wiring diagram. E.g.

$s_1 s_2 s_3 s_1 s_2 s_1 =$

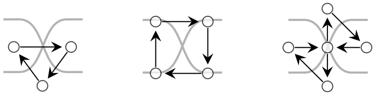


This is a reduced expression for the longest element of S^4

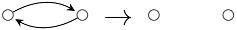
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

To a wiring diagram we assign three quivers, which we call the triangle quiver, square quiver and butterfly quiver:

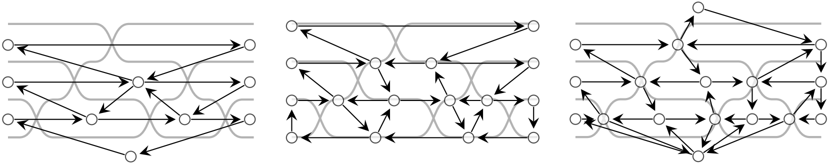
1. Around each crossing place vertices and arrows as follows:



triangle square butterfly

2. Delete 2-cycles: 
3. Label vertices.

E.g. the three quivers assigned to $s_1 s_2 s_3 s_1 s_2 s_1$ (labels omitted):



From now on we only consider the square quiver.

A quiver (or "seed") $\Sigma = (I, \varepsilon)$:

- vertices indexed by a set I
- # arrows $i \xrightarrow{\varepsilon_{ij}} j$ encoded in an antisymmetric matrix ε

A mutation μ_k at $k \in I$ transforms Σ to $\Sigma' = (I, \varepsilon')$:

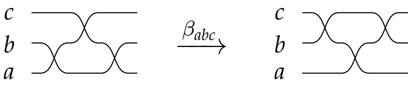
1. For each $i \to k \to j$, draw an arrow $i \to j$.
2. Change $i \to k$ to $i \leftarrow k$.
3. Delete 2-cycles.

An automorphism $\alpha\colon \Sigma \to \Sigma'$ permutes vertices: $\varepsilon'_{\alpha(i)\alpha(j)} = \varepsilon_{ij}$.

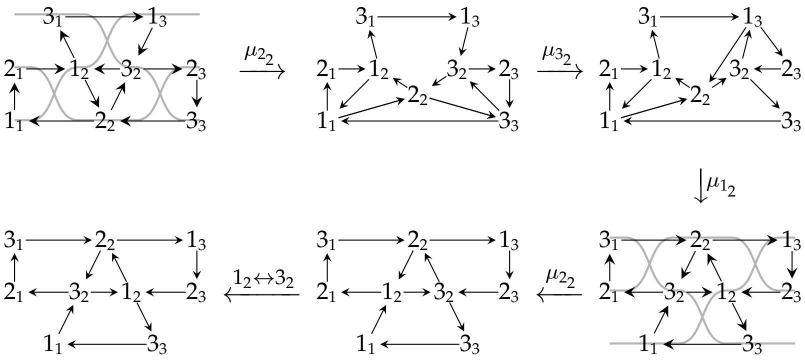
A cluster transformation $\mathbf{c}\colon \Sigma \to \Sigma'$ is a composition of mutations and automorphisms; it can be put in the form

$$\mathbf{c}\colon \Sigma =: \Sigma[1] \xrightarrow{\mu_{k[1]}} \Sigma[2] \xrightarrow{\mu_{k[2]}} \cdots \xrightarrow{\mu_{k[L]}} \Sigma[L+1] \xrightarrow{\alpha} \Sigma' \,.$$

The braid move



induces a cluster transformation on the assigned quiver:

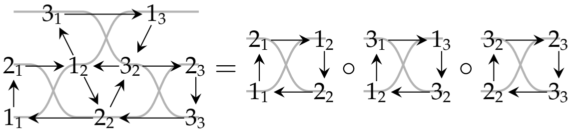


A quiver specifies a 4D $\mathcal{N} = 1$ SUSY gauge theory:

- vertex i: gauge group $\mathrm{SU}(N)_i$
- frozen vertex f: global symmetry group $\mathrm{SU}(N)_f$
- arrow $i \to j$: matter in rep $(\overline{\Box}, \Box)$ of $\mathrm{SU}(N)_i \times \mathrm{SU}(N)_j$

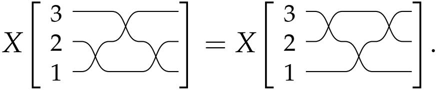
Amalgamation of quivers at frozen vertices
= coupling the corresponding theories by gauging

E.g. the theory specified by $s_1 s_2 s_1$ is constructed from three theories:

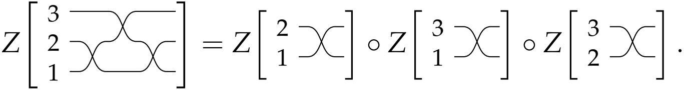


Mutations induce Seiberg duality: theories related by quiver mutations describe the same infrared physics.

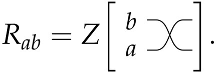
A physical quantity X in dual theories $\mathsf{T} \cong \mathsf{T}^\vee$ gives an equality $X[\mathsf{T}] = X[\mathsf{T}^\vee]$. In particular,



For a nice quantity we have decomposition

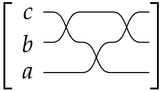


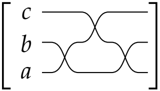
Thus we obtain a solution of YBE



E.g. partition function on $S^1 \times S^3$ gives R with spins in $\mathrm{U}(1)^{N-1}$.

Instead of physical quantities, consider the Hilbert space of states $\mathcal{H}$. Then we get an isomorphism:

$R_{abc} \colon \mathcal{H}$ 

$\xrightarrow{\sim} \mathcal{H}$ 

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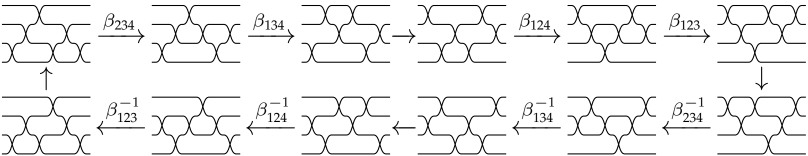
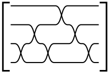
The loop of braid moves

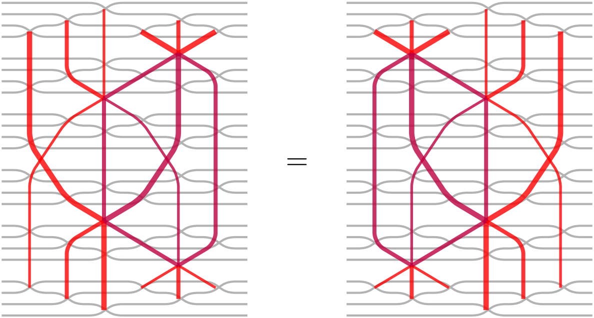


shows

$R_{234}^{-1} R_{134}^{-1} R_{124}^{-1} R_{123}^{-1} R_{234} R_{134} R_{124} R_{123} \in \mathrm{End}\Big(\mathcal{H}$  $\Big)$.

If this endomorphism is trivial, R_{abc} solves TE:

$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234} \, .$$



I don't know if this is the case for 4D SUSY gauge theories.

But this does happen for 1D bosonic QFTs that arise in the context of quantum cluster varieties [Fock–Goncharov].

Quantum torus algebra $\mathbf{D}^q_\Sigma$:

- formal parameter q
- noncommutative variables $(X^q, B^q) = (X^q_i, B^q_i)_{i\in I}$

$$q^{-\varepsilon_{ij}} X^q_i X^q_j = q^{-\varepsilon_{ji}} X^q_j X^q_i\,, \quad q^{-\delta_{ij}} X^q_i B^q_j = q^{\delta_{ij}} B^q_j X^q_i\,, \quad B^q_i B^q_j = B^q_j B^q_i\,.$$

$\mathbf{c}\colon \Sigma \to \Sigma'$ induces a quantum cluster transformation

$$\mathbf{c}^q : \mathbb{D}^q_{\Sigma'} \xrightarrow{\sim} \mathbb{D}^q_\Sigma$$

between the fraction fields of $\mathbf{D}^q_\Sigma$ and $\mathbf{D}^q_{\Sigma'}$.

$\mathbf{c}^q$ quantizes a transition function for the cluster $\mathcal{D}$-variety, equipped with the Poisson structure

$$\{X_i, X_j\} = \varepsilon_{ij} X_i X_j\,, \quad \{X_i, B_j\} = \delta_{ij} X_i B_j\,, \quad \{B_i, B_j\} = 0\,.$$

Heisenberg algebra $\mathbf{H}^{\hbar}_{\Sigma}$:

- formal parameter $\hbar$
- variables $x^{\hbar} = (x^{\hbar}_i)_{i \in I}$, $b^{\hbar} = (b^{\hbar}_i)_{i \in I}$

$$[x^{\hbar}_i, x^{\hbar}_j] = 2\pi \mathrm{i} \hbar \varepsilon_{ij} \,, \quad [x^{\hbar}_i, b^{\hbar}_j] = 2\pi \mathrm{i} \hbar \delta_{ij} \,, \quad [b^{\hbar}_i, b^{\hbar}_j] = 0 \,.$$

We have $\mathbf{D}^{q}_{\Sigma} \hookrightarrow \mathbf{H}^{\hbar}_{\Sigma}$ by

$$q = \exp(\pi \mathrm{i} \hbar) \,, \quad X^{q}_i = \exp(x^{\hbar}_i) \,, \quad B^{q}_i = \exp(b^{\hbar}_i) \,.$$

$\mathbf{H}^{\hbar}_{\Sigma}$ can be represented on the Hilbert space $\mathcal{H}_{\Sigma} = L^2(\mathbb{R}^I)$:

$$\widehat{\ } : x^{\hbar}_i \mapsto \hat{x}_i := \pi \mathrm{i} \hbar \frac{\partial}{\partial a_i} - \sum_{j \in I} \varepsilon_{ij} a_j \,, \quad b^{\hbar} \mapsto \hat{b}_i := 2 a_i \,.$$

For $\hbar \in \mathbb{R}_{>0}$, this assigns a quantum mechanical system to Σ.

$\mathbf{c}\colon \Sigma \to \Sigma'$ induces duality:

- States are mapped by a unitary operator $\mathbf{K_c}\colon \mathcal{H}_{\Sigma'} \to \mathcal{H}_{\Sigma}$.
- Operators are mapped by $\mathbf{K_c}\widehat{A}\mathbf{K_c}^{-1} = \widehat{\mathbf{c}^q(A)}$.

For an automorphism $\mathbf{c} = \alpha$, $\mathbf{K}_\alpha$ relabels coordinates.

For a mutation $\mathbf{c} = \mu_k$, there are two expressions [Kim '21]:

$$\mathbf{K}_{\mu_k} := \mathbf{K}^{\sharp(+)}_{\mu_k}\mathbf{K}'^{(+)}_{\mu_k} = \mathbf{K}^{\sharp(-)}_{\mu_k}\mathbf{K}'^{(-)}_{\mu_k} \,.$$

$\mathbf{K}'^{(\epsilon)}_{\mu_k}\colon \mathcal{H}_{\Sigma'} \to \mathcal{H}_{\Sigma}$ is given by

$$a'_i = \begin{cases} -a_k + \sum_{j\in I}[-\epsilon\varepsilon_{kj}]_+ a_j & \text{if } i = k\,; \\ a_i & \text{if } i \neq k\,. \end{cases}$$

$\mathbf{K}^{\sharp(\epsilon)}_{\mu_k}\colon \mathcal{H}_{\Sigma} \to \mathcal{H}_{\Sigma}$ is a product of *two* noncompact q-dilogs:

$$\mathbf{K}^{\sharp(\epsilon)}_{\mu_k} = \Phi^{\hbar}(\epsilon\hat{x}_k)^{\epsilon}\Phi^{\hbar}(\epsilon\hat{\tilde{x}}_k)^{-\epsilon}\,, \quad \hat{\tilde{x}}_i := \pi \mathrm{i}\hbar\frac{\partial}{\partial a_i} + \sum_{j\in I}\varepsilon_{ij}a_j\,.$$

Let's say $\mathbf{c}\colon \Sigma \to \Sigma$ is trivial if $\mathbf{c}^q = \mathrm{id}_{\mathbb{D}^q_\Sigma}$.

If $\mathbf{c}$ is trivial, $\mathbf{K}_\mathbf{c}$ commutes with $\widehat{X}$ and $\widehat{B}$ by construction.

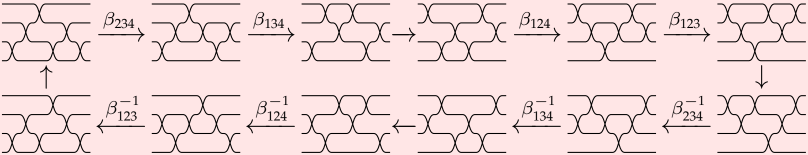
It turns out that $\mathbf{K}_\mathbf{c}$ also commutes with $\widehat{X}^{1/\hbar}$ and $\widehat{B}^{1/\hbar}$, and this implies $\mathbf{K}_\mathbf{c} = \lambda_\mathbf{c}\, \mathrm{id}_{\mathcal{H}_\Sigma}$ for some $\lambda_\mathbf{c} \in \mathrm{U}(1)$ [Fock–Goncharov].

Kim showed $\lambda_\mathbf{c} = 1$ for some important cases. In fact,

If $\mathbf{c}\colon \Sigma \to \Sigma$ is trivial, then $\mathbf{K}_\mathbf{c} = \mathrm{id}_{\mathcal{H}_\Sigma}$.

For $\mathbf{c}\colon \Sigma =: \Sigma[1] \xrightarrow{\mu_{k[1]}} \Sigma[2] \xrightarrow{\mu_{k[2]}} \cdots \xrightarrow{\mu_{k[L]}} \Sigma[L+1] \xrightarrow{\alpha} \Sigma$, we have a choice of signs $(\epsilon[1], \epsilon[2], \dots, \epsilon[L])$. For the tropical sign sequence, the theorem reduces to a noncompact q-dilog identity [Kashaev–Nakanishi] times its complex conjugate.

For the triangle, square and butterfly quivers, the loop



gives rise to a trivial cluster transformation [Sun–Y].

Need only check that **c** acts trivially on the tropical variables [Inoue–Iyama–Keller–Kuniba–Nakanishi].

Therefore, $R_{abc} := \mathbf{K}_{\beta_{abc}}$ solves TE.

The matrix element $\langle a'|\mathbf{K_c}|a\rangle$ can be expressed as an integral of q-dilogs [Kashaev–Nakanishi].

The result coincides with an expression of the partition function of a 3D $\mathcal{N} = 2$ SUSY gauge theory on the squashed 3-sphere

$$S_b^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid b|z_1|^2 + b^{-1}|z_2|^2 = 1\}, \quad b = \sqrt{\hbar}\,.$$

Similar to [Terashima–Yamazaki] but we have twice as many q-dilogs.

This theory is a domain wall in 4D $\mathcal{N} = 2$ SUSY theories.

We expect that TE holds at the level of domain walls.

Related to 3-manifolds, built from tetrahedra attached to triangulated surfaces.

TE is a 3D analog of YBE, important but not well-understood.

YBE & TE (& their higher-dimensional analogs) can be understood in terms of S_n, or wiring diagrams:

- For YBE, R-matrices are adjacent transpositions, satisfying $s_1 s_2 s_1 = s_2 s_1 s_2$.
- For TE, R-matrices are braid moves $s_a s_{a+1} s_a \to s_{a+1} s_a s_{a+1}$.

To wiring diagrams we can assign quivers and QFTs such that braid moves are translated to mutations and dualities:

- Partition functions of dual 4D theories give rise to YBE.
- Isomorphisms between Hilbert spaces of dual Fock–Goncharov QM systems are solutions of TE.

These solutions of TE can be identified with S^3 partition functions of 3D SUSY gauge theories.

Can we produce more solutions of TE?

Can we reproduce known solutions?

Can we understand solutions from 3-manifold viewpoint?

Can we relate this story to wall-crossing of BPS particles in 4D $\mathcal{N} = 2$ SUSY QFTs?

Can we say anything about realistic 3D statistical mechanics systems?